

Weighted Ostrowski type inequalities for functions with one point of nondifferentiability

A. AGLIĆ ALJINOVIĆ^{a,*}, J. PEČARIĆ^b, S. TIPURIĆ-SPUŽEVIĆ^c

^a Department of Applied Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia

^b Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28A, 10000 Zagreb, Croatia

^c Faculty of Science and Education, University of Mostar, Matice Hrvatske BB, 88000 Mostar, Bosnia and Herzegovina

Received 23 May 2012; revised 23 July 2013; accepted 20 August 2013

Available online 31 August 2013

Abstract. We present a weighted generalization involving derivatives of arbitrary order of the recently obtained Ostrowski type inequality for functions with one point of nondifferentiability.

Keywords: Ostrowski type inequality; Sonin's identity

2000 Mathematics Subject Classifications: 26D15; 26D20; 26D99

1. INTRODUCTION

The well known **Ostrowski inequality** states:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty}. \quad (1.1)$$

It holds for every $x \in [a, b]$ whenever $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with bounded derivative. Ostrowski proved it in 1938 in [15] and since

* Corresponding author. Tel.: +385 16129958.

E-mail addresses: andrea.aglic@fer.hr (A. Aglić Aljinović), pecaric@hazu.hr (J. Pečarić), sanja.spuzevic@tel.net.ba (S. Tipurić-Spužević).

Peer review under responsibility of King Saud University.



then it has been generalized in a number of ways. Over the last decades some new inequalities of this type have been intensively considered and applied in Numerical analysis and Probability (see for instance [1,3,5–10,12] or monographs [2,11,13]).

M. Niezgoda in the recent paper [14] considered Ostrowski type inequalities for continuous functions with possibly one point of nondifferentiability. For $c_0 \in [a,b]$, let $\mathcal{D}(c_0)$ be the class of all continuous functions $f: [a,b] \rightarrow \mathbb{R}$ differentiable on the set $\langle a, c_0 \rangle \cup \langle c_0, b \rangle$ and such that

$$M_l = \sup_{x \in (a, c_0)} |f'(x)| < \infty \text{ and } M_r = \sup_{x \in (c_0, b)} |f'(x)| < \infty.$$

In case $c_0 = a$ (resp. $c_0 = b$) we set $M_l = 0$ (resp. $M_r = 0$). M. Niezgoda in [14] established the following Ostrowski type inequality:

Theorem 1. *Let $f: I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior \dot{I} of I , and let $[a, b] \subset \dot{I}$. Suppose that $f' \in \mathcal{D}(c_0)$ for some $c_0 \in [a,b]$. Denote*

$$K_l = \sup_{x \in (a, c_0)} |f''(x)| < \infty \text{ and } K_r = \sup_{x \in (c_0, b)} |f''(x)| < \infty,$$

where for $K_l = 0$ (resp. $K_r = 0$) if $c_0 = a$ (resp. $c_0 = b$). Then for $x \in [a, b]$ we have the following three inequalities

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} \right| \\ & \leq \begin{cases} \frac{b-a}{2^{(p+1)^{1/p}(q+1)^{1/q}}} \{ [K_l(c_0 - a)]^p \frac{c_0-a}{b-a} + [K_r(b - c_0)]^p \frac{b-c_0}{b-a} \}^{1/p} & \text{if } 1 < p < \infty, \\ \frac{1}{4} [K_l(c_0 - a)^2 + K_r(b - c_0)^2] & \text{if } p = 1, \\ \frac{1}{4} (b - a) \max \{ K_l(c_0 - a), K_r(b - c_0) \} & \text{if } p = \infty. \end{cases} \end{aligned}$$

where $1/p + 1/q = 1$.

The aim of this paper is to give a weighted generalization of Theorem 1 involving derivatives of the function f of arbitrary order. This will be done using the following extension of Montgomery identity via the Taylor’s formula recently obtained in [4]:

let $f: I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. If $w: [a,b] \rightarrow [0, \infty)$ is some normalized weighted function. Then the following identity holds

$$\begin{aligned} f(x) &= \int_a^b w(t)f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(t)(t-x)^{i+1} ds \\ &+ \int_a^b K_w^n(x, t) f^{(n)}(t) dt \end{aligned} \tag{1.2}$$

where

$$K_w^n(x, t) = \begin{cases} \frac{1}{(n-1)!} \int_a^t w(u)(u-t)^{n-1} du, & a \leq t \leq x, \\ -\frac{1}{(n-1)!} \int_t^b w(u)(u-t)^{n-1} du, & x < t \leq b. \end{cases}$$

Since we assume $\sum_{i=0}^{-1} = 0$, for $n = 1$, (1.2) reduces to the **weighted Montgomery identity** obtained by J. Pečarić in [16] which states

$$f(x) = \int_a^b w(t) f(t) dt + \int_a^b P_w(x, t) f'(t) dt \tag{1.3}$$

where $P_w(x, t)$ is the weighted Peano kernel

$$P_w(x, t) = \begin{cases} \int_a^t w(u) du, & a \leq t \leq x, \\ \int_a^t w(u) du - 1, & x < t \leq b. \end{cases} \tag{1.4}$$

Here and hereafter the symbol $L^p_{[a, b]}$ ($p \geq 1$) denotes the space of p -power integrable functions on the interval $[a, b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$$

and $L^\infty_{[a, b]}$ denotes the space of essentially bounded functions on $[a, b]$ with the norm

$$\|f\|_\infty = \text{ess sup}_{t \in [a, b]} |f(t)|.$$

2. WEIGHTED GENERALIZATION INVOLVING DERIVATIVES OF THE FUNCTION f OF ARBITRARY ORDER

We denote

$$T_w^n(f; x) = f(x) - \int_a^b w(t) f(t) dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(t) (t-x)^{i+1} dt.$$

In the next theorem we will obtain a bound for the $T_w^n(f; x)$ for functions f such that $f^{(n-1)} \in \mathcal{D}(c_0)$.

Theorem 2. *Let $f: I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior \mathring{I} of I , and let $[a, b] \subset \mathring{I}$. Suppose that $f^{(n-1)} \in \mathcal{D}(c_0)$ for some $c_0 \in [a, b]$ and $w: [a, b] \rightarrow [0, \infty)$ is a normalized weighted function. Denote*

$$M_l(n) = \sup_{x \in (a, c_0)} |f^{(n)}(x)| < \infty \text{ and } M_r(n) = \sup_{x \in (c_0, b)} |f^{(n)}(x)| < \infty.$$

Then for $x \in [a, b]$ we have

$$|T_w^n(f; x)| \leq \begin{cases} \|K_w^n(x, \cdot)\|_q [M_l(n)^p (c_0 - a) + M_r(n)^p (b - c_0)]^{1/p} \\ \|K_w^n(x, \cdot)\|_1 \max\{M_l(n), M_r(n)\} \end{cases} \tag{2.1}$$

where $1 \leq p < \infty$ and $1/p + 1/q = 1$.

Proof. Since $f^{(n-1)}$ is absolutely continuous, from (1.2) we have

$$T_w^n(f; x) = \int_a^b K_w^n(x, t)f^{(n)}(t)dt = \int_a^{c_0} K_w^n(x, t)f^{(n)}(t)dt + \int_{c_0}^b K_w^n(x, t)f^{(n)}(t)dt.$$

By taking the absolute value and applying triangle inequality and Hölder inequality we get

$$\begin{aligned} |T_w^n(f; x)| &\leq \left| \int_a^{c_0} K_w^n(x, t)f^{(n)}(t)dt \right| + \left| \int_{c_0}^b K_w^n(x, t)f^{(n)}(t)dt \right| \\ &\leq \|K_w^n(x, \cdot)\|_{q, [a, c_0]} \|f^{(n)}\|_{p, [a, c_0]} + \|K_w^n(x, \cdot)\|_{q, [c_0, b]} \|f^{(n)}\|_{p, [c_0, b]}. \end{aligned}$$

In case $1 \leq p < \infty$ we apply discrete Hölder inequality

$$\begin{aligned} &\|K_w^n(x, \cdot)\|_{q, [a, c_0]} \|f^{(n)}\|_{p, [a, c_0]} + \|K_w^n(x, \cdot)\|_{q, [c_0, b]} \|f^{(n)}\|_{p, [c_0, b]} \\ &\leq \|K_w^n(x, \cdot)\|_{q, [a, c_0]} M_l(n)(c_0 - a)^{1/p} + \|K_w^n(x, \cdot)\|_{q, [c_0, b]} M_r(n)(b - c_0)^{1/p} \\ &\leq [M_l(n)^p(c_0 - a) + M_r(n)^p(b - c_0)]^{1/p} \\ &\quad \cdot \left[\left(\|K_w^n(x, \cdot)\|_{q, [a, c_0]} \right)^q + \left(\|K_w^n(x, \cdot)\|_{q, [c_0, b]} \right)^q \right]^{1/q} \\ &= [M_l(n)^p(c_0 - a) + M_r(n)^p(b - c_0)]^{1/p} \cdot \|K_w^n(x, \cdot)\|_q. \end{aligned}$$

Thus we have

$$|T_w^n(f; x)| \leq [M_l(n)^p(c_0 - a) + M_r(n)^p(b - c_0)]^{1/p} \cdot \|K_w^n(x, \cdot)\|_q.$$

In case $p = \infty$ ($q = 1$) we have

$$\begin{aligned} &\|K_w^n(x, \cdot)\|_{1, [a, c_0]} \|f^{(n)}\|_{\infty, [a, c_0]} + \|K_w^n(x, \cdot)\|_{1, [c_0, b]} \|f^{(n)}\|_{\infty, [c_0, b]} \\ &\leq \left[\|K_w^n(x, \cdot)\|_{1, [a, c_0]} + \|K_w^n(x, \cdot)\|_{1, [c_0, b]} \right] \cdot \max\{M_l(n), M_r(n)\} \\ &= \|K_w^n(x, \cdot)\|_1 \max\{M_l(n), M_r(n)\} \end{aligned}$$

and

$$|T_w^n(f; x)| \leq \max\{M_l(n), M_r(n)\} \|K_w^n(x, \cdot)\|_1$$

which completes the proof. \square

If we take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ in the last theorem we obtain the next corollary.

Corollary 1. *Suppose that all assumptions of Theorem 2 hold. Then for $x \in [a, b]$ we have*

$$|T^n(f; x)| \leq \begin{cases} \|K^n(x, \cdot)\|_q [M_l(n)^p(c_0 - a) + M_r(n)^p(b - c_0)]^{1/p} \\ \|K^n(x, \cdot)\|_1 \max\{M_l(n), M_r(n)\} \end{cases} \tag{2.2}$$

where $1 \leq p < \infty$ and $1/p + 1/q = 1$,

$$T^n(f; x) = f(x) - \frac{1}{(b-a)} \int_a^b f(t)dt + \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)}$$

and

$$K^n(x, t) = \begin{cases} \frac{-1}{n!(b-a)}(a-t)^n, & a \leq t \leq x, \\ \frac{-1}{n!(b-a)}(b-t)^n, & x < t \leq b. \end{cases}$$

Corollary 2. *Suppose that all assumptions of Theorem 2 hold. Then for $x \in [a, b]$ we have*

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} [M_l(1)^p(c_o - a) + M_r(1)^p(b - c_o)]^{1/p}, \end{aligned}$$

where $1/p + 1/q = 1$, while for $p = 1$ we have

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt \right| \\ & \leq \left(\frac{1}{2} + \frac{1}{b-a} \left| x - \frac{b+a}{2} \right| \right) [M_l(1)(c_o - a) + M_r(1)(b - c_o)], \end{aligned}$$

and for $p = \infty$

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt \right| \\ & \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \max\{M_l(1), M_r(1)\}. \end{aligned}$$

Proof. We take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a, b]$, and $n = 1$ in (2.2). Thus we have

$$\begin{aligned} T^1(f; x) &= f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt \\ K^1(x, t) &= \begin{cases} \frac{1}{b-a}(t-a), & a \leq t \leq x, \\ \frac{1}{b-a}(t-b) & x < t \leq b. \end{cases} \end{aligned}$$

and for $1 \leq q < \infty$

$$\begin{aligned} \|K^1(x, \cdot)\|_q &= \frac{1}{b-a} \left(\int_a^x |t-a|^q dt + \int_x^b |b-t|^q dt \right)^{\frac{1}{q}} \\ &= \frac{1}{b-a} \left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)} \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\|K^1(x, \cdot)\|_\infty = \frac{1}{b-a} \max\{x-a, b-x\} = \frac{1}{2} + \frac{1}{b-a} \left| x - \frac{b+a}{2} \right|.$$

Taking all this in (2.2) the proof follows. \square

Corollary 3. *Suppose that all assumptions of Theorem 2 hold. Then for $x \in [a, b]$ we have*

$$\begin{aligned} & \left| \frac{1}{(b-a)} \int_a^b f(t)dt - f(x) + f'(x) \frac{(b-x)^2 - (a-x)^2}{2(b-a)} \right| \\ & \leq \frac{1}{2(b-a)} \left(\frac{(x-a)^{2q+1} + (b-x)^{2q+1}}{(2q+1)} \right)^{\frac{1}{q}} [M_l(2)^p (c_o - a) + M_r(2)^p (b - c_o)]^{1/p}, \end{aligned}$$

where $1/p + 1/q = 1$, while for $p = 1$ we have

$$\begin{aligned} & \left| \frac{1}{(b-a)} \int_a^b f(t)dt - f(x) + f'(x) \frac{(b-x)^2 - (a-x)^2}{2(b-a)} \right| \\ & \leq \frac{1}{2(b-a)} \max \left\{ (x-a)^2, (b-x)^2 \right\} [M_l(2)(c_o - a) + M_r(2)(b - c_o)] \end{aligned}$$

and for $p = \infty$

$$\begin{aligned} & \left| \frac{1}{(b-a)} \int_a^b f(t)dt - f(x) + f'(x) \frac{(b-x)^2 - (a-x)^2}{2(b-a)} \right| \\ & \leq \frac{(x-a)^3 + (b-x)^3}{6(b-a)} \max \{ M_l(2), M_r(2) \}. \end{aligned}$$

Proof. We take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a, b]$, and $n = 2$ in (2.2). Thus we have

$$T^2(f; x) = f(x) - \frac{1}{(b-a)} \int_a^b f(t)dt + f'(x) \frac{(b-x)^2 - (a-x)^2}{2(b-a)},$$

$$K^2(x, t) = \begin{cases} \frac{-1}{2(b-a)} (a-t)^2, & a \leq t \leq x, \\ \frac{-1}{2(b-a)} (b-t)^2, & x < t \leq b. \end{cases}$$

and for $1 \leq q < \infty$

$$\begin{aligned} \|K^2(x, \cdot)\|_q &= \frac{1}{2(b-a)} \left(\int_a^x |t-a|^{2q} dt + \int_x^b |b-t|^{2q} dt \right)^{\frac{1}{q}} \\ &= \frac{1}{2(b-a)} \left(\frac{(x-a)^{2q+1} + (b-x)^{2q+1}}{(2q+1)} \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\|K^2(x, \cdot)\|_\infty = \frac{1}{2(b-a)} \max \left\{ (x-a)^2, (b-x)^2 \right\}.$$

Taking all this in (2.2) the proof follows. \square

Our next goal is to obtain a bound for

$$T_w^n(f; x) - \frac{1}{b-a} \int_a^b K_w^n(x, t) dt \int_a^b f^{(n)}(t) dt$$

for functions f such that $f^{(n)} \in \mathcal{D}(c_0)$. We will show that this is a weighted generalization of Theorem 1 involving derivatives of the function f of arbitrary order.

For that purpose we shall need **Sonin's identity**(see [17])

$$\begin{aligned} & \int_a^b w(t)f(t)g(t)dt - \left(\int_a^b w(t)f(t)dt \right) \left(\int_a^b w(t)g(t)dt \right) \\ &= \int_a^b w(t)(g(t) - \lambda) \left(f(t) - \int_a^b w(t)f(t)dt \right) dt \end{aligned}$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are two Lebesgue integrable functions, $w:[a,b] \rightarrow [0,\infty)$ is some normalized weighted function and λ is an arbitrary real number.

Theorem 3. *Let $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior \dot{I} of I , and let $[a, b] \subset \dot{I}$. Suppose that $f^{(n)} \in \mathcal{D}(c_0)$ for some $c_0 \in [a, b]$ and $w:[a, b] \rightarrow [0, \infty)$ is a normalized weighted function. Then for $x \in [a, b]$ and $1 \leq p < \infty$ we have*

$$\begin{aligned} & \left| T_w^n(f; x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b K_w^n(x, t) dt \right| \\ & \leq \frac{1}{(p+1)^{1/p}} [M_l(n+1)^p (c_0 - a)^{p+1} + M_r(n+1)^p (b - c_0)^{p+1}]^{1/p} \|K_w^n(x, \cdot) - \mu\|_q, \end{aligned} \tag{2.3}$$

where $1/p + 1/q = 1$, while for $p = \infty$ we have

$$\begin{aligned} & \left| T_w^n(f; x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b K_w^n(x, t) dt \right| \\ & \leq \max \{ M_l(n+1)(c_0 - a), M_r(n+1)(b - c_0) \} \|K_w^n(x, \cdot) - \mu\|_1 \end{aligned} \tag{2.4}$$

where $\mu = \frac{1}{b-a} \int_a^b K_w^n(x, t) dt$.

Proof. By using the identity (1.2) we have

$$\begin{aligned} & T_w^n(f; x) - \frac{1}{b-a} \int_a^b K_w^n(x, t) dt \int_a^b f^{(n)}(t) dt \\ &= \int_a^b K_w^n(x, t) f^{(n)}(t) dt - \frac{1}{b-a} \int_a^b K_w^n(x, t) dt \int_a^b f^{(n)}(t) dt \end{aligned}$$

and further by applying the Sonin’s identity for the functions $K_w^n(x, t)$ and $f^{(n)}(t)$ with $\lambda = f^{(n)}(c_0)$ and $w(t) = \frac{1}{b-a}, t \in [a, b]$, we obtain

$$\begin{aligned} & \int_a^b K_w^n(x, t) f^{(n)}(t) dt - \frac{1}{b-a} \int_a^b K_w^n(x, t) dt \int_a^b f^{(n)}(t) dt \\ &= \int_a^b (f^{(n)}(t) - f^{(n)}(c_0)) \left(K_w^n(x, t) - \frac{1}{b-a} \int_a^b K_w^n(x, t) dt \right) dt \\ &= \int_a^{c_0} (f^{(n)}(t) - f^{(n)}(c_0)) (K_w^n(x, t) - \mu) dt \\ & \quad + \int_{c_0}^b (f^{(n)}(t) - f^{(n)}(c_0)) (K_w^n(x, t) - \mu) dt. \end{aligned}$$

It is obvious that $\int_a^b f^{(n)}(t) dt = f^{(n-1)}(b) - f^{(n-1)}(a)$. By taking the absolute value and then applying the triangle inequality and Hölder’s inequality we get

$$\begin{aligned} & \left| T_w^n(f; x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b K_w^n(x, t) dt \right| \\ & \leq \left| \int_a^{c_0} (f^{(n)}(t) - f^{(n)}(c_0)) (K_w^n(x, t) - \mu) dt \right| \\ & \quad + \left| \int_{c_0}^b (f^{(n)}(t) - f^{(n)}(c_0)) (K_w^n(x, t) - \mu) dt \right| \\ & \leq \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [a, c_0]} \|K_w^n(x, \cdot) - \mu\|_{q, [a, c_0]} \\ & \quad + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [c_0, b]} \|K_w^n(x, \cdot) - \mu\|_{q, [c_0, b]}. \end{aligned}$$

In case $1 \leq p < \infty$ we apply discrete Hölder inequality

$$\begin{aligned} & \left| T_w^n(f; x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b K_w^n(x, t) dt \right| \\ & \leq \left[\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [a, c_0]}^p + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [c_0, b]}^p \right]^{\frac{1}{p}} \\ & \quad \cdot \left[\|K_w^n(x, \cdot) - \mu\|_{q, [a, c_0]}^q + \|K_w^n(x, \cdot) - \mu\|_{q, [c_0, b]}^q \right]^{\frac{1}{q}}. \end{aligned}$$

Since the mean value theorem guarantees the existence of $\xi \in \langle t, c_0 \rangle$ such that $f^{(n)}(t) - f^{(n)}(c_0) = f^{(n+1)}(\xi)(t - c_0)$ we have

$$f^{(n)}(t) - f^{(n)}(c_0) = M_l(n+1)(t - c_0)$$

and thus

$$\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [a, c_0]}^p \leq M_l(n+1)^p \int_a^{c_0} (c_0 - t)^p dt = M_l(n+1)^p \frac{(c_0 - a)^{p+1}}{p+1}.$$

The same reasoning leads to

$$\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [c_0, b]}^p \leq M_r(n+1)^p \int_{c_0}^b (t - c_0)^p dt = M_r(n+1)^p \frac{(b - c_0)^{p+1}}{p+1}.$$

Finally for $1 < q < \infty$ we have

$$\left[\|K_w^n(x, \cdot) - \mu\|_{q,[a,c_0]}^q + \|K_w^n(x, \cdot) - \mu\|_{q,[c_0,b]}^q \right]^{\frac{1}{q}} = \|K_w^n(x, \cdot) - \mu\|_{q,[a,b]}$$

and

$$\begin{aligned} \|K_w^n(x, \cdot) - \mu\|_{\infty,[a,c_0]} &\leq \|K_w^n(x, \cdot) - \mu\|_{\infty,[a,b]} \\ \|K_w^n(x, \cdot) - \mu\|_{\infty,[c_0,b]} &\leq \|K_w^n(x, \cdot) - \mu\|_{\infty,[a,b]}, \end{aligned}$$

which implies the first inequality.

In case $p = \infty$

$$\begin{aligned} &\left| T_w^n(f; x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a} \int_a^b K_w^n(x, t) dt \right| \\ &\leq \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{\infty,[a,c_0]} \|K_w^n(x, \cdot) - \mu\|_{1,[a,c_0]} \\ &\quad + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{\infty,[c_0,b]} \|K_w^n(x, \cdot) - \mu\|_{1,[c_0,b]} \\ &\leq M_l(n+1)(c_0 - a) \|K_w^n(x, \cdot) - \mu\|_{1,[a,c_0]} \\ &\quad + M_r(n+1)(b - c_0) \|K_w^n(x, \cdot) - \mu\|_{1,[c_0,b]} \\ &\leq \max\{M_l(n+1)(c_0 - a), M_r(n+1)(b - c_0)\} \|K_w^n(x, \cdot) - \mu\|_{1,[a,b]} \end{aligned}$$

and the second inequality is proved too. \square

Corollary 4. *Let $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior \dot{I} of I , and let $[a, b] \subset \dot{I}$. Suppose that $f' \in \mathcal{D}(c_0)$ for some $c_0 \in [a, b]$. Then if $x \in [a, b]$ and $1 < p < \infty$ we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \\ &\leq \frac{(b-a)^{1/q}}{2(p+1)^{1/p}(q+1)^{1/q}} \left[M_l(2)^p (c_0 - a)^{p+1} + M_r(2)^p (b - c_0)^{p+1} \right]^{1/p}, \end{aligned} \tag{2.5}$$

while for $p = 1$ we have

$$\begin{aligned} &\left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \\ &\leq \frac{1}{4} \left[M_l(2)(c_0 - a)^2 + M_r(2)(b - c_0)^2 \right], \end{aligned} \tag{2.6}$$

and for $p = \infty$ we have

$$\begin{aligned} &\left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \\ &\leq \frac{1}{4} (b-a) \max\{M_l(2)(c_0 - a), M_r(2)(b - c_0)\}. \end{aligned} \tag{2.7}$$

Proof. If we take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ and $n = 1$ in (2.3) we have

$$\begin{aligned} \int_a^b K^1(x, t) dt &= \left(\int_a^x \frac{t-a}{b-a} dt - \int_x^b \frac{b-t}{b-a} dt \right) = \frac{(x-a)^2 - (b-x)^2}{2(b-a)} \\ &= \left(x - \frac{a+b}{2} \right), \end{aligned}$$

$$\mu = \frac{1}{b-a} \int_a^b K^1(x, t) dt = \frac{1}{b-a} \left(x - \frac{a+b}{2} \right),$$

$$\begin{aligned} T^1(f; x) - \frac{f(b) - f(a)}{b-a} \int_a^b K^1(x, t) dt \\ = f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \end{aligned}$$

and for $1 \leq q < \infty$

$$\begin{aligned} \|K^1(x, \cdot) - \mu\|_q &= \left\| K^1(x, \cdot) - \frac{1}{b-a} \left(x - \frac{a+b}{2} \right) \right\|_q \\ &= \frac{1}{(b-a)} \left(\int_a^x \left| (t-a) - \left(x - \frac{a+b}{2} \right) \right|^q dt + \int_x^b \left| (t-b) - \left(x - \frac{a+b}{2} \right) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

First we observe $I_1 = \int_a^x \left| (t-a) - \left(x - \frac{a+b}{2} \right) \right|^q dt$. In case $x \in [a, \frac{a+b}{2}]$ we have

$$I_1 = \int_a^x \left(t - x + \frac{b-a}{2} \right)^q dt = \frac{\left(\frac{b-a}{2} \right)^{q+1} - \left(\frac{a+b}{2} - x \right)^{q+1}}{q+1},$$

while in case $x \in \langle \frac{a+b}{2}, b \rangle$

$$\begin{aligned} I_1 &= \int_a^{x-(b-a)/2} \left(-t + x - \frac{b-a}{2} \right)^q dt + \int_{x-(b-a)/2}^x \left(t - x + \frac{b-a}{2} \right)^q dt \\ &= \frac{\left(x - \frac{a+b}{2} \right)^{q+1} + \left(\frac{b-a}{2} \right)^{q+1}}{q+1}. \end{aligned}$$

Now, observing $I_2 = \int_x^b \left| (t-b) - \left(x - \frac{a+b}{2} \right) \right|^q dt$ in case $x \in [a, \frac{a+b}{2}]$ we have

$$\begin{aligned} I_2 &= \int_x^{x+(b-a)/2} \left(-t + x + \frac{b-a}{2} \right)^q dt + \int_{x+(b-a)/2}^b \left(t - x - \frac{b-a}{2} \right)^q dt \\ &= \frac{\left(\frac{b-a}{2} \right)^{q+1} + \left(\frac{a+b}{2} - x \right)^{q+1}}{q+1}, \end{aligned}$$

while in case $x \in \langle \frac{a+b}{2}, b \rangle$

$$I_2 = \int_x^b \left(-t + x + \frac{b-a}{2} \right)^q dt = \frac{-\left(x - \frac{a+b}{2} \right)^{q+1} + \left(\frac{b-a}{2} \right)^{q+1}}{q+1}.$$

Finally, in both cases we have $I_1 + I_2 = \frac{2}{q+1} \left(\frac{b-a}{2}\right)^{q+1}$ and (2.5) is proved. Since all this is also valid for $q = 1$ ($p = \infty$) we have

$$\|K^1(x, \cdot) - \mu\|_1 = \frac{1}{4}(b - a)$$

and (2.4) implies (2.7). Last case for $q = \infty$ ($p = 1$)

$$\begin{aligned} \|K^1(x, \cdot) - \mu\|_\infty &= \sup_{t \in [a,b]} |K^1(x, t) - \mu| \\ &= \frac{1}{(b-a)} \max \left\{ \sup_{t \in [a,x]} \left| (t-a) - \left(x - \frac{a+b}{2}\right) \right|, \sup_{t \in [x,b]} \left| (t-b) - \left(x - \frac{a+b}{2}\right) \right| \right\} \\ &= \frac{1}{(b-a)} \max \left\{ \left(x - \frac{a+b}{2}\right), \frac{b-a}{2} \right\} = \frac{1}{2} \end{aligned}$$

follows from (2.5). \square

Remark 1. Inequalities from the last Corollary are identical to those from the Theorem 1 since

$$\begin{aligned} &\frac{(b-a)^{1/q}}{2(p+1)^{1/p}(q+1)^{1/q}} \left[M_l(2)^p(c_0 - a)^{p+1} + M_r(2)^p(b - c_0)^{p+1} \right]^{1/p} \\ &= \frac{(b-a)^{1-1/p}}{2(p+1)^{1/p}(q+1)^{1/q}} \left[M_l(2)^p(c_0 - a)^{p+1} + M_r(2)^p(b - c_0)^{p+1} \right]^{1/p} \\ &= \frac{b-a}{2(p+1)^{1/p}(q+1)^{1/q}} \left\{ [M_l(2)(c_0 - a)]^p \frac{c_0 - a}{b-a} + [M_r(2)(b - c_0)]^p \frac{b - c_0}{b-a} \right\}^{1/p}. \end{aligned}$$

The last goal is to obtain a bound for

$$T_w^n(f; x) - f^{(n)}(c_0) \int_a^b K_w^n(x, t) dt$$

for functions f such that $f^{(n)} \in \mathcal{D}(c_0)$.

Theorem 4. Suppose that all assumptions of the Theorem 3 hold. Then for $x \in [a,b]$ we have

$$\begin{aligned} &\left| T_w^n(f; x) - f^{(n)}(c_0) \int_a^b K_w^n(x, t) dt \right| \\ &\leq \left\{ \frac{1}{(p+1)^{1/p}} \left[M_l(n+1)^p(c_0 - a)^{p+1} + M_r(n+1)^p(b - c_0)^{p+1} \right]^{1/p} \|K_w^n(x, \cdot)\|_q \right. \\ &\quad \left. \max \{ M_l(n+1)(c_0 - a), M_r(n+1)(b - c_0) \} \|K_w^n(x, \cdot)\|_1 \right\} \end{aligned} \tag{2.8}$$

Proof. By applying the identity (1.2) we have

$$T_w^n(f; x) - f^{(n)}(c_0) \int_a^b K_w^n(x, t) dt = \int_a^b K_w^n(x, t) (f^{(n)}(t) - f^{(n)}(c_0)) dt$$

and further by taking the absolute value, applying the triangle inequality and Hölder’s inequality we get

$$\begin{aligned} & \left| T_w^n(f; x) - f^{(n)}(c_0) \int_a^b K_w^n(x, t) dt \right| \\ & \leq \left| \int_a^{c_0} K_w^n(x, t) (f^{(n)}(t) - f^{(n)}(c_0)) dt \right| + \left| \int_{c_0}^b K_w^n(x, t) (f^{(n)}(t) - f^{(n)}(c_0)) dt \right| \\ & \leq \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [a, c_0]} \|K_w^n(x, \cdot)\|_{q, [a, c_0]} \\ & \quad + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [c_0, b]} \|K_w^n(x, \cdot)\|_{q, [c_0, b]}. \end{aligned}$$

In case $1 \leq p < \infty$ we apply discrete Hölder’s inequality

$$\begin{aligned} & \left| T_w^n(f; x) - f^{(n)}(c_0) \int_a^b K_w^n(x, t) dt \right| \\ & \leq \left[\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [a, c_0]}^p + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [c_0, b]}^p \right]^{\frac{1}{p}} \\ & \quad \cdot \left[\|K_w^n(x, \cdot)\|_{q, [a, c_0]}^q + \|K_w^n(x, \cdot)\|_{q, [c_0, b]}^q \right]^{\frac{1}{q}}. \end{aligned}$$

The same reasoning as in the proof of the Theorem 3 leads to

$$\begin{aligned} & \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [a, c_0]}^p + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p, [c_0, b]}^p \\ & \leq M_l(n+1)^p \frac{(c_0 - a)^{p+1}}{p+1} + M_r(n+1)^p \frac{(b - c_0)^{p+1}}{p+1}, \end{aligned}$$

which implies the first inequality. In case $p = \infty$

$$\begin{aligned} & \left| T_w^n(f; x) - f^{(n)}(c_0) \int_a^b K_w^n(x, t) dt \right| \\ & \leq \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{\infty, [a, c_0]} \|K_w^n(x, \cdot)\|_{1, [a, c_0]} \\ & \quad + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{\infty, [c_0, b]} \|K_w^n(x, \cdot)\|_{1, [c_0, b]} \\ & \leq M_l(n+1)(c_0 - a) \|K_w^n(x, \cdot)\|_{1, [a, c_0]} \\ & \quad + M_r(n+1)(b - c_0) \|K_w^n(x, \cdot)\|_{1, [c_0, b]} \\ & \leq \max\{M_l(n+1)(c_0 - a), M_r(n+1)(b - c_0)\} \|K_w^n(x, \cdot)\|_{1, [a, b]} \end{aligned}$$

and the second inequality is proved too. \square

Corollary 5. *Suppose that all assumptions of the Theorem 4 hold. Then if $x \in [a, b]$ and $1 < p < \infty$ we have*

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - f'(c_0) \left(x - \frac{a+b}{2} \right) \right| \\ & \leq \frac{((x-a)^{q+1} + (b-x)^{q+1})^{1/q}}{(b-a)(p+1)^{1/p}(q+1)^{1/q}} [M_l(2)^p (c_0 - a)^{p+1} + M_r(2)^p (b - c_0)^{p+1}]^{1/p}, \end{aligned} \tag{2.9}$$

while for $p = 1$ we have

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - f'(c_0) \left(x - \frac{a+b}{2} \right) \right| \\ & \leq \left(\frac{1}{2} + \frac{1}{b-a} \left| \frac{b+a}{2} - x \right| \right) [M_l(2)(c_0 - a)^2 + M_r(2)(b - c_0)^2] \end{aligned} \tag{2.10}$$

and for $p = \infty$ we have

$$\begin{aligned} & \left| f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt - f'(c_0) \left(x - \frac{a+b}{2} \right) \right| \\ & \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \max\{M_l(2)(c_0 - a), M_r(2)(b - c_0)\}. \end{aligned} \tag{2.11}$$

Proof. If we take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $n = 1$ in (2.8), as in the proof of the Corollary 4 we have

$$\begin{aligned} T^1(f; x) &= f(x) - \frac{1}{(b-a)} \int_a^b f(t) dt, \\ \int_a^b K^1(x, t) dt &= \left(x - \frac{a+b}{2} \right) \end{aligned}$$

and

$$\begin{aligned} \|K^1(x, \cdot)\|_q &= \frac{1}{(b-a)} \left(\int_a^x |(t-a)^q dt + \int_x^b |(t-b)^q dt \right)^{\frac{1}{q}} \\ &= \frac{1}{(b-a)} \left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right)^{\frac{1}{q}}, \end{aligned}$$

which proves (2.9), while

$$\|K^1(x, \cdot)\|_1 = \frac{(x-a)^2 + (b-x)^2}{2(b-a)}$$

proves (2.11) and finally

$$\begin{aligned} \|K^1(x, \cdot)\|_\infty &= \max \left\{ \sup_{t \in [a,x]} \left| \frac{t-a}{b-a} \right|, \sup_{t \in [x,b]} \left| \frac{t-b}{b-a} \right| \right\} \\ &= \max \left\{ \frac{x-a}{b-a}, \frac{b-x}{b-a} \right\} = \left(\frac{1}{2} + \frac{1}{b-a} \left| \frac{b+a}{2} - x \right| \right) \end{aligned}$$

proves (2.10). \square

ACKNOWLEDGEMENT

This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 036-1170889-1054 (first author) and 117-1170889-0888 (second author).

REFERENCES

- [1] G.A. Anastassiou, Univariate Ostrowski inequalities, Revisited, *Monatshefte für Mathematik* 135 (2002) 175–189.
- [2] A. Aglič Aljinović, A. Čivljak, S. Kovač, J. Pečarić, M. Ribičić Penava, *General Integral Identities and Related Inequalities*, Element, Zagreb, 2013.
- [3] A. Aglič Aljinović, M. Matić, J. Pečarić, Improvements of some Ostrowski type inequalities, *J. Comp. Anal. Appl.* 7 (3) (2005) 289–304.
- [4] A. Aglič Aljinović, J. Pečarić, On some Ostrowski type inequalities via Montgomery identity and Taylor’s formula, *Tamkang J. Math.* 36 (3) (2005) 199–218.
- [5] M. Bohner, T. Matthews, Ostrowski inequalities on time scales, *JIPAM* 9 (1) (2008), article 6.
- [6] A. Čivljak, Lj. Dedić, M. Matić, On Ostrowski and Euler–Grüss type inequalities involving measures, *J. Math. Inequal.* 1 (1) (2007) 65–81.
- [7] Lj. Dedić, M. Matić, J. Pečarić, On generalizations of Ostrowski inequality via some Euler-type identities, *Math. Inequal. Appl.* 3 (3) (2000) 337–353.
- [8] S.S. Dragomir, N.S. Barnett, P. Cerone, An Ostrowski type inequality for double integrals in terms of L_p -norms and applications in numerical integration, *Rev. D’anal. Numer. Theor. De L’Approx.* 32 (2) (2003) 161–169.
- [9] S.S. Dragomir, P. Cerone, N.S. Barnett, J. Roumeliotis, An inequality of the Ostrowski type for double integrals and applications for cubature formulae, *Tamsui Oxf. J. Math. Sci.* 16 (1) (2000) 1–16.
- [10] S.S. Dragomir, J.E. Pečarić, S. Wang, The unified treatment of trapezoid, Simpson and Ostrowski type inequalities for monotonic mappings and applications, *Math. Comp. Modelling* 31 (No 6/7) (2001) 61–70.
- [11] I. Franjić, J. Pečarić, I. Perić, A. Vukelić, *Euler Integral Identity, Quadrature Formule and Error Estimations*, Element, Zagreb, 2011.
- [12] M. Matić, J. Pečarić, N. Ujević, Generalizations of weighted version of Ostrowski’s inequality and some related results, *J. Inequal. Appl.* 5 (6) (2000) 639–666.
- [13] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.
- [14] M. Niezgodna, Grüss and Ostrowski type inequalities, *Appl. Math. Comput.* 217 (23) (2011) 9779–9789.
- [15] A. Ostrowski, Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.* 10 (1938) 226–227.
- [16] J. Pečarić, On the Čebyšev inequality, *Bul. Inst. Politehn. Timisoara* 25 (39) (1980) 10–11.
- [17] N.J. Sonin, O nekotoryh neravnenstvah, odnosjascihsja k opredelennym intagralam, *Zap. Imp. Akad. Nauk po fiziko-matem. otd. t. 6* (1898) 1–54.