ARAB JOURNAL OF
MATHEMATICAL SCIENCES

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Arab J Math Sci 23 (2017) 242–257

Weighted modulus S_θ-convergence of order α in probability[☆]

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Received 23 June 2016; received in revised form 19 March 2017; accepted 20 March 2017
Available online 31 March 2017

Abstract. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence, ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $t_n > \delta$, $\forall\, n \in \mathbb{N}$ (where δ is a fixed positive real number) and $T_n = t_1 + t_2 + \cdots + t_n$ (where $n \in \mathbb{N}$ and $T_0 = 0$). A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus S_θ-convergent of order α in probability (where $0 < \alpha \leq 1$) to a random variable X (like Ghosal(2014)) if for any $\varepsilon, \delta > 0$,

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha} \left|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}\right| = 0.$$

The results are applied to build the probability distribution for weighted modulus N_θ-convergence of order α. Also these methods are compared with the convergence of weighted modulus statistical convergence of order α and weighted modulus strong Cesàro convergence of order α respectively. If $\limsup_{r\to\infty} \frac{T_{k_r}}{T^\alpha_{k_{r-1}}} < \infty$, then weighted modulus S_θ-convergence of order α in probability implies weighted modulus statistical convergence of order α in probability and weighted modulus N_θ-convergence of order α implies weighted modulus strong Cesàro convergence of order α in probability except the condition $\limsup_{r\to\infty} \frac{T_{k_r}}{T^\alpha_{k_{r-1}}} = \infty$. So our main objective is to interpret the above exceptional condition and produce a relational behavior of above mentioned four convergences. This is also used to prove the uniqueness

[☆] This research work is supported by Jadavpur University, Kolkata-700032, West Bengal, India.
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Peer review under responsibility of King Saud University.

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http://dx.doi.org/10.1016/j.ajmsc.2017.03.001

of limit value of weighted lacunary statistical convergence and improve the definition of weighted lacunary statistical convergence.

2010 Mathematics Subject Classification: 40A35; 40G15; 60B10

Keywords: Weighted lacunary statistical convergence of order α; weighted modulus statistical convergence of order α; weighted modulus strong Cesàro convergence of order α; weighted modulus S_θ-convergence of order α; weighted modulus N_θ-convergence of order α

1. Introduction

The notion of statistical convergence was introduced by Fast [11], Steinhaus [31] and Schonenberg [28] and other authors independently. Now the idea of statistical convergence has turned out to be one of the most active areas of research in summability theory after the works of Šalát [24], Fridy [13]. It has several generalizations and applications like:

(i) statistical convergence of order α by Çolak [4],
(ii) lacunary statistical convergence by Fridy and Orhan [14,15],
(iii) lacunary statistical convergence of order α by Sengül and Et [30],
(iv) pointwise and uniform statistical convergences of order α by Cinar et al. [5],
(v) $\alpha\beta$-statistical convergence of order γ by Aktuğlu [1],
(vi) weighted statistical convergence of order α by Ghosal [18],
(vii) weighted $\alpha\beta$-statistical convergence of order γ by Ghosal [17],
(viii) statistical convergence for sequence of functions by Balcerzak et al. [2],
(ix) statistical convergence in probability for a sequence of random functions by Şencimen [29],
(x) λ-statistical convergence of order α of sequences of functions Et et al. [10],
(xi) statistical convergence of order α in probability by Das et al. [8],
(xii) probabilistic norm and convergence of random variables by Lafuerza-Guillén et al. [20], and many other, different fields of mathematics.

In another direction, the history of strong p-Cesàro summability, being longer, is not so clear. As per author's knowledge in [6], it has been shown that if a sequence is strongly p-Cesàro summable (for $0 < p < \infty$) to x, then the sequence must be statistically convergent to the same limit. Both the authors Fast [11] and Schonenberg [28] noted that if a bounded sequence is statistically convergent to x, then it is strongly Cesàro summable to x. In [12], the relation between strongly Cesàro summable and N_θ-convergence was established. Also other relations are established.

In particular, in probability theory, a new type of convergence called 'weighted statistical convergence of order α in probability' was introduced in [18], as follows: Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$, for all $n \in \mathbb{N}$ and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space $\mathcal{W}$ (for each n) with respect to a given class of events $\triangle$ and a given probability function $P : \triangle \to \mathbb{R}$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent of order α in probability (where $0 < \alpha \le 1$) to a random variable $X : \mathcal{W} \to \mathbb{R}$ if for any $\varepsilon, \delta > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} |\{k \le T_n : t_k P(|X_k - X| \ge \varepsilon) \ge \delta\}| = 0.$$

In this case, $X_n \xrightarrow{(S^{\alpha}_{\overline{N}}, P, t_n)} X$ and the class of all weighted modulus statistical convergence sequences of order α in probability is denoted by $(S^{\alpha}_{\overline{N}}, P, t_n)$. One can also see [7,9,16] for related works.

The aim of this paper is to show that the definition of weighted lacunary statistical convergence, introduced by Başarir and Konca [3], is not well in the sense that each bounded real sequence is weighted lacunary statistically convergent to any real number. So some problems are still there; therefore it will be modified in this paper and the question of uniqueness of limit value is proved. Using it we develop some definitions of the convergence of a sequence of random variables in probability namely, weighted modulus S_{θ}-convergence of order α and weighted modulus N_{θ}-convergence of order α. Also these methods are compared with the convergence of weighted modulus statistical convergence of order α and weighted modulus strong Cesàro convergence of order α respectively.

The following definitions and notions will be needed in sequel.

Definition 1.1 (*See [18]*)**.** Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$, for all $n \in \mathbb{N}$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted statistically convergent of order α (where $0 < \alpha \leq 1$) to x if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} \left|\{k \leq T_n : t_k|x_k - x| \geq \varepsilon\}\right| = 0.$$

In this case we write $x_n \xrightarrow{(S^{\alpha}_{\overline{N}}, t_n)} x$. The class of all weighted statistically convergent sequences of order α is denoted by $(S^{\alpha}_{\overline{N}}, t_n)$. For $\alpha = 1$, we say that $\{x_n\}_{n\in\mathbb{N}}$ is weighted statistical convergence to x and this is denoted by $x_n \xrightarrow{(S^{1}_{\overline{N}}, t_n)} x$.

Definition 1.2 (*See [19,22]*)**.** Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$, where $n \in \mathbb{N}$ and $T_n \to \infty$, as $n \to \infty$. Then the sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted strong Cesàro convergent (or strongly $(\overline{N}, t_n)$-summable) to a real number x if

$$\lim_{n\to\infty} \frac{1}{T_n} \sum_{k=1}^{n} t_k|x_k - x| = 0.$$

In this case we write $x_n \xrightarrow{|\overline{N}, t_n|} x$. The set of all strongly $(\overline{N}, t_n)$-summable real sequences is denoted by $|\overline{N}, t_n|$.

Definition 1.3 (*See [21,23]*)**.** A modulus function ϕ is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $\phi(x) = 0$ if and only if $x = 0$,
(ii) $\phi(x + y) \leq \phi(x) + \phi(y)$, for all $x, y > 0$,
(iii) ϕ is increasing,
(iv) ϕ is continuous from the right at zero.

A modulus function may be bounded or unbounded. Savaş [25,26], Tripathy and Sarma [32] and other authors used modulus function to construct new sequence spaces. Recently Savaş and Patterson [27] have defined and studied some sequence spaces by using a modulus function.

Definition 1.4 (*See [18]*)**.** Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\liminf_{n\to\infty} t_n > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$, for all $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus statistically convergent of order α in probability (where $0 < \alpha \leq 1$) to a random variable $X : \mathcal{W} \to \mathbb{R}$ if for any $\varepsilon, \delta > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} |\{k \leq T_n : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}| = 0.$$

In this case, $X_n \xrightarrow{(S_{\overline{N}}^\alpha, P^\phi, t_n)} X$ and the class of all weighted modulus statistically convergent sequences of random variables of order α in probability is denoted by $(S_{\overline{N}}^\alpha, P^\phi, t_n)$.

Definition 1.5 (*See [18]*)**.** Let ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$, where $n \in \mathbb{N}$ and $T_n \to \infty$, as $n \to \infty$. Then the sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus strong Cesàro convergent of order α in probability (where $0 < \alpha \leq 1$) to a random variable X if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} \sum_{k=1}^{n} t_k\phi(P(|X_k - X| \geq \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(\overline{N}^\alpha, P^\phi, t_n)} X$ and the class of all weighted modulus strong Cesàro convergent sequences of random variables of order α in probability, is denoted by $(\overline{N}^\alpha, P^\phi, t_n)$.

2. Main results

Recently, Başarir and Konca [3] have defined the concept of weighted lacunary statistical convergence as follows: Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $T_n = \sum_{k=1}^n t_k$ for all $n \in \mathbb{N}$, $T_n \to \infty$ as $n \to \infty$, $H_r = \sum_{k\in I_r} t_k$ (where $k_0 = 0$ and $I_r = (k_{r-1}, k_r]$ $\forall\, r \in \mathbb{N}$), $T_{k_r} = \sum_{k\in(0,k_r]} t_k$, $T_{k_{r-1}} = \sum_{k\in(0,k_{r-1}]} t_k$, $Q_r = \frac{T_{k_r}}{T_{k_{r-1}}}$, $T_0 = 0$ and $I'_r = (T_{k_{r-1}}, T_{k_r}]$. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted lacunary statistically convergent to a real number x if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{H_r} |\{k \in I'_r : t_k|x_k - x| \geq \varepsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{S_{(\overline{N},\theta)}} x$. We denote the set of all weighted lacunary statistically convergent sequences by $S_{(\overline{N},\theta)}$.

But the above definition is not well defined in general. This follows from the following example.

Example 2.1. Let $\theta = \{r^2\}_{r\in\mathbb{N}\cup\{0\}}$ be any lacunary sequence and $t_n = \frac{1}{\sqrt[4]{n}} > 0$, for all $n \in \mathbb{N}$, then $T_n = t_1 + t_2 + t_3 + \cdots + t_n = 1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \cdots + \frac{1}{\sqrt[4]{n}} \to \infty$, as $n \to \infty$ (since $\sum_{n=1}^\infty \frac{1}{n^p}$ is properly divergent if $0 < p \leq 1$).

Again

$$\begin{aligned}
T_{2^2} - T_{1^1} &= 1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \frac{1}{\sqrt[4]{4}} - 1 \\
&= \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \frac{1}{\sqrt[4]{4}} > \frac{3}{\sqrt[4]{4}} = \frac{3}{\sqrt[4]{2^2}}, \\
T_{3^2} - T_{2^2} &= \left(1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \cdots + \frac{1}{\sqrt[4]{9}}\right) - \left(1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \frac{1}{\sqrt[4]{4}}\right) \\
&= \frac{1}{\sqrt[4]{5}} + \frac{1}{\sqrt[4]{6}} + \frac{1}{\sqrt[4]{7}} + \frac{1}{\sqrt[4]{8}} + \frac{1}{\sqrt[4]{9}} > \frac{5}{\sqrt[4]{9}} = \frac{5}{\sqrt[4]{3^2}}, \\
T_{4^2} - T_{3^2} &= \left(1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \cdots + \frac{1}{\sqrt[4]{16}}\right) - \left(1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \cdots + \frac{1}{\sqrt[4]{9}}\right) \\
&= \frac{1}{\sqrt[4]{10}} + \frac{1}{\sqrt[4]{11}} + \frac{1}{\sqrt[4]{12}} + \frac{1}{\sqrt[4]{13}} + \frac{1}{\sqrt[4]{14}} + \frac{1}{\sqrt[4]{15}} + \frac{1}{\sqrt[4]{16}} > \frac{7}{\sqrt[4]{4^2}}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
T_{(r+1)^2} - T_{r^2} &= \frac{1}{\sqrt[4]{r^2+1}} + \frac{1}{\sqrt[4]{r^2+2}} + \frac{1}{\sqrt[4]{r^3+3}} + \cdots + \frac{1}{\sqrt[4]{(r+1)^2}} \\
&> \frac{2r+1}{\sqrt[4]{(r+1)^2}} > \frac{2r}{\sqrt{r+1}} > \sqrt{r}, \text{ for } r \in \mathbb{N}
\end{aligned}$$

(since $\frac{r}{r+1} \geq \frac{1}{2}\ \forall\, r \geq 1$, so $\frac{2r}{\sqrt{r+1}} = 2\sqrt{r}\sqrt{\frac{r}{r+1}} \geq 2\sqrt{r}\frac{1}{\sqrt{2}} = \sqrt{2r} > \sqrt{r}$).

So $H_r = T_{k_r} - T_{k_{r-1}} = T_{r^2} - T_{(r-1)^2} > \sqrt{r-1} \to \infty$, as $r \to \infty$.

Now define a sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ by,

$$x_n = \begin{cases} 0, & \text{if } n \text{ is a even integer,} \\ 1, & \text{otherwise.} \end{cases}$$

Then for any $\varepsilon > 0$ and x be any fixed real number, $\{k \in I'_r : t_k|x_k - x| \geq \varepsilon\} \subseteq \{k \in (T_{k_{r-1}}, T_{k_r}] : \frac{(1+|x|)}{\sqrt[4]{k}} \geq \varepsilon\} \subseteq \{1, 2, 3, \ldots, [(\frac{1+|x|}{\varepsilon})^4] + 1\}$ = Finite subset of $\mathbb{N}$ (since $|x_k - x| \leq |x_k| + |x| \leq 1 + |x|$, by definition of $\{x_n\}_{n\in\mathbb{N}}$).

So $\lim\limits_{r\to\infty} \frac{1}{H_r}\ |\{k \in I'_r : t_k|x_k - x| \geq \varepsilon\}| = \lim\limits_{r\to\infty} \frac{1}{H_r}$ |Finite subset of $\mathbb{N}$| $= 0$, (since $H_r \to \infty$, as $r \to \infty$).

This implies $x_n \xrightarrow{S_{(\overline{N},\theta)}} x$ where x be any real number, i.e., any bounded sequence of real numbers is weighted lacunary statistically convergent to any real number. Hence the definition is not well defined. So the definition of weighted lacunary statistical convergence needs to be modified.

Now, we are going to modify the definition of weighted lacunary statistical convergence as follows:

Definition 2.1. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $t_n > \delta,\ \forall\, n \in \mathbb{N}$ (where δ is a fixed positive real number) and $T_n = t_1 + t_2 + \cdots + t_n$ (where $n \in \mathbb{N}$ and $T_0 = 0$). A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted lacunary statistically convergent (or weighted S_θ-convergent) to a real number x if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})}\ |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x_k - x| \geq \varepsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{(S_\theta, t_n)} x$. The class of all weighted S_θ-convergent sequences of real numbers is denoted by (S_θ, t_n).

Throughout the paper $\{t_n\}_{n\in\mathbb{N}}$ denote the weighted sequence, such that $t_n > \delta,\ \forall\, n \in \mathbb{N}$ (where δ is a fixed positive real number).

Remark 2.1. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ so $T_{k_r} = t_1 + t_2 + \cdots + t_{k_r}$, for all $r \in \mathbb{N}$. This implies $T_{k_r} - T_{k_{r-1}} = t_{k_{r-1}+1} + t_{k_{r-1}+2} + \cdots + t_{k_r} > \delta(k_r - k_{r-1})$, for all $r \in \mathbb{N}$.

Theorem 2.1. *If $x_n \xrightarrow{(S_\theta, t_n)} x$ and $x_n \xrightarrow{(S_\theta, t_n)} y$ then $x = y$.*

Proof. If possible let $x \neq y$. Choose $\varepsilon = \frac{1}{2}|x - y| > 0$ and $t_n > \delta,\ \forall\, n \in \mathbb{N}$. Then $|x - y| \le |x - x_k| + |x_k - y| \Rightarrow t_k|x - y| \le t_k|x - x_k| + t_k|x_k - y|\ \forall\, k \in \mathbb{N}$.

So $\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x - y| \ge \delta\varepsilon\} \subseteq \{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x_k - x| \ge \frac{\varepsilon\delta}{2}\} \cup \{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x_k - y| \ge \frac{\varepsilon\delta}{2}\}$ and $|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x - y| \ge \delta\varepsilon\}| = T_{k_r} - T_{k_{r-1}}$ (since $|x - y| = 2\varepsilon > \varepsilon$ and $t_k > \delta$ so $t_k|x - y| > \delta\varepsilon,\ \forall\, k \in \mathbb{N}$).

$1 = \frac{1}{T_{k_r} - T_{k_{r-1}}}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x - y| \ge \delta\varepsilon\}| \le \frac{1}{(T_{k_r} - T_{k_{r-1}})}\ |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x_k - x| \ge \frac{\varepsilon\delta}{2}\}| + \frac{1}{(T_{k_r} - T_{k_{r-1}})}\ |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x_k - y| \ge \frac{\varepsilon\delta}{2}\}|$ which is impossible because the right hand limit is equal to zero but not left hand. $\square$

Next we would like to introduce the definitions of weighted modulus S_θ-convergence of order α for a sequence of real numbers and weighted modulus S_θ-convergence of order α of a sequence of random variables in probability:

Definition 2.2. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence, ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $t_n > \delta,\ \forall\, n \in \mathbb{N}$ (where δ is a fixed positive real number) and $T_n = t_1 + t_2 + \cdots + t_n$ (where $n \in \mathbb{N}$ and $T_0 = 0$). A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus lacunary statistically convergent (or, weighted modulus S_θ-convergent) of order α (where $0 < \alpha \le 1$) to a real number x if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha}\ |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(|x_k - x|) \ge \varepsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{(S_\theta^\alpha, \phi, t_n)} x$. The class of all weighted modulus S_θ-convergent sequences of real numbers of order α is denoted by $(S_\theta^\alpha, \phi, t_n)$. For $\phi(u) = u$, $u \in [0, \infty)$ we say that $\{x_n\}_{n\in\mathbb{N}}$ is weighted S_θ-convergence (or, weighted lacunary statistical convergence) of order α to x and this is denoted by $x_n \xrightarrow{(S_\theta^\alpha, t_n)} x$.

In Theorem 2.1 we can easily shown that if $x_n \xrightarrow{(S_\theta^\alpha, \phi, t_n)} x$ and $x_n \xrightarrow{(S_\theta^\alpha, \phi, t_n)} y$ then $x = y$.

Definition 2.3. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence, ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $t_n > \delta,\ \forall\, n \in \mathbb{N}$ (where δ is a fixed positive real number) and $T_n = t_1 + t_2 + \cdots + t_n$ (where $n \in \mathbb{N}$ and $T_0 = 0$). A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus S_θ-convergent of order α in probability (where $0 < \alpha \le 1$) to a random variable X if for any $\varepsilon, \delta > 0$,

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha}\ |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}| = 0.$$

In this case we write $X_n \xrightarrow{(S_\theta^\alpha, P^\phi, t_n)} X$. The class of all weighted modulus S_θ-convergent sequences of random variables of order α in probability is denoted by $(S_\theta^\alpha, P^\phi, t_n)$.

Theorem 2.2. *If $X_n \xrightarrow{(S_\theta^\alpha, P^\phi, t_n)} X$ and $X_n \xrightarrow{(S_\theta^\beta, P^\phi, t_n)} Y$ then $P\{X = Y\} = 1$ for any α, β where $0 < \alpha, \beta \leq 1$.*

Proof. Without loss of generality we assume $\beta \leq \alpha$. If possible let $P\{X = Y\} \neq 1$. Then there exist three positive real numbers ε, η and δ such that $\phi(P(|X - Y| \geq \varepsilon)) > \eta$ and $t_n > \delta\ \forall\, n \in \mathbb{N}$. Then we have

$$\eta\delta \leq t_k\phi(P(|X - Y| \geq \varepsilon)) \leq t_k\phi\left(P\left(|X_k - X| \geq \frac{\varepsilon}{2}\right)\right) + t_k\phi\left(P\left(|X_k - Y| \geq \frac{\varepsilon}{2}\right)\right).$$

$$\begin{aligned} \Rightarrow \frac{T_{k_r} - T_{k_{r-1}}}{(T_{k_r} - T_{k_{r-1}})^\alpha} &\leq \frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha} \\ &\times \left|\left\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi\left(P\left(|X_k - X| \geq \frac{\varepsilon}{2}\right)\right) \geq \frac{\eta\delta}{2}\right\}\right| \\ &+ \frac{1}{(T_{k_r} - T_{k_{r-1}})^\beta}\left|\left\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi\left(P\left(|X_k - Y| \geq \frac{\varepsilon}{2}\right)\right) \geq \frac{\eta\delta}{2}\right\}\right| \end{aligned}$$

which is impossible because the left hand limit is not 0 whereas the right hand limit is 0. So $P\{X = Y\} = 1$. □

Theorem 2.3. *Let $0 < \alpha \leq \beta \leq 1$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. If $X_n \xrightarrow{(S_\theta^\alpha, P^\phi, t_n)} X$ and $P(|X| \geq y) = 0$ for some positive real number y, then $g(X_n) \xrightarrow{(S_\theta^\beta, P^\phi, t_n)} g(X)$.*

Proof. Since g is uniformly continuous on $[-y, y]$, then in this interval, for each $\varepsilon > 0$ there exists δ_0 such that

$$|g(x_n) - g(x)| < \varepsilon \text{ if } |x_n - x| < \delta_0 \text{ and } |x| < y.$$

Let $A = \{|X| < y\}$, $B = \{|X_n - X| < \delta_0\}$ and $C = \{|g(X_n) - g(X)| < \varepsilon\}$. Then $A \cap B \subset C$, $\Rightarrow P(C^c) \leq P(A^c) + P(B^c)$ ('c' stands for the complement). So $P(|g(X_n) - g(X)| \geq \varepsilon) \leq P(|X_n - X| \geq \delta_0) + P(|X| \geq y)$. $\Rightarrow P(|g(X_n) - g(X)| \geq \varepsilon) \leq P(|X_n - X| \geq \delta_0)$ (since $P(|X| \geq y) = 0$).

It follows that

$$\phi(P(|g(X_n) - g(X)| \geq \varepsilon)) \leq \phi(P(|X_n - X| \geq \delta_0)).$$

Then for $\eta > 0$,

$$\begin{aligned} &\frac{1}{(T_{k_r} - T_{k_{r-1}})^\beta}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|g(X_k) - g(X)| \geq \varepsilon)) \geq \eta\}| \\ &\leq \frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \geq \delta_0)) \geq \eta\}|. \quad \square \end{aligned}$$

Corollary 2.1. *Let $0 < \alpha \le \beta \le 1$, $X_n \xrightarrow{(S_\theta^\alpha, P^\phi, t_n)} x$ (where x is a real constant) and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(X_n) \xrightarrow{(S_\theta^\beta, P^\phi, t_n)} g(x)$.*

Proof. Here x can be regarded as a random variable with one element x in the corresponding spectrum. So it is a one point distribution i.e., $P(X = x) = 1, \Rightarrow P(|X| \ge |x| + 1) = 0$. Then from Theorem 2.3, we get $g(X_n) \xrightarrow{(S_\theta^\beta, P^\phi, t_n)} g(x)$.

The $(S_\theta^\alpha, P^\phi, t_n)$-limit of a sequence of random variables for two different lacunary sequences may not be equal which can be shown from the following example.

Example 2.2. Let us take two lacunary sequences $\theta_1 = \{k_r\}_{r\in\mathbb{N}\cup\{0\}} = \{(2r)^{(2r)}\}_{r\in\mathbb{N}}$ (where $k_0 = 0$) and $\theta_2 = \{k'_r\}_{r\in\mathbb{N}\cup\{0\}} = \{(2r+1)^{(2r+1)}\}_{r\in\mathbb{N}}$ (where $k'_0 = 0$), $t_n = n$, $\forall\, n \in \mathbb{N}$, and $\phi(u) = \sqrt{u}$, $\forall\, u \in [0, \infty)$.

Let $\psi : (0, \infty) \to \mathbb{N}$ be defined by,

$$\psi(x) = n, \quad \text{if } T_{n^n} < x \le T_{(n+1)^{(n+1)}}, \quad \text{where } n \in \mathbb{N}.$$

A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ be defined by,

$$X_n \in \begin{cases} \{-1, 1\}, & \text{with p.m.f } P(X_n = -1) = \dfrac{1}{n^4},\ P(X_n = 1) = 1 - \dfrac{1}{n^4}, \\ & \text{if } \psi(n) \text{ is even,} \\ \{0, 1\}, & \text{with p.m.f } P(X_n = 0) = 1 - \dfrac{1}{n^4},\ P(X_n = 1) = \dfrac{1}{n^4}, \\ & \text{if } \psi(n) \text{ is odd.} \end{cases}$$

For $0 < \varepsilon, \delta < 1$ and the lacunary sequence θ_1,

$$P(|X_n - 0| \ge \varepsilon) = \begin{cases} 1, & \text{if } \psi(n) \text{ is even,} \\ \dfrac{1}{n^4}, & \text{if } \psi(n) \text{ is odd.} \end{cases}$$

Then

$$\begin{aligned} T_{k_{r+1}} - T_{k_r} &= (1 + 2 + \cdots + k_{r+1}) - (1 + 2 + \cdots + k_r) \\ &= \frac{k_{r+1}(k_{r+1}+1)}{2} - \frac{k_r(k_r+1)}{2} \\ &= \frac{1}{2}(k_{r+1} - k_r)(k_{r+1} + k_r + 1) \\ &= \frac{1}{2}\{(2r+2)^{(2r+2)} - (2r)^{(2r)}\}.\{(2r+2)^{(2r+2)} + (2r)^{(2r)} + 1\} \\ &= \frac{1}{2}(2r)^{(2r+2)}.(2r)^{(2r+2)}.\left\{\left(1 + \frac{1}{r}\right)^{(2r+2)} - \frac{1}{(2r)^2}\right\} \\ &\quad .\left\{\left(1 + \frac{1}{r}\right)^{(2r+2)} + \frac{1}{(2r)^2} + \frac{1}{(2r)^{(2r+2)}}\right\} \end{aligned}$$

and

$$|\{k \in (T_{k_r}, T_{k_{r+1}}] : t_k\phi(P(|X_k - 0| \ge \varepsilon)) \ge \delta\}| = T_{(2r+1)^{(2r+1)}} - T_{(2r)^{(2r)}}.$$

So

$$\frac{1}{(T_{k_{r+1}}-T_{k_r})^{\alpha}}\ |\{k\in(T_{k_r},T_{k_{r+1}}]:t_k\phi(P(|X_k-0|\geq\varepsilon))\geq\delta\}|$$
$$\leq M_1\frac{(2r)^{(4r+2)}}{\{(2r)^{(4r+4)}\}^{\alpha}}\to 0 \text{ as } r\to\infty,\ \text{and } \alpha=1,$$

where M_1 is a positive constant. This shows that $X_n \xrightarrow{(S^{\alpha}_{\theta_1},P^{\phi},t_n)} 0$.

Similarly for the lacunary sequence θ_2,

$$P(|X_n-1|\geq\varepsilon)=\begin{cases}\dfrac{1}{n^4}, & \text{if } \psi(n) \text{ is even},\\ 1-\dfrac{1}{n^4}, & \text{if } \psi(n) \text{ is odd}.\end{cases}$$

Then

$$\begin{aligned}T_{k'_{r+1}}-T_{k'_r} &= \frac{1}{2}(k'_{r+1}-k'_r)(k'_{r+1}+k'_r+1)\\ &= \frac{1}{2}\{(2r+3)^{(2r+3)}-(2r+1)^{(2r+1)}\}.\{(2r+3)^{(2r+3)}+(2r+1)^{(2r+1)}+1\}\\ &= \frac{1}{2}(2r+1)^{(4r+6)}.\left\{\left(1+\frac{2}{2r+1}\right)^{(2r+3)}-\frac{1}{(2r+1)^2}\right\}\\ &\quad .\left\{\left(1+\frac{2}{2r+1}\right)^{(2r+3)}+\frac{1}{(2r+1)^2}+\frac{1}{(2r+1)^{(2r+3)}}\right\}\end{aligned}$$

and

$$|\{k\in(T_{k'_r},T_{k'_{r+1}}]:t_k\phi(P(|X_k-1|\geq\varepsilon))\geq\delta\}|=T_{(2r+2)^{(2r+2)}}-T_{(2r+1)^{(2r+1)}}.$$

So

$$\frac{1}{(T_{k'_{r+1}}-T_{k'_r})^{\alpha}}\ |\{k\in(T_{k'_r},T_{k'_{r+1}}]:t_k\phi(P(|X_k-1|\geq\varepsilon))\geq\delta\}|$$
$$\leq M_2\frac{(2r+1)^{(4r+2)}}{\{(2r)^{(4r+6)}\}^{\alpha}}=\left(1+\frac{1}{2r}\right)^{(4r+2)}.\frac{1}{(2r)^4}\to 0 \text{ as } r\to\infty,\ \text{and } \alpha=1,$$

where M_2 is a positive constant. This shows that $X_n \xrightarrow{(S^{\alpha}_{\theta_2},P^{\phi},t_n)} 1$.

Theorem 2.4. *For any* $0<\alpha\leq\beta\leq 1$, $(S^{\alpha}_{\theta},P^{\phi},t_n)\subset(S^{\beta}_{N},P^{\phi},t_n)$ *if* $\limsup_{r\to\infty}\frac{T_{k_r}}{T^{\alpha}_{k_{r-1}}}<\infty$.

If $\limsup_{r\to\infty}\frac{T_{k_r}}{T^{\alpha}_{k_{r-1}}}=\infty$, *then there exists a sequence of random variables* $\{X_n\}_{n\in\mathbb{N}}$ *which is* $(S^{\alpha}_{\theta},P^{\phi},t_n)$ *but not* $(S^{\beta}_{N},P^{\phi},t_n)$.

Proof. Let $X_n \xrightarrow{(S^{\alpha}_{\theta},P^{\phi},t_n)} X$ and $\limsup_{r\to\infty}\frac{T_{k_r}}{T^{\alpha}_{k_{r-1}}}=c$. Then for each $\varepsilon,\delta,\eta>0$, there exists a natural number l such that

$$\frac{1}{(T_{k_r}-T_{k_{r-1}})^{\alpha}}\ |\{k\in(T_{k_{r-1}},T_{k_r}]:t_k\phi(P(|X_k-X|\geq\varepsilon))\geq\delta\}|<\frac{\eta}{2(c+1)}\ \forall\, r>l$$

and

$$(T_{k_r} - T_{k_{r-1}}) > 1 \;\forall\, r > l.$$

Let $M_r = \frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha} \; |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}|$ for all $r \in \mathbb{N}$ and $M = \max\{M_1, M_2, \ldots, M_l, \frac{\eta}{2(c+1)}\}$.

Since $T_{k_{r-1}} \to \infty$ as $r \to \infty$ and Ml is a fixed positive real number so there exists a natural number $q(> l)$ such that $\frac{1}{T^\alpha_{k_{r-1}}} < \frac{\eta}{2Ml}$ for all $r \geq q$.

Let n be a sufficiently large integer, then there exists $r \in \mathbb{N}$ such that $T_{k_{r-1}} < T_n \leq T_{k_r}$.

Now

$$\begin{aligned}
&\frac{1}{T_n^\beta} \; |\{k \leq T_n : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}| \\
&\quad\leq \frac{1}{T^\alpha_{k_{r-1}}} \; |\{k \leq T_{k_r} : t_k\phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}| \\
&\quad= \frac{(T_{k_1} - T_{k_0})^\alpha}{T^\alpha_{k_{r-1}}} M_1 + \frac{(T_{k_2} - T_{k_1})^\alpha}{T^\alpha_{k_{r-1}}} M_2 + \cdots + \frac{(T_{k_l} - T_{k_{l-1}})^\alpha}{T^\alpha_{k_{r-1}}} M_l \\
&\qquad+ \frac{(T_{k_{l+1}} - T_{k_l})^\alpha}{T^\alpha_{k_{r-1}}} M_{l+1} + \cdots + \frac{(T_{k_r} - T_{k_{r-1}})^\alpha}{T^\alpha_{k_{r-1}}} M_r \\
&\quad\leq \frac{Ml}{T^\alpha_{k_{r-1}}} + \frac{(T_{k_{l+1}} - T_{k_l})}{T^\alpha_{k_{r-1}}} M_{l+1} + \cdots + \frac{(T_{k_r} - T_{k_{r-1}})}{T^\alpha_{k_{r-1}}} M_r
\end{aligned}$$

(since $T_{k_0} = T_0 = 0, (T_{k_r} - T_{k_{r-1}}) \leq 1 \;\forall\, 1 \leq r \leq l \Rightarrow\; (T_{k_r} - T_{k_{r-1}})^\alpha \leq 1 \;\forall\, 1 \leq r \leq l$ $\Rightarrow\; \sum_{r=1}^{l}(T_{k_r} - T_{k_{r-1}})^\alpha \leq l$ and $u > 1 \;\Rightarrow\; u \geq u^\alpha, \forall\, 0 < \alpha \leq 1$)

$$\leq \frac{Ml}{T^\alpha_{k_{r-1}}} + \frac{\eta}{2(c+1)} \frac{T_{k_r}}{T^\alpha_{k_{r-1}}} \leq Ml \frac{1}{T^\alpha_{k_{r-1}}} + \frac{\eta}{2(c+1)}.(c+1) \leq \eta, \forall\, r > q.$$

This shows that $X_n \xrightarrow{(S^\beta_{\overline{N}}, P^\phi, t_n)} X$.

For second part, take a lacunary sequence $\theta = \{k_r\}_{r \in \mathbb{N}\cup\{0\}} = \{r^2\}_{r \in \mathbb{N}\cup\{0\}}$, $t_k = 2^{k-1}$, $\forall\, k \in \mathbb{N}$, $0 < \alpha \leq 1$ and $\phi(u) = \sqrt{u}$, $\forall\, u \in [0, \infty)$, then $T_n = 2^n - 1$, for all $n \in \mathbb{N}$.

Then

$$\begin{aligned}
&\frac{T_{k_r}}{T^\alpha_{k_{r-1}}} \geq \frac{T_{k_r}}{T_{k_{r-1}}} = \frac{2^{r^2} - 1}{2^{(r-1)^2} - 1} = \frac{2^{r^2}\left(1 - \frac{1}{2^{r^2}}\right)}{2^{(r-1)^2}(1 - \frac{1}{2^{(r-1)^2}})} > 2^{2r-1} > r \text{ for all } r \in \mathbb{N}, \\
&\Rightarrow\; \frac{T_{k_r}}{T_{k_{r-1}}} > r \text{ for all } r \in \mathbb{N} \\
&\Rightarrow\; T_{k_r} > 2T_{k_{r-1}} \text{ for all } r \in \mathbb{N} \setminus \{1\} \\
&\Rightarrow\; T_{k_r} > T_{k_{r-1}} + (2^{(r-1)^2} - 1) > T_{k_{r-1}} + (2^{(r-1)^2} - 1)^\alpha \text{ for all } r \in \mathbb{N} \setminus \{1\} \\
&\Rightarrow\; T_{k_r} - T_{k_{r-1}} > (2^{(r-1)^2} - 1)^\alpha \text{ for all } r \in \mathbb{N} \setminus \{1\}.
\end{aligned}$$

So the length of the interval $(T_{k_{r-1}}, T_{k_r}]$ is strictly greater than $(2^{(r-1)^2} - 1)^\alpha$ for all $r \geq 2$.

We consider a sequence of random variables:

$$X_n \in \begin{cases} \{-1, 1\}, \text{ with p.m.f } P(X_n = 1) = P(X_n = -1), \\ \quad \text{if } n \text{ is the first } [(2^{(r-1)^2} - 1)^\alpha] \text{ integer in the interval } (T_{k_{r-1}}, T_{k_r}], \\ \{0, 1\}, \text{ with p.m.f } P(X_n = 0) = 1 - \dfrac{1}{(2^n)^4}, \\ \quad P(X_n = 1) = \frac{1}{(2^n)^4}, \text{ otherwise,} \end{cases}$$

(where $[x]$ is the greatest integer not greater than x).

For $0 < \varepsilon, \delta < 1$ we get

$$\phi(P(|X_n - 0| \geq \varepsilon)) = \begin{cases} 1, \text{ if } n \text{ is the first } [(2^{(r-1)^2} - 1)^\alpha] \\ \text{ integer in the interval } (T_{k_{r-1}}, T_{k_r}], \\ \dfrac{1}{(2^n)^2}, \text{ otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha} \, |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \\ &\quad \leq \frac{(2^{(r-1)^2} - 1)^\alpha}{(2^{r^2} - 2^{(r-1)^2})^\alpha} = \left[\frac{1 - \frac{1}{2^{(r-1)^2}}}{2^{2r-1}\{1 - \frac{1}{2^{(2r-1)}}\}}\right]^\alpha \leq \left(\frac{1}{2^{2r-2}}\right)^\alpha. \end{aligned}$$

Since

$$\begin{aligned} T_{k_{r-1}} + (2^{(r-1)^2} - 1)^\alpha &= (2^{(r-1)^2} - 1) + (2^{(r-1)^2} - 1)^\alpha \\ &\leq (2^{(r-1)^2} - 1) + (2^{(r-1)^2} - 1) \\ &= 2(2^{(r-1)^2} - 1) = 2^{r^2-2r+2} - 1 < 2^{r^2-2r+3} - 1 \\ &= T_{r^2-2r+3} \text{ (since } T_n = 2^n - 1, \ \forall\, n \in \mathbb{N}). \end{aligned}$$

So $T_{r^2-2r+3} - T_{k_{r-1}} > (2^{(r-1)^2} - 1)^\alpha$. This implies first $[(2^{(r-1)^2} - 1)^\alpha]$ integers of the interval $(T_{k_{r-1}}, T_{k_r}]$ lie in the interval $(T_{k_{r-1}}, T_{r^2-2r+3}]$ (where $T_{r^2-2r+3} < T_{k_r}$ $\forall\, r \geq 2$).

Next let $n = r^2 - 2r + 3$ then

$$\begin{aligned} &\frac{1}{T_n^\alpha} \, |\{k \leq T_n : t_k\phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \\ &\quad \geq \frac{[(2^{(r-1)^2} - 1)^\alpha]}{(2^{r^2-2r+3} - 1)^\alpha} = \frac{[(2^{r^2-2r+1} - 1)^\alpha]}{(2^{r^2-2r+3} - 1)^\alpha} > \left(\frac{1}{12}\right)^\alpha. \end{aligned}$$

As $\{\frac{1}{T^\alpha_{(r^2-2r+3)}} \, |\{k \leq T_{(r^2-2r+3)} : t_k\phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}|\}_{r\in\mathbb{N}}$ is a subsequence of the sequence $\{\frac{1}{T_n^\alpha} \, |\{k \leq T_n : t_k\phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}|\}_{r\in\mathbb{N}}$, this shows that $\{X_n\}_{n\in\mathbb{N}}$ not $(S^\alpha_{\overline{N}}, P^\phi, t_n)$ to 0 but $X_n \xrightarrow{(S^\alpha_\theta, P^\phi, t_n)} 0$. $\square$

Theorem 2.5. *If* $0 < \alpha \le \beta \le 1$ *then* $(S^{\alpha}_{\overline{N}}, P^{\phi}, t_n) \subset (S^{\beta}_{\theta}, P^{\phi}, t_n)$ *if* $\liminf_{r\to\infty} \frac{T_{k_r}}{T_{k_{r-1}}} > 1$.

Proof. Let $\liminf_{r\to\infty} \frac{T_{k_r}}{T_{k_{r-1}}} > 1$ and $X_n \xrightarrow{(S^{\alpha}_{\overline{N}}, P^{\phi}, t_n)} X$. Then for each $\delta > 0$, there exists a natural number l such that $\frac{T_{k_r}}{T_{k_{r-1}}} \ge 1 + \delta$ forall $r \ge l$. Let $\varepsilon, \delta > 0$,

$$\begin{aligned}
&\frac{1}{(T_{k_r})^{\alpha}} \, |\{k \le T_{k_r} : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}| \\
&\quad \ge \frac{1}{(T_{k_r})^{\alpha}} \, |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}| \\
&\quad = \left(\frac{T_{k_r} - T_{k_{r-1}}}{T_{k_r}}\right)^{\alpha} \frac{1}{(T_{k_r} - T_{k_{r-1}})^{\alpha}} \, |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}| \\
&\quad \ge \left(\frac{\delta}{1+\delta}\right)^{\alpha} \frac{1}{(T_{k_r} - T_{k_{r-1}})^{\beta}} \, |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\phi(P(|X_k - X| \ge \varepsilon)) \ge \delta\}|.
\end{aligned}$$

(Since $\frac{T_{k_r}}{T_{k_r} - T_{k_{r-1}}} \le \frac{1+\delta}{\delta}$).

This shows that $X_n \xrightarrow{(S^{\beta}_{\theta}, P^{\phi}, t_n)} X$. $\square$

Now we like to introduce the definitions of weighted modulus N_θ-convergence of order α of a sequence of real numbers and weighted modulus N_θ-convergence of order α in probability of a sequence of random variables from [17] as follows:

Definition 2.4. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence, ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$ (assuming $T_0 = 0$). A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus N_θ-convergent of order α (where $0 < \alpha \le 1$) to a real number x if

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})^{\alpha}} \sum_{k\in I_r} t_k\phi(|x_k - x|) = 0.$$

In this case we write $x_n \xrightarrow{(\overline{N}^{\alpha}_{\theta}, \phi, t_n)} x$ and the class of all weighted modulus N_θ-convergent sequences of real numbers of order α is denoted by $(\overline{N}^{\alpha}_{\theta}, \phi, t_n)$. For $\phi(u) = u, u \in [0, \infty)$ we say that $\{x_n\}_{n\in\mathbb{N}}$ is weighted N_θ-convergence of order α to x and this is denoted by $x_n \xrightarrow{(\overline{N}^{\alpha}_{\theta}, t_n)} x$.

Definition 2.5. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence, ϕ be a modulus function and $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \cdots + t_n$ where $n \in \mathbb{N}$ and $T_n \to \infty$ as $n \to \infty$ (assuming $T_0 = 0$). A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be weighted modulus N_θ-convergent of order α (where $0 < \alpha \le 1$) in probability to a random variable X if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})^{\alpha}} \sum_{k\in I_r} t_k\phi(P(|X_k - X| \ge \varepsilon)) = 0.$$

In this case we write $X_n \xrightarrow{(\overline{N}_\theta^\alpha, P^\phi, t_n)} X$ and the class of all weighted modulus N_θ-convergent sequences of random variables of order α in probability is denoted by $(\overline{N}_\theta^\alpha, P^\phi, t_n)$.

If $X_n \xrightarrow{(\overline{N}_\theta^\alpha, P^\phi, t_n)} X$ and $X_n \xrightarrow{(\overline{N}_\theta^\beta, P^\phi, t_n)} Y$ then $P\{X = Y\} = 1$ for any α, β where $0 < \alpha, \beta \leq 1$.

Theorem 2.6. *For any* $0 < \alpha \leq \beta \leq 1$, $X_n \xrightarrow{(\overline{N}_\theta^\alpha, P^\phi, t_n)} X$ *implies* $X_n \xrightarrow{(\overline{N}^\beta, P^\phi, t_n)} X$ *if* $\limsup_{r\to\infty} \frac{T_{k_r}}{T_{k_{r-1}}^\alpha} < \infty$.

If $\limsup_{r\to\infty} \frac{T_{k_r}}{T_{k_{r-1}}^\alpha} = \infty$, *then there exists a sequence of random variables* $\{X_n\}_{n\in\mathbb{N}}$ *which is* $(\overline{N}_\theta^\alpha, P^\phi, t_n)$ *but not* $(\overline{N}^\beta, P^\phi, t_n)$.

Proof. The first part of this theorem is parallel to Theorem 2.4, so omitted. For the second part we will give an example.

Let $\theta = \{2^{r^2}\}_{r\in\mathbb{N}\cup\{0\}}$, $t_n = 1, \forall\, n \in \mathbb{N}$, $0 < \alpha \leq 1$ and $\phi(u) = u$, for $u \in [0, \infty)$ then $\frac{k_r}{k_{r-1}^\alpha} \geq \frac{k_r}{k_{r-1}} = \frac{2^{r^2}}{2^{r^2-2r+1}} = 2^{2r-1} > r, \forall\, r \geq 1$.

We consider a sequence of random variables:

$$X_n \in \begin{cases} \{1, -1\}, \text{ with p.m.f } P(X_n = 1) = P(X_n = -1), \text{ if } n \text{ is the first } [\{2^{(r-1)^2}\}^\alpha] \\ \text{integer in the interval } I_r, \\ \{0\}, \text{ with p.m.f } P(X_n = 0) = 1, \text{ otherwise.} \end{cases}$$

Then

$$\frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha} \sum_{k\in I_r} t_k \phi(P(|X_k - 0| \geq \varepsilon)) \leq \frac{\{2^{(r-1)^2}\}^\alpha}{\{2^{r^2} - 2^{(r-1)^2}\}^\alpha}$$

$$= \frac{\{2^{(r-1)^2}\}^\alpha}{(2^{r^2})^\alpha \{1 - \frac{1}{2^{2r-1}}\}^\alpha} = \frac{1}{(2^{(2r-1)})^\alpha} \cdot \frac{1}{(1 - \frac{1}{2^{(2r-1)}})^\alpha} \leq 2\left(\frac{1}{2^{2r-1}}\right)^\alpha.$$

Since $2^{r^2} > 2^{r^2-2r+2} = 2.2^{(r-1)^2} \geq 2^{(r-1)^2} + \{2^{(r-1)^2}\}^\alpha$. This implies $2^{r^2-2r+2} - 2^{(r-1)^2} > \{2^{(r-1)^2}\}^\alpha$. So first $[(2^{(r-1)^2})^\alpha]$ integers of the interval I_r lie in the interval $(2^{(r-1)^2}, 2^{r^2-2r+2}]\ \forall\, r \geq 2$.

Next let $m = 2^{r^2-2r+2}$ then

$$\frac{1}{T_m^\alpha} \sum_{k=1}^m t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \frac{[\{2^{(r-1)^2}\}^\alpha]}{\{2^{(r^2-2r+2)}\}^\alpha} > \frac{[\{2^{(r-1)^2}\}^\alpha]}{[\{2^{(r^2-2r+2)}\}^\alpha] + 1} = \frac{1}{[2^\alpha] + 1} > 0.$$

This shows that $X_n \xrightarrow{(\overline{N}_\theta^\alpha, P^\phi, t_n)} 0$ but $\{X_n\}_{n\in\mathbb{N}}$ is not $(\overline{N}^\alpha, P^\phi, t_n)$ to 0. □

Theorem 2.7.

(i) *If* $0 < \alpha \leq \beta \leq 1$ *then* $X_n \xrightarrow{(\overline{N}^\alpha, P^\phi, t_n)} X$ *implies* $X_n \xrightarrow{(\overline{N}_\theta^\beta, P^\phi, t_n)} X$ *if* $\liminf_{r\to\infty} \frac{T_{k_r}}{T_{k_{r-1}}} > 1$.

(ii) *If* $X_n \xrightarrow{(\overline{N}^\alpha, P^\phi, t_n)} X$ *and* $X_n \xrightarrow{(\overline{N}_\theta^\alpha, P^\phi, t_n)} Y$ *then* $P\{X = Y\} = 1$.

Proof. (i) Proof is parallel to Theorem 2.5, so omitted.

(ii) If possible, let $P\{X = Y\} \neq 1$, then there exists a positive real number ε such that $P(|X - Y| \geq \epsilon) > 0$. Then

$$\phi(P(|X - Y| \geq \varepsilon)) \leq \frac{1}{T_{k_r}^\alpha} \sum_{k=1}^{k_r} t_k \phi(P(|X_k - X| \geq \varepsilon)) + \frac{1}{T_{k_r}^\alpha} \sum_{k=1}^{k_r} t_k \phi(P(|X_k - Y| \geq \varepsilon)).$$

Taking limit both sides we get,

$$\lim_{r\to\infty} \frac{1}{T_{k_r}^\alpha} \sum_{k=1}^{k_r} t_k \phi(P(|X_k - Y| \geq \varepsilon)) \geq \phi(P(|X - Y| \geq \varepsilon)) > 0. \tag{1}$$

Next

$$\begin{aligned}
&\frac{1}{(T_{k_r} - T_{k_{r-1}})^\alpha} \sum_{k\in I_r} t_k \phi(P(|X_k - Y| \geq \varepsilon)) \\
&= \frac{T_{k_r}^\alpha}{(T_{k_r} - T_{k_{r-1}})^\alpha} \left\{ \frac{1}{T_{k_r}^\alpha} \sum_{k=1}^{k_r} t_k \phi(P(|X_k - Y| \geq \varepsilon)) \right\} \\
&- \frac{T_{k_{r-1}}^\alpha}{(T_{k_r} - T_{k_{r-1}})^\alpha} \left\{ \frac{1}{T_{k_{r-1}}^\alpha} \sum_{k=1}^{k_{r-1}} t_k \phi(P(|X_k - Y| \geq \varepsilon)) \right\}.
\end{aligned}$$

Since

$$\lim_{r\to\infty} \frac{1}{T_{k_r}^\alpha} \sum_{k=1}^{k_r} t_k \phi(P(|X_k - Y| \geq \varepsilon)) = \lim_{r\to\infty} \frac{1}{T_{k_{r-1}}^\alpha} \sum_{k=1}^{k_{r-1}} t_k \phi(P(|X_k - Y| \geq \varepsilon)) > 0$$

and left hand limit converges to 0. So

$$\lim_{r\to\infty} \frac{T_{k_r}}{(T_{k_r} - T_{k_{r-1}})} = \lim_{r\to\infty} \frac{T_{k_{r-1}}}{(T_{k_r} - T_{k_{r-1}})}$$

which is possible only when $1 < \lim_{r\to\infty} \frac{T_{k_r}}{T_{k_{r-1}}} < \infty$ and

$$\lim_{r\to\infty} \frac{1}{T_{k_r}^\alpha} \sum_{k=1}^{k_r} t_k \phi(P(|X_k - Y| \geq \varepsilon)) = 0$$

which contradict the Eq. (1). So $P\{X = Y\} = 1$. $\square$

ACKNOWLEDGMENTS

The authors are thankful to Prof. Pratulananda Das (Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India), the Editor and Referees for their several valuable suggestions.

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