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Vector implicit quasi complementarity problems

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Abstract. In this work, we establish some existence theorems for solutions to a new class of vector implicit quasi complementarity problems and the corresponding vector implicit quasi variational inequality problems. Further we introduce the notion of a local non-positivity of a pair of mappings (F, Q) and consider the existence and properties of solutions for vector implicit quasi variational inequality problems and the corresponding vector implicit quasi complementarity problems in the neighborhood of a given point belonging to an underlined domain K.

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1. Introduction

The complementarity problem theory was introduced and studied by Lemke [19] and Cottle and Dantzig [3] in 1960. Also, in the 1960s, variational inequality was introduced by Hartman and Stampacchia [9] and Browder [1]. In 1971, Karamardian [14] firstly considered the equivalence of some scalar complementarity problems with solution sets $C(F,K) = \{x \in K\colon F(x) \in K^{\star}, \langle x, F(x)\rangle = 0\}$ and some scalar variational type problems with solution sets $V(F,K) = \{x \in K\colon \langle u - x, F(x)\rangle \geqslant 0 \text{ for all } u \in K\}$ for a mapping F defined on a closed convex cone K in a locally convex Hausdorff topological vector space X to a vector space Y. Since then, there have been much research

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[2,4,6–8,11,12,15–18,20–24] on the equivalence between a range of many kinds of complementarity problems and corresponding variational inequality problems under suitable different conditions. Solutions of these class of problems have extensive and important applications in vector optimization, optimal control, mathematical programing, operations research, and equilibrium problem of economics. Inspired and motivated by the above applications, various generalized variational inequality problems, the generalized complementarity problems have become important developed directions of variational inequality theory (for example, see [3,4,14]).

In 2001, Yin et al. [25] introduced a class of F-complementarity problems, which consist in finding $x \in K$ such that

$$\langle Tx, x\rangle + F(x) = 0, \quad \langle Tx, y\rangle + F(y) \geqslant 0, \quad \forall y \in K,$$

where X is a Banach space with topological dual X^*, and $\langle .,. \rangle$ duality pairing between them, K a closed convex cone of X, and T: $K \to X^*$, $F : K \to \mathbb{R}$. They obtained an existence theorem for solving F-complementarity problems and also proved that if F is positively homogeneous (i.e. $F(tx) = tF(x)$ for all $t > 0$ and $x \in K$), the F-complementarity problem is equivalent to the following generalized variational inequality problem which consists in finding $x \in K$ such that

$$\langle Tx, y - x\rangle + F(y) - F(x) \geqslant 0, \quad \forall y \in K.$$

In 2003, Fang and Huang [6] introduced a class of vector F-complementarity problems and investigated the solvability of the class for demipseudomonotone mappings and finite-dimensional continuous mappings in reflexive Banach spaces. Later, Huang and Li [12] introduced a class of scalar F-implicit complementarity problems and the corresponding variational inequality problems in Banach spaces. In 2006, Li and Huang [20] extended the result in [12] to the vector case and presented the equivalence between the vector F-implicit complementarity problems and the corresponding vector F-implicit variational inequality problems. They obtained some existence theorems for solutions for their problems.

Recently, Lee et al. [18] extended Li and Huang's results in the setting of set-valued mapping. They studied a class of vector F-implicit complementarity problems and established some existence results in topological vector spaces without considering the continuity or the monotonicity on mappings. Recently, Wu and Huang [22] introduced a class of mixed vector F-implicit complementarity problems and the corresponding mixed vector F-implicit variational inequality problems. They derived some existence theorems of solutions for the mixed vector F-implicit complementarity problems and the mixed vector F-implicit variational inequality problems by using Fan-KKM theorem under some suitable assumptions without the monotonicity in the neighborhood of a given point belonging to an underlined domain K of the set-valued mappings, where the neighborhood is contained in K.

Very recently, Khan [16] introduced and studied the following vector implicit quasi complementarity problem of finding $x \in K$ such that

$$\langle N(Ax, Tx), g(x)\rangle + F(g(x), g(x)) = 0$$

and

$$\langle N(Ax, Tx), h(y)\rangle + F(h(y), g(x)) \in P(x), \quad \forall y \in K,$$

and the following vector implicit quasi variational inequality problem of finding $x \in K$ such that

$$\langle N(Ax, Tx), h(y) - g(x)\rangle + F(h(y), g(x)) - F(g(x), g(x)) \in P(x), \quad \forall y \in K,$$

where K is a nonempty closed and convex subset of a Banach space X, P: $K \to 2^Y$ is a set-valued mapping with nonempty convex cone values, $L(X, Y)$ is the space of all continuous linear mappings from X into Y and N: $L(X, Y) \times L(X, Y) \to L(X, Y)$, A, T: $K \to L(X, Y)$, g, h: $K \to K$ and F: $K \times K \to Y$ are the mappings. He investigated the nonemptiness and closeness of solution sets of the problems and proved that solution sets of both the problems are equivalent to each other under some suitable conditions.

Motivated by the recent work going in this direction, in this work we introduce a class of vector implicit quasi variational inequality problem and the corresponding vector implicit quasi complementarity problems in real Banach spaces. Further by using Fan-KKM theorem, we investigate the nonemptiness and closeness of solution sets of those problems. Furthermore, we introduce the notion of a local non-positivity of two mappings (F, Q) and consider the existences and properties of solutions for vector implicit quasi variational inequality problems and the corresponding vector implicit quasi complementarity problems in the neighborhood of a point belonging to an underlined domain K. The results presented in this work improve and generalize some recent results due to Wu et al. [22], Khan [16], Farajzadeh et al. [24], Lee et al. [18].

2. Preliminaries

Throughout this paper unless otherwise specified, let X and Y be real Banach spaces and K be a nonempty convex subset of X. A nonempty subset $P \subseteq Y$ is said to be *cone* if (i) $P + P = P$, (ii) $\lambda P \subseteq P$, for all $\lambda \geqslant 0$. A cone P is said to be *pointed* whenever $P \cap (-P) = \{0\}$. An *ordered Banach space* (Y, P) is a real Banach space Y with an ordering defined by a cone $P \subseteq Y$ with an apex at the origin in the form of

$$x \leqslant y \Longleftrightarrow y - x \in P.$$

Let g, h: $K \to K$ be mappings, Q, F: $K \times K \to Y$ be bi-mappings and P: $K \to 2^Y$ be a set-valued mapping with nonempty convex cone values. In this paper, we consider the following *vector implicit quasi complementarity problem* (VIQCP) of finding $x \in K$ such that

$$Q(x, g(x)) + F(g(x), g(x)) = 0 \text{ and } Q(x, h(y)) + F(h(y), g(x)) \in P(x), \quad \forall y \in K.$$

Some special cases:

(i) If $Q : K \times K \to \mathbb{R}$, $F : K \to \mathbb{R}$ and $P(x) = \mathbb{R}_+, \forall x \in K$, then (VIQCP) reduces to the following F-implicit complementarity problem; finding $x \in K$ such that

$$Q(x, g(x)) + F(g(x)) = 0 \text{ and } Q(x, h(y)) + F(h(y)) \geqslant 0, \quad \forall y \in K.$$

which was considered by Wu et al. [22];

(ii) Let N: $L(X, Y) \times L(X, Y) \to L(X, Y)$, A, T: $K \to L(X, Y)$, g, h: $K \to K$ and F: $K \times K \to Y$ be the mappings. If we set $Q(x, y) = \langle N(Tx, Ax), y \rangle$, then (VIQCP) reduces the following vector implicit quasi complementarity problem; finding $x \in K$ such that

$$\langle N(Ax, Tx), g(x) \rangle + F(g(x), g(x)) = 0$$

and

$$\langle N(Ax, Tx), h(y) \rangle + F(h(y), g(x)) \in P(x), \quad \forall y \in K,$$

which was considered by Khan [16]; if A, T, h and g are the identity mappings, then we have the following vector quasi complementarity problem; finding $x \in K$ such that

$$\langle N(x, x), x \rangle + F(x, x) = 0$$

and

$$\langle N(x, x), y \rangle + F(y, x) \in P(x), \quad \forall y \in K,$$

which was considered by Khan [15];

(iii) If g, h are identity mappings and $P(x) = \mathbb{R}_+, \forall x \in K$, then (VIQCP) reduces to the complementarity problem (CP) which consists of finding $x \in K$ such that

$$Q(x, x) + F(x, x) = 0 \text{ and } Q(x, y) + F(y, x) \geqslant 0, \quad \forall y \in K,$$

which appears to be new.

Also we consider the following corresponding *vector implicit quasi variational inequality problem* (VIQVIP) of finding $x \in K$ such that

$$Q(x, h(y)) - Q(x, g(x))) + F(h(y), g(x)) - F(g(x), g(x)) \in P(x), \quad \forall y \in K.$$

In the rest of this section, we recall some definitions and a preliminary result which is used in the next section.

Definition 2.1. Let K be a nonempty convex subset of a vector space X and M: $K \to Y$ be a mapping. M is said to be λ-positively homogenous of degree r if $F(\lambda x) = \lambda^r F(x)$, $\forall x \in K$ and for some $\lambda \geqslant 0$.

Definition 2.2. Let K be a nonempty subset of a topological vector space X. A set-valued mapping T: $K \to 2^X$ is said to be a KKM mapping, if every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K, $co\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i)$, where co denotes the convex hull.

Theorem 2.3. [5] *Let K be a nonempty subset of topological vector space X. Let T: $K \to 2^X$ be a KKM-mapping such that for any $y \in K$, $T(y)$ is closed and $T(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in T(y)$ for all $y \in K$.*

Lemma 2.4. *Let (Y, P) be an ordered Banach space induced by a pointed, closed and convex cone P. Then $x \in P$ and $y \in P$ imply that $x + y \in P$ for all $x,y \in Y$.*

3. Main results

The following result due to Karamardian [14] is the first work on the equivalence of complementarity problems and the corresponding variational inequality problems.

Theorem 3.1. *Let K be a closed convex cone in a locally convex Hausdorff topological vector space X and Y be a vector space. Let F: $K \to Y$ be a mapping and $K^\star = \{y \in Y$: $\langle x, y\rangle \geqslant 0$ for all $x \in K\}$ be the polar of K in Y. Then the solution set $C(F,K) = \{x \in K$: $F(x) \in K^\star, \langle x, F(x)\rangle \geqslant 0\}$ for complementarity problems and the solution set $V(F,K) = \{x \in K$: $\langle u - x, F(x)\rangle \geqslant 0$ for all $u \in K\}$ for the corresponding variational inequality problems are the same.*

3.1. Equivalence of (VIQVIP) and (VIQCP)

Now we establish the equivalence between (VIQCP) and (VIQVIP) under some suitable conditions.

Theorem 3.2.

(i) *If x solves (VIQCP), then x solves (VIQVIP).*
(ii) *Assume that Q, F: $K \times K \to Y$ are 2-positively homogeneous of degree 1 in the second variable and in the first variable, respectively, and the mapping h is onto. If x solves (VIQVIP), then x solves (VIQCP).*

Proof

(i) Let $x \in K$ be the solution of (VIQCP), then there exists $x \in K$ such that

$$Q(x, g(x)) + F(g(x), g(x)) = 0 \text{ and } Q(x, h(y)) + F(h(y), g(x)) \in P(x), \quad \forall y \in K.$$

Now

$$\begin{aligned} &Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) - F(g(x), g(x)) \\ &\quad = Q(x, h(y)) + F(h(y), g(x)) - [Q(x, g(x)) + F(g(x), g(x))] \in P(x), \end{aligned}$$

for all $y \in K$. Thus $x \in K$ is the solution of (VIQVIP).

(ii) Now, let $x \in K$ be the solution of (VIQVIP), then

$$\begin{aligned} &Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) - F(g(x), g(x)) \\ &\quad \in P(x), \quad \forall y \in K. \end{aligned} \tag{3.1.1}$$

Since $Q(x, 2y) = 2Q(x, y)$ and $F(2x, y) = 2F(x, y)$, for all $x, y \in K$, it follows that $F(0, y) = 0$ and $Q(x, 0) = 0$ and since h is onto, therefore there exists $y, y' \in K$ such that $h(y) = 0$, $h(y') = 2g(x)$. By substituting $h(y) = 0$ and $h(y') = 2g(x)$ in (3.1.1), we get

$$Q(x, g(x)) + F(g(x), g(x)) \in -P(x),$$
$$Q(x, g(x)) + F(g(x), g(x)) \in P(x),$$

and hence

$$Q(x, g(x)) + F(g(x), g(x)) \in P(x) \cap -P(x).$$

Since $P(x)$ is a pointed cone, we have

$$Q(x, g(x)) + F(g(x), g(x)) = 0. \tag{3.1.2}$$

Thus by (3.1.1) and (3.1.2), we have

$$\begin{aligned} Q(x, h(y)) + F(h(y), g(x)) &= Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) \\ &\quad - F(g(x), g(x)) + Q(x, g(x)) + F(g(x), g(x)) \\ &= Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) \\ &\quad - F(g(x), g(x)) \\ &\in P(x), \end{aligned}$$

which implies that x solves (VIQCP). This completes the proof. $\square$

Remark 3.3. K need not be convex in the proof process. In fact, $K = \mathbb{N} \cup \{0\}$, as a subset of $X = \mathbb{R}$, is not convex, but it satisfies that $0 \in K$ and $2K \subset K$.

The following example shows that the assumption that h is onto in Theorem 3.2 is essential.

Example 3.4. Let $X = Y = K = \mathbb{R}$, $P(x) = [0, \infty)$ for all $x \in K$, $h(x) = g(x) = 1$, $Q(x,y) = x^2y^2$, $F(x,y) = x + y$ for all $x,y \in K$. Then

$$Q(x, g(x)) + F(g(x), g(x)) = x^2 + 2 = 0,$$

which shows that (VIQCP) does not have any solution, while every member of K is a solution of (VIQVIP).

The following example shows that the positive homogeneity of Q and F are essential.

Example 3.5. Let $X = Y = \mathbb{R}$, $K = \mathbb{R}$, $F(x,y) = 1$ and $g(x) = h(x) = x$, for all x, $y \in K$, and Q: $K \times K \to Y$ be a mapping defined by

$$Q(x,y) = \begin{cases} 0, & x = 0, \\ 1, & \text{otherwise}. \end{cases}$$

Obviously, $x = 0$ is a solution of (VIQVIP) but is not a solution of (VIQCP).

The following example shows that Q is 2-positive homogeneous of degree 1. Hence it satisfies the assumption of Theorem 3.2, but does not satisfy the assumption of the corresponding results in [8,10,13,18,20].

Example 3.6. Let $X = Y = \mathbb{R}$, $K = \mathbb{R}$, $g(x) = h(x) = x$, for all $x \in K$ and $F(x,y) = 0$, for all x, $y \in K$. Let Q: $K \times K \to Y$ be defined by

$$Q(x,y) = \begin{cases} y, & \text{if } x \text{ rational}, \\ 0, & \text{if } x \text{ irrational}. \end{cases}$$

3.2. The existence of solutions to (VIQVIP) and (VIQCP)

Now, by using the Fan-KKM Theorem 2.3, we have the following existence result for (VIQVIP).

Theorem 3.7. *Let K be the nonempty closed and convex subset of X. Assume that*

(a) *the mappings Q,F: $K \times K \to Y$ and g,h: $K \to K$ are continuous;*
(b) *there exists a mapping H: $K \times K \to Y$ such that*

  (i) $H(x,x) \in P(x)$, $\forall x \in K$;
  (ii) $Q(x,h(y)) - Q(x,g(x)) + F(h(y),g(x)) - F(g(x),g(x)) - H(x,y) \in P(x)$, *for all $x,y \in K$;*
  (iii) *the set $\{y \in K: H(x,y) \notin P(x)\}$ is convex for all $x \in K$;*

(c) *there exists a nonempty, compact set D of K such that for each $x \in K\backslash D$, there exists $y \in D$ such that*

$$Q(x,h(y)) - Q(x,g(x)) + F(h(y),g(x)) - F(g(x),g(x)) \notin P(x).$$

Then the solution set of (VIQVIP) is nonempty and closed.

Proof. We define a set-valued mapping G: $K \to 2^K$ by

$$G(y) = \{x \in D : Q(x,h(y)) - Q(x,g(x)) + F(h(y),g(x)) - F(g(x),g(x)) \\ \in P(x)\}, \quad \forall y \in K.$$

By the assumption (a), for any $y \in K$, $G(y)$ is closed in D. Since every element $x \in \cap_{y \in K} G(y)$ is a solution of (VIQVIP), we have to show that $\cap_{y \in K} G(y) \neq \emptyset$. Since D is compact, it is sufficient to prove that the family $\{G(y)\}_{y \in K}$ has the finite intersection property. Let $\{y_1, y_2, \ldots, y_n\}$ be a finite subset of K and set $B := \overline{co}(D \cup \{y_1, y_2, \ldots, y_n\})$, where $\overline{co}$ denotes the closure of co. Then B is a compact and convex subset of K.

Define two set-valued mappings F_1,F_2: $B \to 2^B$ by

$$F_1(y) = \{x \in B : Q(x,h(y)) - Q(x,g(x)) + F(h(y),g(x)) - F(g(x),g(x)) \in P(x)\}$$

and

$$F_2(y) = \{x \in B : H(x,y) \in P(x)\}, \quad \text{for all } y \in B.$$

From the conditions (i) and (ii) of (b), we have $H(y,y) \in P(y)$ and

$$Q(y,h(y)) - Q(y,g(y)) + F(h(y),g(y)) - F(g(y),g(y)) - H(y,y) \in P(y).$$

Now Lemma 2.4 implies

$$Q(y,h(y)) - Q(y,g(y)) + F(h(y),g(y)) - F(g(y),g(y)) \in P(y)$$

and so $F_1(y)$ is nonempty. Similarly, we can prove that for any $y \in B$,$F_1(y)$ is closed. Since $F_1(y)$ is a closed subset of a compact set B, we know that $F_1(y)$ is compact.

To show that $F_2(y)$ is a KKM-mapping. Suppose that there exists a finite subset $\{u_1, u_2, \ldots, u_n\}$ of B and $\lambda_i \geqslant 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} F_2(u_j).$$

Since $H(u, u_j) \notin P(u)$ for $j = 1, 2, \ldots, n$ and $\{y \in K: H(x, y) \notin P(x)\}$ is convex, it follows that

$$H\left(u, \sum_{i=1}^{n} \lambda_i u_i\right) = H(u, u) \notin P(u),$$

which is a contradiction to the assumption (i) of (b). Therefore $F_2(y)$ is a KKM-mapping. On the other hand, from the assumption (ii) of (b) and the fact that $P(x)$ is a cone, we have $F_2(y) \subset F_1(y)$ for all $y \in B$. Hence F_1 is also a KKM mapping. Since $F_1(y)$ is a closed subset of a compact set B and thus $F_1(y)$ is compact. By Theorem 2.3

$$\bigcap_{y \in B} F_1(y) \neq \emptyset.$$

By assumption (c), each element of $\cap_{y \in B} F_1(y)$ cannot belong to $K \backslash D$ but to D. Therefore $\cap_{y \in B} F_1(y) \subset G(y_i)$ for $i = 1, 2, \ldots, n$, that is, $\bigcap_{i=1}^{n} G(y_i) \neq \emptyset$. Hence $\{G(y): y \in K\}$ is a family of closed subsets of the compact subset D, having the finite intersection property. Therefore $\cap_{y \in K} G(y) \neq \emptyset$ and it is a compact subset of K. That is, there exists $x \in K$ such that

$$Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) - F(g(x), g(x)) \in P(x), \quad \forall y \in K.$$

Since Q, F, g and h are continuous, the solution set of (VIQVIP) is obviously closed.

Let $D = K$ in the condition (c) of Theorem 3.7, then we have the following result. $\square$

Theorem 3.8. *Let K be a nonempty, compact and convex subset of X and assume that the conditions (a) and (b) of Theorem* 3.7 *hold, then (VIQVIP) has a solution.*

Proof. The conclusion follows directly from Theorem 3.7. $\square$

Theorem 3.9. *Assume that Q, F: $K \times K \to Y$ are 2-positively homogeneous of degree 1 in the second variable and in the first variable, respectively, and h is onto. If all the assumptions of Theorem* 3.7 *are satisfied, then (VIQCP) has a solution. Furthermore, the solution set of (VIQCP) is closed.*

Proof. The conclusion follows directly from Theorems 3.2 and 3.7. $\square$

3.3. The existence results in the neighborhood of a given point

Usually, it is not easy to find the exact solution to given complementarity problems and variational inequality problems. However, in this section, we try to find some neighborhood of a given point, in which the solutions exist by considering the notion of the local non-positivity of a pair (F, Q) of mapping Q, F: $K \times K \rightarrow Y$.

Now we introduce a concept of the local non-positivity for a pair (F, Q) of two bi-mappings Q, F: $K \times K \rightarrow Y$.

Definition 3.10. Let K be a nonempty subset of X. If g, h: $K \rightarrow K$ are mappings and Q, F: $K \times K \rightarrow Y$ bi-mappings, then (F, Q) is said to be *locally non-positive* at $x_0 \in K$ with respect to (g, h) if there exists a neighborhood $N(x_0)$ of x_0 and $z_0 \in K \cap \operatorname{int} N(x_0)$ such that

$$Q(x, h(z_0)) - Q(x, g(x)) + F(h(z_0), g(x)) - F(g(x), g(x)) \in -P(x), \quad \forall x \in K \cap \partial N(x_0),$$

where $\partial N(x_0)$ denotes the boundary of $N(x_0)$.

Example 3.11. Let $X = Y = \mathbb{R}$, $K = [0, 1)$ and $P(x) = [0, \infty)$ for all $x \in K$. Define mappings g, h: $K \rightarrow K$ by $g(x) = \frac{x}{5}$ and $h(x) = \frac{x}{3}$, and bi-mappings Q, F: $K \times K \rightarrow Y$ by $Q(x, y) = xy$ and $F(x, y) = x + y$, then (F, Q) is locally non-positive at $x_0 = 0 \in K$ with respect to (g, h). If we take a neighborhood $N(0) = \left(-\frac{1}{3}, \frac{1}{3}\right)$ of x_0 and $z_0 = \frac{1}{5} \in K \cap$ int $N(0) = \left[0, \frac{1}{3}\right)$ then for the unique element $x = \frac{1}{3}$ of $K \cap \partial N(0) = \left\{\frac{1}{3}\right\}$ we have

$$Q(x, h(z_0)) - Q(x, g(x)) + F(h(z_0), g(x)) - F(g(x), g(x)) = \frac{xz_0}{3} - \frac{x^2}{5} + \frac{z_0}{3} - \frac{x}{5} = 0 \in -P(x).$$

Theorem 3.12. *Let K be the nonempty closed and convex subset of X. Assume that*

(a) *the mappings Q, F: $K \times K \rightarrow Y$ and g, h: $K \rightarrow K$ are continuous;*
(b) *there exists a mapping H: $K \times K \rightarrow Y$ such that*

  (i) *$H(x, x) \in P(x)$, $\forall x \in K$;*
  (ii) *$Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) - F(g(x), g(x)) - H(x, y) \in P(x)$, for all $x, y \in K$;*
  (iii) *the set $\{y \in K$: $H(x, y) \notin P(x)\}$ is convex for all $x \in K$;*

(c) *(F, Q) is locally non-positive at $x_0 \in K$ with respect to (g, h) and there exists a nonempty compact set D of $K \cap N(x_0)$ such that for all $x \in (K \cap N(x_0)) \backslash D$ there exists $y \in D$ such that*

$$Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) - F(g(x), g(x)) \in -P(x);$$

(d) *the set $\{y \in K$: $Q(x, h(y)) - Q(x, g(x)) + F(h(y), g(x)) - F(g(x), g(x)) \in P(x)\}$ is convex for all $x \in K$.*

Then (VIQVIP) has a solution in the neighborhood of x_0, that is, there exists $x^\star \in (K \cap N(x_0)) \backslash D$ such that

$$Q(x^\star, h(y)) - Q(x^\star, g(x^\star)) + F(h(y), g(x^\star)) - F(g(x^\star), g(x^\star)) \in P(x^\star), \quad \forall y \in K.$$

Proof. Since (F,Q) is locally non-positive at $x_0 \in K$ with respect to (g,h), without loss of generality, we can assume that $N(x_0)$ is a closed and convex set. Since $K \backslash N(x_0)$ is also closed and convex, from Theorem 3.7, (VIQVIP) has a solution $x \in K \cap \ N(x_0)$ such that

$$Q(x^\star, h(y)) - Q(x^\star, g(x^\star)) + F(h(y), g(x^\star)) - F(g(x^\star), g(x^\star)) \\ \in P(x^\star), \quad \forall y \in K \cap N(x_0) \tag{3.3.1}$$

Now we prove

$$Q(x^\star, h(y)) - Q(x^\star, g(x^\star)) + F(h(y), g(x^\star)) - F(g(x^\star), g(x^\star)) \\ \in P(x^\star), \quad \forall y \in K.$$

If $x^\star \in K \cap \mathrm{int} N(x_0)$, then $N(x_0) \backslash \{x^\star\}$ is a neighborhood of the origin and so it is absorbing. For any $y \in K$, there exists $t \in (0,1)$ such that $t(y - x^\star) \in N(x_0) \backslash \{x^\star\}$ and so $y_t = ty + (1-t)x^\star \in K \cap N(x_0)$. It follows from (3.3.1) that

$$Q(x^\star, h(y_t)) - Q(x^\star, g(x^\star)) + F(h(y_t), g(x^\star)) - F(g(x^\star), g(x^\star)) \in P(x^\star).$$

From the condition (d), we have

$$Q(x^\star, h(y)) - Q(x^\star, g(x^\star)) + F(h(y), g(x^\star)) - F(g(x^\star), g(x^\star)) \in P(x^\star).$$

Note that (F,Q) is locally non-positive at $x_0 \in K$ with respect to g, h. If $x^\star \in K \cap \partial N(x_0)$, then there exists $z_0 \in K \cap \mathit{int}\ N(x_0)$ such that

$$Q(x^\star, h(z_0)) - Q(x^\star, g(x^\star)) + F(h(z_0), g(x^\star)) - F(g(x^\star), g(x^\star)) \\ \in -P(x^\star). \tag{3.3.2}$$

Similarly, for any $y \in K$, there exists $t \in (0,1)$ such that $t(y - z_0) \in N(x_0) \backslash \{z_0\}$ and so $z_t = ty + (1-t)z_0 \in K \ \cap N(x_0)$. It follows from (3.3.1) that

$$Q(x^\star, h(z_t)) - Q(x^\star, g(x^\star)) + F(h(z_t), g(x^\star)) - F(g(x^\star), g(x^\star)) \\ \in P(x^\star). \tag{3.3.3}$$

Letting $t \to 0$ in (3.3.3), we obtain

$$Q(x^\star, h(z_0)) - Q(x^\star, g(x^\star)) + F(h(z_0), g(x^\star)) - F(g(x^\star), g(x^\star)) \\ \in P(x^\star). \tag{3.3.4}$$

Since P is a pointed cone, (3.3.4) with (3.3.2) implies that

$$Q(x^\star, h(z_0)) - Q(x^\star, g(x^\star)) + F(h(z_0), g(x^\star)) - F(g(x^\star), g(x^\star)) = 0. \tag{3.3.5}$$

From the condition (d), we have

$$tQ(x^\star, h(y)) + (1-t)Q(x^\star, h(z_0)) - Q(x^\star, g(x^\star)) + tF(h(y), g(x^\star)) \\ + (1-t)F(h(z_0), g(x^\star)) - F(g(x^\star), g(x^\star)) \\ \in P(x^\star). \tag{3.3.6}$$

From the inclusions (3.3.5) and (3.3.6), we have

$$tQ(x^\star, h(y)) - tQ(x^\star, g(x^\star)) + tF(h(y), g(x^\star)) - tF(g(x^\star), g(x^\star)) \in P(x^\star).$$

Therefore

$$Q(x^\star, h(y)) - Q(x^\star, g(x^\star)) + F(h(y), g(x^\star)) - F(g(x^\star), g(x^\star)) \in P(x^\star).$$

This completes the proof. □

Letting $D = K$ in the condition (c) of Theorem 3.12, we have the following result.

Theorem 3.13. *Let K be a nonempty, compact and convex subset of a real Banach space X, and assume that the conditions (a), (b) and (d) of Theorem* 3.12 *and the following condition hold:*

(c′) *(F,Q) is locally non-positive at $x_0 \in K$ with respect to (g,h). Then $(VIQVIP)$ has a solution in the neighborhood of x_0, that is, there exists $x^\star \in K \cap N(x_0)$ such that*

$$Q(x^\star, h(y)) - Q(x^\star, g(x^\star)) + F(h(y), g(x^\star)) - F(g(x^\star), g(x^\star)) \in P(x^\star), \quad \forall y \in K.$$

Theorem 3.14. *Assume that Q, F: $K \times K \to Y$ are 2-positively homogeneous of degree 1 in the second variable and in the first variable, respectively, and h is onto. If all the assumptions of Theorem* 3.12 *are satisfied, then $(VIQCP)$ has a solution in the neighborhood of x_0, that is, there exists $x^\star \in (K \cap N(x_0)) \backslash D$ such that*

$$Q(x^\star, g(x^\star)) + F(g(x^\star), g(x^\star)) = 0$$

and

$$Q(x^\star, h(y)) + F(h(y), g(x^\star)) \in P(x^\star), \quad \forall y \in K.$$

Proof. The conclusion follows directly from Theorems 3.2 and 3.12. □

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References

[1] F.E. Browder, The fixed point theory of multivalued mappings in topological vector spaces, Math. Ann. 177 (1968) 283–301.
[2] G.Y. Chen, X.Q. Yang, The vector complementarity problem and its equivalances with the weak minimal element in ordered spaces, J. Math. Anal. Appl. 153 (1990) 136–158.
[3] R.W. Cottle, G.B. Dantzig, Complementary pivot theory of mathematical programming, Linear Algebra Appl. 1 (1968) 163–165.
[4] I.C. Dolcetta, U. Mosco, Implicit complementarity problems and quasivariational inequalities, in: R.W. Cottle, R. Giannessi, J.L. Lions (Eds.), Variational Inequalities and Complementarity Problems, Theory and Applications, John Wiley and Sons, 1990, pp. 75–87.
[5] K. Fan, A generalization of Tychonoff's fixed point theorem, Math. Ann. 142 (1961) 305–310.
[6] Y.P. Fang, N.J. Huang, The vector *F*-complementarity problems with demipseudomonotone mappings in Banach spaces, Appl. Math. Lett. 16 (2003) 1019–1024.
[7] A.P. Farajzadeh, M.A. Noor, S. Zainab, Mixed quasi complementarity problems in topological vector spaces, J. Global Optim. 45 (2009) 229–235.

[8] A.P. Farajzadeh, J. Zafarani, Vector *F*-implicit complementarity problems in topological vector spaces, Appl. Math. Lett. 20 (2007) 1075–1081.
[9] P. Hartman, G. Stampacchia, On some nonlinear elliptic differential functional equation, Acta Math. 115 (1966) 271–310.
[10] N.J. Huang, Y.P. Fang, Strong vector *F*-complementarity problem and least element problem of feasible set, Nonlinear Anal. 61 (2005) 901–918.
[11] N.J. Huang, C.J. Gao, Some generalized vector variational inequalities and complementarity problems for multivalued mappings, Appl. Math. Lett. 16 (2003) 1003–1010.
[12] N.J. Huang, J. Li, F-implicit complementarity problems in Banach spaces, Z. Anal. Anwend. 23 (2004) 293–302.
[13] N.J. Huang, J. Li, D.O. Regan, Generalized *F*-complementarity problems in Banach spaces, Nonlinear Anal. 68 (2008) 3828–3840.
[14] S. Karamardian, Generalized complementarity problem, J. Optim. Theory 8 (1971) 161–168.
[15] S.A. Khan, Generalized vector complementarity-type problems in topological vector spaces, Comput. Math. Appl. 59 (2010) 3595–3602.
[16] S.A. Khan, Generalized vector implicit quasi complementarity problems, J. Global Optim. 49 (2011) 695–705.
[17] B.S. Lee, A.P. Farajzadeh, Generalized vector implicit complementarity problems with corresponding variational inequality problems, Appl. Math. Lett. 21 (2008) 1095–1100.
[18] B.S. Lee, M.F. Khan, Salahuddin, Vector *F*-implicit complementarity problems with corresponding variational inequality problems, Appl. Math. Lett. 20 (2007) 433–438.
[19] C.E. Lemke, Bimatrix equilibrium points and mathematical programming, Manage. Sci. 11 (1965) 681–689.
[20] J. Li, N.J. Huang, Vector *F*-implicit complementarity problems in Banach spaces, Appl. Math. Lett. 19 (2006) 464–471.
[21] R.U. Verma, Nonlinear variational inequalities on convex subsets of Banach spaces, Appl. Math. Lett. 10 (1997) 25–27.
[22] K.Q. Wu, N.J. Huang, General mixed vector *F*-implicit complementarity problems in Banach spaces, J. Appl. Anal. 14 (2008) 73–88.
[23] X.Q. Yang, Vector complementarity and minimal problems, J. Optim. Theory Appl. 77 (1993) 483–495.
[24] J.C. Yao, Existence of generalized variational inequalities, Oper. Res. Lett. 15 (1994) 35–40.
[25] H.Y. Yin, C.X. Xu, Z.X. Zhang, The *F*-complementarity problem and its equivalence with the least element problem, Acta Math. Sin. 44 (2001) 679–686.