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Values shared by meromorphic functions and their derivatives

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Abstract. In this paper we deal with the problem of uniqueness of meromorphic functions as well as their power which share a small function with their derivatives and obtain some results which improve and generalize the recent results due to Zhang and Yang (2009) and Sheng and Zongsheng (2012).

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1. INTRODUCTION DEFINITIONS AND RESULTS

In this paper, by a meromorphic function we will always mean a meromorphic function in the complex plane $\mathbb{C}$. We adopt the standard notations of Nevanlinna theory of meromorphic functions as explained in [4]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \not\in E$.

Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{k)}(r,a;f)$ to denote counting function of a-points of f with multiplicity $\leq k$, $N_{(k+1}(r,a;f)$ to denote counting function of a-points of f with multiplicity $>k$. Similarly $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ are their reduced functions respectively.

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$and $g-a$

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have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, $r \notin E$.

During the last four decades uniqueness theory of entire and meromorphic functions has become a prominent branch of value distribution theory (see [12]).

Rubel–Yang [6] proposed to investigate uniqueness of an entire function f under the assumption that f and its derivative f' share two complex values. Subsequently, related to one or two value sharing similar considerations have been made with respect to higher derivatives and more general (linear) differential expressions by Brück [1], Gundersen [2], Mues–Steinmetz [5], Yang [8].

In this direction an interesting problem still open is the following conjecture proposed by Brück [1]:

Conjecture 1.1. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f^{'} - a}{f - a} = c$$

for some non-zero constant c.

The case that $a = 0$ and that $N(r,0;f') = S(r,f)$ had been proved by Brück [1] while the case that f is of finite order had been proved by Gundersen–Yang [3]. However, the corresponding conjecture for meromorphic functions fails in general (see [3]).

To the knowledge of the author perhaps Yang–Zhang [10] (see also [13]) were the first to consider uniqueness of a power of a meromorphic (entire) function $F = f^n$ and its derivative F' when they share a certain value as this type of consideration gives the most specific form of the function.

As a result during the last decade, growing interest has been devoted to this setting of meromorphic functions. Improving all the results obtained in [10], Zhang [13] proved the following theorem.

Theorem A *([13]). Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and*

$$(n-k-1)(n-k-4) > 3k+6, \tag{1.1}$$

then $f^n \equiv (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

In 2009 Zhang and Yang [14] further improved the above result in the following manner.

Theorem B *([14]). Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and*

$$n > k + 1 + \sqrt{k+1}. \tag{1.2}$$

Then the conclusion of Theorem A holds.

Theorem C *([14]). Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and*

$$n > 2k + 3 + \sqrt{(2k+3)(k+3)}. \tag{1.3}$$

Then the conclusion of Theorem A holds.

Corollary A *([14]). Let f be a non-constant meromorphic function and $n \geq 4$ be an integer. Denote $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Recently Sheng and Zongsheng [7] proved the following results.

Theorem D. *Let f be a non-constant meromorphic function such that $\overline{N}(r, \infty; f) = S(r, f)$. Denote $F = f^n$. Suppose that F and F' share 1 CM. If (1) $n \geq 3$, or (2) $n = 2$ and $N(r, 0; f) = O(N_{(3}(r, 0; f))$, then $F \equiv F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Theorem E. *Let f be a non-constant meromorphic function and $a(z) \not\equiv 0$ be a rational function. If $f^n - a$ and $(f^n)' - a$ share the value 0 IM and*

$$n > 4 + 2\sqrt{3}, \tag{1.4}$$

then $f^n \equiv (f^n)'$, and f assumes the form

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Now observing the above results the following question is inevitable.

Question 1. Can the lower bounds of n given in (1.2)–(1.4) in Theorems B, C and E respectively be further reduced ?

In this paper, taking a possible answer of the above question into the background we obtain the following results.

Henceforth we suppose $m\,(\geq 0), n\,(\geq 1)$ and $k\,(\geq 1)$ are three integers and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is a nonzero polynomial.

Theorem 1.1. *Let f be a non-constant meromorphic function and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n P(f) - a$ and $[f^n P(f)]^{(k)} - a$ share the value 0 CM and*

$$n > k + 2,$$

then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; and $f^{n+i} \equiv (f^{n+i})^{(k)}$, where f assumes the form

$$f(z) = c e^{\frac{\lambda}{n+i} z},$$

where c is a nonzero constant and $\lambda^k = 1$.

Theorem 1.2. *Let f be a non-constant meromorphic function such that $N_{1)}(r, \infty; f) = S(r, f)$ and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n P(f) - a$ and $[f^n P(f)]^{(k)} - a$ share the value 0 CM and*

$$n > k + 1.$$

Then the conclusion of Theorem 1.1 holds.

Theorem 1.3. *Let f be a non-constant meromorphic function and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n P(f) - a$ and $[f^n P(f)]^{(k)} - a$ share the value 0 IM and*

$$n > 2k + m + 2.$$

Then the conclusion of Theorem 1.1 holds.

Theorem 1.4. *Let f be a non-constant meromorphic function such that $\overline{N}(r, \infty; f) = S(r, f)$, $N(r, 0; f) = O(N_{(2}(r, 0; f))$ and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n P(f) - a$ and $[f^n P(f)]^{(k)} - a$ share the value 0 IM and*

$$n > k + m.$$

Then the conclusion of Theorem 1.1 holds.

Remark 1.1. Clearly Theorems 1.2 and 1.4 improve and generalize Theorem D.

Remark 1.2. It is easy to see that conditions

$$\overline{N}(r, \infty; f) = S(r, f) \quad \text{and} \quad N(r, 0; f) = O(N_{(2}(r, 0; f))$$

in Theorem 1.4 are sharp by the following example.

Example 1.1. *Let*

$$f(z) = \frac{1}{2} - \frac{\sqrt{5}}{2}\, i \tan\left(\frac{\sqrt{5}}{4} iz\right).$$

Then f^2 *and* $(f^2)'$ *share the value* 1 *IM,* $\overline{N}(r,\infty;f) \neq S(r,f)$ *and* $N(r,0;f) \neq O(N_{(2}(r,0;f))$ *but* $f^2 \not\equiv (f^2)'$.

We now explain the following definitions and notations which will be used in the paper.

Definition 1.1 (*[6]*)**.** Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.
$N(r,a;f \mid \geq p)$ ($\overline{N}(r,a;f \mid \geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

Definition 1.2 (*[11]*)**.** For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid \geq 2) + \cdots + \overline{N}(r,a;f \mid \geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

2. Lemmas

In this section we present following lemmas which will be needed in the sequel.

Lemma 2.1 (*[9]*)**.** *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, $\ldots$, $a_0(z)$ *be meromorphic functions such that* $T(r,a_i(z)) = S(r,f)$ *for* $i = 0,1,2,\ldots,n$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.2. *Let* f *be a non-constant meromorphic function and* $n \geq k+1$. *If* $f^n P(f) \equiv [f^n P(f)]^{(k)}$ *then* $P(z)$ *reduces to a nonzero monomial, namely* $P(z) = a_i z^i \not\equiv 0$ *for some* $i \in \{0,1,\ldots,m\}$*; and* $f^{n+i} \equiv (f^{n+i})^{(k)}$, *where* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n+i} z},$$

where c *is a nonzero constant and* $\lambda^k = 1$.

Proof. Suppose

$$f^n P(f) \equiv [f^n P(f)]^{(k)}. \tag{2.1}$$

We now prove that $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0,1,\ldots,m\}$. If not we may assume that $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where at least two of $a_0, a_1, \ldots, a_{m-1}, a_m$ are nonzero. Without loss of generality, we assume that $a_s, a_t \neq 0$, where $s \neq t$, $s,t = 0,1,2,\ldots,m$.

From (2.1) it is clear that f is an entire function. Also since $n \geq k+1$, it follows from (2.1) that 0 is a Picard Exceptional Value of f. So we can take $f = e^{\alpha}$, where α is a non-constant entire function. Then by induction we get

$$a_i[f^{n+i} - (f^{n+i})^{(k)}] = t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}) e^{(n+i)\alpha}, \tag{2.2}$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)})$ $(i = 0, 1, 2, \ldots, m)$ are differential polynomials in $\alpha', \alpha'', \ldots, \alpha^{(k)}$.

From (2.1) and (2.2) we obtain

$$t_m(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{m\alpha} + \cdots + t_1(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{\alpha} + t_0(\alpha', \alpha'', \ldots, \alpha^{(k)}) \equiv 0. \tag{2.3}$$

Since $T(r, t_i) = S(r, f)(i = 0, 1, \ldots, m)$, by the Borel unicity theorem {see, e.g. [12, Theorem 1.52]}, (2.3) gives $t_i \equiv 0\,(i = 0, 1, \ldots, m)$. As $a_s, a_t \neq 0$, from (2.2) we have

$$f^{n+s} \equiv (f^{n+s})^{(k)} \quad \text{and} \quad f^{n+t} \equiv (f^{n+t})^{(k)}$$

which is a contradiction. Actually in this case we get two different forms of $f(z)$ simultaneously. Hence $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. So from (2.1) we get

$$f^{n+i} \equiv [f^{n+i}]^{(k)},$$

where $i \in \{0, 1, \ldots, m\}$. Clearly f assumes the form

$$f(z) = ce^{\frac{\lambda}{n+i}z},$$

where c is a nonzero constant and $\lambda^k = 1$. $\square$

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F = \frac{f^n P(f)}{a}$ and $G = \frac{[f^n P(f)]^{(k)}}{a}$. Clearly F and G share 1 CM except for zeros and poles of $a(z)$ and so

$$\overline{N}(r, 1; F) = \overline{N}(r, 1; G) + S(r, f).$$

$$\begin{aligned}\Phi_1 &= \frac{1}{F}\left(\frac{G'}{G-1} - \frac{F'}{F-1}\right)\\ &= \frac{G}{F}\left(\frac{G'}{G-1} - \frac{G'}{G}\right) - \left(\frac{F'}{F-1} - \frac{F'}{F}\right).\end{aligned} \tag{3.1}$$

We now consider the following two cases:

Case 1: Let $\Phi_1 \equiv 0$.

On integration we get

$$F - 1 \equiv c(G - 1), \tag{3.2}$$

where c is a nonzero constant.

This implies that $\overline{N}(r, \infty; f) = S(r, f)$. Let $c \neq 1$.

Then from (3.2) we get

$$\frac{1}{F} \equiv \frac{1}{c-1}\left(c\frac{G}{F} - 1\right). \tag{3.3}$$

Now using (3.3) and Lemma 2.1 we get

$$
\begin{aligned}
(n+m)\,T(r,f) = T(r,F) + O(1) &\le T\left(r, \frac{G}{F}\right) + S(r,f) \\
&= N\left(r,\infty; \frac{(f^nP(f))^{(k)}}{f^nP(f)}\right) + S(r,f) \\
&\le N_k(r,0; f^nP(f)) + k\overline{N}(r,\infty; f) + S(r,f) \\
&\le k\,\overline{N}(r,0;f) + m\,T(r,f) + S(r,f),
\end{aligned}
$$

which is impossible since $n > k+1$.

Hence $c = 1$. From (3.2) we get $F \equiv G$, i.e., $f^nP(f) \equiv [f^nP(f)]^{(k)}$ and so the result follows from Lemma 2.2.

Case 2: Let $\Phi_1 \not\equiv 0$.

Clearly $F \not\equiv G$. From (3.1) we get $m(r, \Phi_1) = S(r,f)$ and

$$
m(r,F) \le m\left(r, \frac{1}{\Phi_1}\right) + S(r,f). \tag{3.4}
$$

Then from (3.1) we get

$$
\begin{aligned}
N(r,\infty;F) - \overline{N}(r,\infty;F) &\le N(r,0;\Phi_1) + S(r,f) \\
&\le T\left(r,\frac{1}{\Phi_1}\right) - m\left(r,\frac{1}{\Phi_1}\right) + S(r,f) \\
&\le T(r,\Phi_1) - m\left(r,\frac{1}{\Phi_1}\right) + S(r,f) \\
&= N(r,\infty;\Phi_1) + m(r,\Phi_1) - m\left(r,\frac{1}{\Phi_1}\right) + S(r,f) \\
&\le N_{k+1}(r,0;F) - m\left(r,\frac{1}{\Phi_1}\right) + S(r,f) \\
&\le (k+1)\,\overline{N}(r,0;f) + m\,T(r,f) \\
&\quad - m\left(r,\frac{1}{\Phi_1}\right) + S(r,f).
\end{aligned} \tag{3.5}
$$

Now using (3.4), (3.5) and Lemma 2.1 we get

$$
\begin{aligned}
(n+m)\,T(r,f) &= T(r,F) + O(1) \\
&\le (k+1)\overline{N}(r,0;f) + m\,T(r,f) + \overline{N}(r,\infty;f) + S(r,f).
\end{aligned} \tag{3.6}
$$

Let

$$
\Phi_2 = F\left(\frac{F'}{F-1} - \frac{G'}{G-1}\right)^k. \tag{3.7}
$$

Clearly $\Phi_2 \not\equiv 0$. Since $n > k+1$, from (3.7) we get

$$\begin{aligned} &N(r,0;F) + k\,\overline{N}(r,0;f) \\ &\leq N(r,0;\Phi_2) + S(r,f) \\ &\leq T\left(r,\frac{1}{\Phi_2}\right) - m\left(r,\frac{1}{\Phi_2}\right) + S(r,f) \\ &\leq T(r,\Phi_2) - m\left(r,\frac{1}{\Phi_2}\right) + S(r,f) \\ &= N(r,\infty;\Phi_2) + m(r,\Phi_2) - m\left(r,\frac{1}{\Phi_2}\right) + S(r,f) \\ &\leq N(r,\infty;F) + k\,\overline{N}(r,\infty;f) + m(r,F) - m\left(r,\frac{1}{\Phi_2}\right) + S(r,f) \\ &\leq T(r,F) + k\,\overline{N}(r,\infty;f) - m\left(r,\frac{1}{\Phi_2}\right) + S(r,f). \end{aligned} \tag{3.8}$$

Also from (3.7) we get

$$m\left(r,\frac{1}{F}\right) \leq m\left(r,\frac{1}{\Phi_2}\right) + S(r,f). \tag{3.9}$$

Now using (3.8), (3.9) we get

$$\overline{N}(r,0;f) \leq \overline{N}(r,\infty;f) + S(r,f). \tag{3.10}$$

Then using (3.6), (3.10) we get

$$n\,T(r,f) \leq (k+2)\,\overline{N}(r,\infty;f) + S(r,f), \tag{3.11}$$

which is impossible since $n > k+2$. $\square$

Proof of Theorem 1.2. We omit the proof since it can be carried out in the line of proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. Let $F = \frac{f^n P(f)}{a}$ and $G = \frac{[f^n P(f)]^{(k)}}{a}$. Clearly F and G share 1 IM except for zeros and poles of $a(z)$ and so

$$\overline{N}(r,1;F) = \overline{N}(r,1;G) + S(r,f).$$

First we suppose $F \not\equiv G$.

Note that

$$\begin{aligned} \overline{N}(r,1;F) &\leq \overline{N}\left(r,1;\frac{G}{F}\right) + S(r,f) \\ &\leq T\left(r,\frac{G}{F}\right) + S(r,f) \\ &\leq N\left(r,\infty;\frac{G}{F}\right) + m\left(r,\infty;\frac{G}{F}\right) + S(r,f) \end{aligned} \tag{3.12}$$

$$= N\left(r,\infty;\frac{[f^nP(f)]^{(k)}}{f^nP(f)}\right) + m\left(r,\infty;\frac{[f^nP(f)]^{(k)}}{f^nP(f)}\right) + S(r,f)$$
$$\leq k\,\overline{N}(r,\infty;f) + N_k(r,0;f^nP(f)) + S(r,f)$$
$$\leq k\,\overline{N}(r,\infty;f) + k\,\overline{N}(r,0;f) + m\,T(r,f) + S(r,f).$$

Now using (3.12) and Lemma 2.1, we get from the second fundamental theorem that

$$\begin{aligned}(n+m)\,T(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) + S(r,F) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f^nP(f)) + \overline{N}(r,1;F) + S(r,f) \\ &\leq (k+1)\,\overline{N}(r,\infty;f) + (k+1)\,\overline{N}(r,0;f) \\ &\quad + 2m\,T(r,f) + S(r,f),\end{aligned} \tag{3.13}$$

which is impossible since $n > 2k + m + 2$.

Hence $F \equiv G$, i.e., $f^nP(f) \equiv [f^nP(f)]^{(k)}$. The remaining part follows from Lemma 2.2. □

Proof of Theorem 1.4. Let $F = \frac{f^nP(f)}{a}$ and $G = \frac{[f^nP(f)]^{(k)}}{a}$. Clearly F and G share 1 IM except for zeros and poles of $a(z)$ and so

$$\overline{N}(r,1;F) = \overline{N}(r,1;G) + S(r,f).$$

Let

$$\Phi_3 = \frac{F'(F-G)}{F(F-1)}. \tag{3.14}$$

We now consider the following two cases.

Case 1: Let $\Phi_3 \not\equiv 0$.

Clearly $F \not\equiv G$. From (3.14) we have $T(r,\Phi_3) = S(r,f)$ and

$$N_{(2}(r,0;f) \leq N(r,0;\Phi_3) + S(r,f) = S(r,f).$$

Therefore by the given condition we have

$$N(r,0;f) = O(N_{(2}(r,0;f)) = S(r,f).$$

Then from (3.13) we arrive at a contradiction.

Case 2: Let $\Phi_3 \equiv 0$.

This gives $F \equiv G$, i.e., $f^nP(f) \equiv [f^nP(f)]^{(k)}$. The remaining part follows from Lemma 2.2. □

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