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GEOMETRY OF DEGENERATE HYPERSURFACES

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ABSTRACT. In this paper we investigate the geometry of degenerate hypersurfaces of semi-Riemannian manifolds. It is found that such hypersurfaces carry a special type of distribution (the screen distribution) which plays an important role in determining the geometry of the hypersurface. A fundamental existence theorem for degenerate hypersurfaces is proved and several results on the degenerate hypersurfaces of Lorentz space are obtained. In particular, it is shown that the geometry of degenerate hypersurfaces of Lorentz space mainly reduces to the study of the geometry of the Riemannian foliation defined by the canonical screen distribution.

INTRODUCTION

The geometry of submanifolds in manifolds endowed with some geometrical structures has been intensively studied and several important results have been obtained (see [2],[7], [11], [13], [30], [31] and the references therein). In case the ambient space is a semi-Riemannian manifold, degenerate submanifolds have been introduced and investigated (cf. [3]-[7], [8],[10], [14]-[16],[23]-[25]). The study of a degenerate submanifold is essentially different from the one of a non-degenerate submanifold because of the lack of a canonical transversal bundle which has to replace the normal bundle from the classical theory of Riemannian submanifolds.

The purpose of the present paper is to present to the reader the author's point of view with respect to the differential geometry of degenerate hypersurfaces. In the first section we introduce the reader to the general theory of semi-Riemannian and degenerate manifolds. In section 2 we construct the transversal vector bundle (see [4] for the general case), give examples and define the induced geometric objects on a degenerate

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hypersurface. The main ingredient in our study is the screen distribution on a degenerate hypersurface. In case the ambient space is either a time-orientable Lorentz manifold or a semi- Euclidean space, we construct a canonical screen distribution. In section 3 we present the Gauss - Codazzi structure equations and prove the Fundamental Theorem for degenerate immersions. Section 4 is devoted to the study of totally geodesic and totally umbilical degenerate hypersurfaces. The canonical screen distribution is used in the last section for new results on degenerate hypersurfaces of a Lorentz space. Here we show that the study of differential geometry of a degenerate hypersurface M in a Lorentz space R_1^{m+2}, mainly reduces to the study of differential geometry of the Riemannian foliation defined by the canonical screen distribution $S(TM)$.

1. SEMI-RIEMANNIAN MANIFOLDS AND DEGENERATE MANIFOLDS

Let $\bar{M}$ be a real $(m+2)$-dimensional smooth manifold and $\bar{g}$ a symmetric tensor field of type (0,2) on $\bar{M}$. Thus $\bar{g}$ assigns smoothly to each point x of $\bar{M}$, a symmetric bilinear form $\bar{g}_x$ on the tangent space $T_x\bar{M}$. The dimension of the largest subspace $W_x \subset T_x\bar{M}$, on which $\bar{g}_x$ is negative definite is called the *index* of $\bar{g}_x$ on $T_x\bar{M}$. In the present paper we suppose $\bar{g}_x$ is non-degenerate on $T_x\bar{M}$ and the index of $\bar{g}_x$ is the same for all $x \in \bar{M}$. Thus each $T_x\bar{M}$ becomes a $(m+2)$- dimensional semi-Euclidean space. The tensor field $\bar{g}$ satisfying the above conditions is called a *semi-Riemannian (pseudo-Riemannian) metric* and $\bar{M}$ endowed with $\bar{g}$ is called a *semi-Riemannian (pseudo-Riemannian) manifold.*

The geometry of semi-Riemannian manifolds and its applications to relativity theory is very well presented in [19]. We only introduce here the main concepts from the geometry of semi-Riemannian manifolds, which are necessary for developing a theory of degenerate hypersurfaces.

As $T_x\bar{M}$ is a semi-Euclidean space, a tangent vector $u \in T_x\bar{M}$ is said to be

spacelike, if $\bar{g}_x(u,u) > 0 \quad$ or $\quad u = 0$, *timelike*, if $\bar{g}_x(u,u) < 0$,

lightlike (null), if $\bar{g}_x(u,u) = 0$ and $u \neq 0$.

We keep the same terminology for vector fields on $\bar{M}$ and even for vector fields on degenerate hypersurfaces.

Suppose q is the index of $\bar{M}$, that is, q is the common value of the index of $\bar{g}_x$ for any $x \in \bar{M}$. In case $q = 1$, $\bar{M}$ (resp. $\bar{g}$) is called a *Lorentz manifold* (resp. *Lorentz metric*). In the present paper we suppose $0 < q < m + 2$, that is, $\bar{g}$ cannot be positive or negative definite.

Denote by $\mathcal{F}(\bar{M})$ the algebra of smooth real functions on $\bar{M}$ and by $\Gamma(T\bar{M})$ the $\mathcal{F}(\bar{M})$-module of smooth vector fields on $\bar{M}$. Similarly, for any vector bundle E over $\bar{M}$, denote by $\Gamma(E)$ the $\mathcal{F}(\bar{M})$-module of smooth sections of E. Throughout the paper we keep the above notation for any other vector bundle.

Next, consider a linear connection $\bar{\nabla}$ on the semi-Riemannian manifold $(\bar{M}, \bar{g})$. We say that $\bar{\nabla}$ is a *metric connection (Riemannian connection)* if the metric tensor field $\bar{g}$ is parallel with respect to $\bar{\nabla}$, i.e., if we have

$$(1.1) \quad (\bar{\nabla}_X \bar{g})(Y,Z) = X(\bar{g}(Y,Z)) - \bar{g}(\bar{\nabla}_X Y, Z) - \bar{g}(Y, \bar{\nabla}_X Z) = 0,$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. The following result is of great importance for the geometry of semi-Riemannian manifolds.

Theorem 1.1 ([19], p. 61). *On a semi-Riemannian manifold there exists a unique torsion-free metric connection.*

The metric connection from the above theorem is called the *Levi-Civita connection* and it is given by

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= X(\bar{g}(Y,Z)) + Y(\bar{g}(X,Z)) - Z(\bar{g}(X,Y)) + \bar{g}([X,Y],Z) \\ (1.2) \qquad & +\bar{g}([Z,X],Y) - \bar{g}([Y,Z],X), \quad \forall\, X,Y,Z \in \Gamma(T\bar{M}). \end{aligned}$$

In case $\bar{M}$ is a semi-Riemannian manifold of constant sectional curvature c it is denoted by $\bar{M}(c)$. Then the curvature tensor field $\bar{R}$ of $\bar{M}(c)$

is given by

$$(1.3)\quad \bar{R}(X,Y)Z = c(\bar{g}(Y,Z)X - \bar{g}(X,Z)Y), \quad \forall\, X,Y,Z \in \Gamma(T\bar{M}).$$

Example 1.1. On R^{m+2} we consider the semi-Euclidean metric

$$(1.4)\qquad \bar{g}(x,y) = -\sum_{i=0}^{q-1} x^i y^i + \sum_{a=q}^{m+1} x^a y^a,$$

of index $0 < q < m+2$. Denote by R_q^{m+2} the semi-Euclidean space $(R^{m+2}, \bar{g})$ with $\bar{g}$ given by (1.4). The Levi-Civita connectin on R_q^{m+2} is defined by

$$(1.5)\qquad \bar{\nabla}_X Y = \sum_{A=0}^{m+1} X(Y^A)\frac{\partial}{\partial x^A},$$

where $Y = Y^A(\partial/\partial x^A)$. Finally, the curvature tensor field $\bar{R}$ of $\bar{\nabla}$ vanishes and therefore R_q^{m+2} is of constant curvature $c = 0$.

Example 1.2. In a semi-Euclidean space R_q^{m+2} define the *pseudosphere* of radius $r > 0$ by

$$S_q^{m+1}(r) = \left\{ x \in R_q^{m+2} : -\sum_{i=0}^{q-1}(x^i)^2 + \sum_{a=q}^{m+1}(a^a)^2 = r^2 \right\},$$

and the *pseudohyperbolic space* of radius $r > 0$ by

$$H_{q-1}^{m+1}(r) = \left\{ x \in R_q^{m+2} : -\sum_{i=0}^{q-1}(x^i)^2 + \sum_{a=q}^{m+1}(a^a)^2 = -r^2 \right\}.$$

Both $S_q^{m+1}(r)$ and $H_{q-1}^{m+1}(r)$ are totally umbilical semi- Riemannian submanifolds of index q and $q-1$ of constant curvature $c = 1/r^2$ and $c = -1/r^2$, respectively.

Next, suppose M is a real $(m+1)$-dimensional smooth manifold and g is a symmetric tensor field of type (0,2) on M such that g_x is of constant index q on T_xM for any $x \in M$. Define the *radical subspace* of T_xM by (cf. [1])

$$Rad\, T_xM = \{ v \in T_xM : g(v,u) = 0, \quad \forall\, u \in T_xM \}$$

.

We say that (M, g) is an *r-degenerate manifold* if the mapping $Rad\,TM$ assigns to each $x \in M$ the radical subspace $Rad\,T_xM$ defines a smooth distribution of rank $r > 0$ on M. In this case $Rad\,TM$ is called the *radical distribution* on M . By the above definition we have

$$(1.6) \qquad g(V, X) = 0, \quad \forall\, X \in \Gamma(TM), \quad V \in \Gamma(Rad\,TM).$$

Moreover, it is easy to see that (M, g) is r-degenerate if and only if g has a constant rank $m - r + 1$ on M.

In literature such manifolds have been studied under several names: singular Riemannian spaces ([14],[18],[28],[29]), degenerate Riemannian manifolds ([12], [20]), degenerate pseudo-Riemannian manifolds ([26]), degenerate manifolds ([17]), isotropic spaces ([25]) and lightlike manifolds ([7]).

Now, suppose $Rad\,TM$ is an integrable distribution. Then there exists a coordinate system $(\mathcal{U}; x^1, \ldots, x^{m+1})$ on M such that (x^α), $\alpha \in \{1, \ldots, r\}$ are the coordinates on a leaf M' of $Rad\,TM$ and $x^i = c^i$, $i \in \{r+1, \ldots, m+1\}$ are the local equations of M'. As g is r-degenerate on TM, from (1.6) we derive

$$g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^i}\right) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right) = 0.$$

Thus the matrix of g with respect to the natural frame field becomes

$$(1.7) \quad [g] = \begin{bmatrix} O_{r,r} & O_{r,m-r+1} \\ O_{m-r+1,r} & g_{ij}(x^1, \ldots, x^{m+1}) \end{bmatrix}, \quad i, j \in \{r+1, \ldots, m+1\}.$$

In case, with respect to the above coordinate system we have

$$(1.8) \qquad \frac{\partial g_{ij}}{\partial x^\alpha} = 0, \quad \alpha \in \{1, \ldots, r\}, \quad i, j \in \{r+1, \ldots, m+1\},$$

we say that M is a *Reinhart degenerate manifold.* We use the name of Reinhart because the class of Riemannian foliations satisfying (1.8) *(bundle-like metrics)* introduced in 1959 by Reinhart [21], were later named as Reinhart spaces (cf. [27]).

A vector field X on the degenerate manifold M is said to be a *Killing vector field* if

$$(1.9) \quad (L_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g([X, Z], Y) = 0$$

for any $Y, Z \in \Gamma(TM)$. A distribution D on M is said to be a *Killing distribution* if each vector field that belongs to D is a Killing vector field.

An important research matter in the geometry of degenerate manifolds is the study of the existence of some particular linear connections on such manifolds. In this respect we state the following result.

Theorem 1.2. *Let (M, g) be a degenerate manifold. Then the following assertions are equivalent:*

(i) (M, g) is a Reinhart degenerate manifold.

(ii) $Rad\,TM$ is a Killing distribution.

(iii) There exists a torsion-free linear connection ∇ on M such that g is parallel with respect to ∇.

Proof. (i) $\Rightarrow$ (ii). Suppose M is a Reinhart degenerate manifold. As $Rad\,TM$ is integrable we may consider a coordinate system $(\mathcal{U}; x^1, \ldots, x^{m+1})$ such that any $X \in \Gamma(Rad\,TM)$ is locally expressed as $X = X^\alpha(\partial/\partial x^\alpha)$. In this way, $L_X g = 0$ is equivalent to

$$(1.10) \quad X^\alpha \left\{ \frac{\partial g(Y, Z)}{\partial x^\alpha} - g([\partial/\partial x^\alpha, Y], Z) - g([\partial/\partial x^\alpha, Z], Y) \right\} = 0,$$

for any $Y, Z \in \Gamma(TM)$. By using (1.6) and taking into account that $Rad\ TM$ is integrable, it is easy to check that when at least one of vector fields Y and Z belongs to $Rad\,TM$, (1.10) is identically satisfied. Now take $Y = \partial/\partial x^i$ and $Z = \partial/\partial x^j$, $i, j \in \{r+1, \ldots, m+1\}$, and (1.10) follows by using (1.8). Hence $Rad\,TM$ is a Killing distribution.

(ii) $\Rightarrow$ (i). Suppose $Rad\ TM$ is a Killing distribution, that is, (1.9) holds for any $X \in \Gamma(Rad\ TM)$ and $Y, Z \in \Gamma(TM)$. Consider $Y \in \Gamma(Rad\,TM)$ in (1.9) and by using (1.6) obtain $g([X, Y], Z) = 0$, for any

$Z \in \Gamma(TM)$. Hence $[X,Y] \in \Gamma(Rad\, TM)$, that is, $Rad\, TM$ is involutive, and by Frobenius theorem, it is integrable. Finally, take $X = \partial/\partial x^\alpha \in \Gamma(Rad\, TM)$, $Y = \partial/\partial x^i$ and $Z = \partial/\partial x^j$ in (1.9) and obtain (1.8). Hence (M,g) is a Reinhart degenerate manifold.

(iii) $\Rightarrow$ (ii) Suppose there exists a torsion-free linear connection ∇ on M and g is parallel with respect to ∇. Then, by using (1.1), we obtain

$$\begin{aligned}(L_X g)(Y,Z) &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = Y(g(X,Z)) + Z(g(X,Y)) \\ &\quad -g(X,\nabla_Y Z) - g(X, \nabla_Z Y) = 0,\end{aligned}$$

for any $X \in \Gamma(Rad\, TM)$ and $Y, Z \in \Gamma(TM)$. Hence $Rad\, TM$ is a Killing distribution on M.

(ii) $\Rightarrow$ (iii). As (ii) is satisfied, from the proof of (ii) $\Rightarrow$ (i) it follows that $Rad\, TM$ is integrable. Consider $Rad\, TM$ as a $(m+1+r)$-dimensional manifold with local coordinates $(x^\alpha, x^i, y^\alpha)$, where (x^α, x^i) are the local coordinates on M induced by the foliation determined by $Rad\, TM$ and (y^α) are coordinates on fibres of vector bundle $Rad\, TM$. Thus the coordinate transformations on $Rad\, TM$ are given by

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^1, \ldots, x^{m+1}), \tilde{x}^i = \tilde{x}^i(x^{r+1}, \ldots, x^{m+1}), \tilde{y}^\alpha = B^\alpha_\beta(x) y^\beta,$$

where $B^\alpha_\beta(x) = \partial \tilde{x}^\alpha / \partial x^\beta$. Hence we obtain

$$\frac{\partial}{\partial y^\alpha} = B^\beta_\alpha(x) \frac{\partial}{\partial \tilde{y}^\beta}.$$

Thus there exists a vector bundle NM over M, locally spanned by $\{\partial/\partial y^\alpha\}$, $\alpha \in \{1, \ldots, r\}$, and transversal to TM, i.e., we have

$$T(Rad\, TM)_{|M} = TM \oplus NM. \qquad (1.11)$$

Next, as M is assumed to be paracompact, we consider a Riemannian metric G on M and a complementary orthogonal distribution $S(TM)$ to $Rad\, TM$ in TM with respect to G. Thus (1.11) becomes

$$T(Rad\, TM)_{|M} = S(TM) \oplus Rad\, TM \oplus NM. \qquad (1.12)$$

We should note that NM and $Rad\,TM$ are vector bundles of rank r over M such that the transition matrices from $\{\partial/\partial y^\alpha\}$ to $\{\partial/\partial \tilde{y}^\beta\}$ and from $\{\partial/\partial x^\alpha\}$ to $\{\partial/\partial \tilde{x}^\beta\}$ are the same. Hence any section $N = N^\alpha(\partial/\partial y^\alpha)$ of NM defines a section $N^* = N^\alpha(\partial/\partial x^\alpha)$ of $Rad\ TM$. Now, denote by A, B and C the projection morphisms of $T(Rad\,TM)_{|M}$ on $S(TM)$, $Rad\,TM$ and NM respectively, and define

$$(1.13)\quad \bar{g}(\bar{X},\bar{Y}) = g(A\bar{X}, A\bar{Y}) + G(B\bar{X},(C\bar{Y})^*) + G(B\bar{Y},(C\bar{X})^*),$$

for any $\bar{X},\bar{Y} \in \Gamma(T(Rad\ TM)_{|M})$. It is easy to verify that $\bar{g}$ is a semi-Riemannian metric on the manifold $Rad\,TM$ and the degenerate metric g is the restriction of $\bar{g}$ to $\Gamma(TM)$. Denote by $\bar{\nabla}$ the Levi-Civita connection on $(Rad\,TM, \bar{g})$ and set

$$(1.14)\qquad \bar{\nabla}_X Y = \nabla_X Y + B^\alpha(X,Y)\frac{\partial}{\partial y^\alpha},\quad \forall\, X,Y \in \Gamma(TM),$$

where $\nabla_X Y \in \Gamma(TM)$ and $B^\alpha(X,Y) \in \mathcal{F}(M)$. It follows that ∇ is a torsion-free linear connection on M and B^α are symmetric bilinear forms on $\Gamma(TM)$. Moreover, by using (1.9), (1.13) and (1.14), and taking into account that $\bar{g}$ is parallel with respect to $\bar{\nabla}$ we obtain

$$0 = (L_X g)(Y,Z) = -\bar{g}(X, \bar{\nabla}_Y Z + \bar{\nabla}_Z Y) = -2B^\alpha(Y,Z)G\left(X, \frac{\partial}{\partial x^\alpha}\right),$$

for any $X \in \Gamma(Rad\ TM)$ and $Y,Z \in \Gamma(TM)$. Since $r > 0$ and G is a Riemannian metric on the distribution $Rad\,TM$ we deduce $B^\alpha(Y,Z) = 0$, for $\alpha \in \{1,\ldots,r\}$. It follows $\bar{\nabla}_Y Z = \nabla_Y Z$, that is g is parallel with respect to ∇. This completes the proof of the theorem.

We close this section with some remarks on the assertions of Theorem 1.2. The equivalence of (ii) and (iii) was first proved in [20] by using the theory of G- structures. The class of linear connections from (iii) was first considered in [18]. The quadratic forms satisfying (1.8) were studied in [9] and [22]. Finally, we consider the vector bundle $TM^* = TM/Rad\,TM$ and the canonical projection $p : TM \to TM^*$. Then on TM^* there exists a semi- Riemannian metric g^* defined by

$$g^*(pX, pY) = g(X,Y),\ \forall\ X,Y \in \Gamma(TM).$$

In [17] was proved the equivalence of (ii) with

(iv) *There exists a unique metric connection* ∇^* *on* TM^* *such that*

$$\nabla^*_X pY - \nabla^*_Y pX - p([X,Y]) = 0, \quad \forall\, X, Y \in \Gamma(TM).$$

It seems to the author that results from the theory of Riemannian foliations might be of interest for further research into degenerate manifolds. Actually, Vránceanu [29] has already applied his theory of non-holonomic spaces to the study of some invariants in the geometry of degenerate manifolds.

2. THE INDUCED GEOMETRIC OBJECTS ON A DEGENERATE HYPERSURFACE

Let $(\bar{M}, \bar{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold of index $0 < q < m+2$ and M be a hyprsurface of $\bar{M}$. Then $\bar{g}$ induces a symmetric tensor field g of type (0,2) on M. Suppose (M, g) is a degenerate manifold, that is, *Rad TM* defines a distribution of rank $r > 0$ on M. As usual, for any $x \in M$ define

$$T_xM^\perp = \{v \in T_x\bar{M} : \bar{g}(v,u) = 0, \quad \forall\, u \in T_xM\},$$

and consider the vector bundle

$$TM^\perp = \bigcup_{x \in M} T_xM^\perp.$$

It is easy to see that M is degenerate if and only if $Rad\, TM = TM^\perp$. Therefore, in this case $r = 1$, and $TM^\perp$ (the former normal bundle for a non- degenerate hypersurface) becomes a distribution on M and rank $g = m$ on M.

Now suppose M is locally given by the equation

$$F(x^0, \ldots, x^{m+1}) = 0, \tag{2.1}$$

where F is differentiable on a domain $D \subset R^{m+2}$ and rank $[F'_{x^0}, \ldots, F'_{x^{m+1}}] = 1$. Denote $\bar{g}_{AB} = \bar{g}(\partial/\partial x^A, \partial/\partial x^B)$ and consider the gradient vector field of F defined by

$$grad\ F = \bar{g}^{AB} F'_{x^A} F'_{x^B}, \tag{2.2}$$

where $\bar{g}^{AB}$ are the entries of the inverse matrix of $[\bar{g}_{AB}]$. As $grad\ F$ is normal to M we conclude that M is degenerate if and only if $grad\ F$ at any $x \in M$ lies in T_xM, i.e., $(grad\ F)_x$ is a null vector with respect to $\bar{g}$. Thus by using (2.2) we obtain the following characterization for degenerate hypersurfaces.

Theorem 2.1. *M is a degenerate hypersurface of $\bar{M}$ if and only if on any coordinate neighborhood $\mathcal{U} \subset M$ on which M is given by (2.1), F satisfies the partial differential equation*

$$\bar{g}^{AB} F'_{x^A} F'_{x^B} = 0, \tag{2.3}$$

at any point of M.

In order to study the geometry of the degenerate hypersurface M we need a complementary vector bundle to TM in $T\bar{M}$ which is going to replace $TM^{\perp}$ from the classical theory of Riemannian hypersurfaces. To this end we consider a complementary vector bundle $S(TM)$ to $TM^{\perp}$ in TM, i.e., we have

$$TM = S(TM) \perp TM^{\perp}. \tag{2.4}$$

Clearly $S(TM)$ is also orthogonal to $TM^{\perp}$. Throughout the paper $\perp$ and $\oplus$ will denote orthogonal direct sum and a direct but not orthogonal sum respectively. As M is assumed paracompact, there always exists $S(TM)$. Moreover, we show later in this section how to construct a canonical distribution on some special degenerate hypersurfaces. Motivated by the fact that the lightlike rays on the null cone (see Example 2.1) lie on $TM^{\perp}$ and hence fibres of $S(TM)$ are visualized as transversal screens to these rays, we call $S(TM)$ a *screen distribution.*

It is easy to verify that $S(TM)$ is non-degenerate and therefore we have

$$T\bar{M}_{|M} = S(TM) \perp S(TM)^{\perp}, \tag{2.5}$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\bar{M}_{|M}$. Moreover, $S(TM)^{\perp}$ is a vector bundle of rank 2 and $TM^{\perp}$ is a vector subbundle of $S(TM)^{\perp}$. Consider a complementary vector bundle F to $TM^{\perp}$ in $S(TM)^{\perp}$ and take sections $V \in \Gamma(F_{|\mathcal{U}})$ and $\xi \in \Gamma(TM^{\perp}_{|\mathcal{U}})$, where $\mathcal{U}$ is a coordinate neighborhood of M. Note that $\bar{g}(\xi, V) \neq 0$, otherwise $S(TM)^{\perp}$ would be degenerate on $\mathcal{U}$. Finally, define the vector field

$$N = \frac{1}{\bar{g}(\xi, v)} \left\{ V - \frac{\bar{g}(V,V)}{2\bar{g}(\xi, V)} \xi \right\}, \tag{2.6}$$

on $\mathcal{U}$. By direct calculation, it follows that

$$g(N,N) = g(N,W) = 0, \quad \forall\, W \in \Gamma(S(TM)_{|\mathcal{U}}), \tag{2.7}$$

and

$$g(N, \xi) = 1. \tag{2.8}$$

In case we take another coordinate neighborhood $\mathcal{U}^* \subset M$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \phi$, denote by N^* the corresponding vector field given by (2.6). Then $\xi^* = \alpha\xi$ and $N^* = (1/\alpha)N$. Hence we have a line vector bundle over M whose sections satisfy (2.7) and (2.8). Finally, it is easy to check that any line bundle satisfying (2.7) and (2.8) has sections given by (2.6). Clearly N is nowhere tangent to M. Therefore we may state the following result on which is based the theory of degenerate hypersurfaces.

Theorem 2.2. *Let $(M, g, S(TM))$ be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a unique line vector bundle $tr(TM)$ over M such that its sections satisfy (2.7) and (2.8) on any coordinate neighborhood.*

By using (1.1), (1.2) and Theorem 2.2 we obtain

$$T\bar{M}_{|M} = S(TM) \perp (TM \oplus tr(TM)) = TM \oplus tr(TM). \tag{2.9}$$

The last decomposition in (2.9) is a motivation for us to call $tr(TM)$ the *transversal vector bundle* of M with respect to $S(TM)$. For the construction of the transversal vector bundle to a degenerate submanifold of arbitrary codimension see [4].

Example 2.1. In R^4_1 consider the null cone M given by the equation

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0, \quad x^0 \neq 0.$$

It is easy to verify that

$$\xi = \sum_{A=0}^{3} x^A \frac{\partial}{\partial x^A},$$

is a null vector field tangent to M and $\bar{g}(\xi, X) = 0$ for any $X \in \Gamma(TM)$. Hence $TM^{\perp} = Span\ \{\xi\}$. Then consider along M the vector field

$$N = \frac{1}{2(x^0)^2}\left(-x^0\frac{\partial}{\partial x^0} + \sum_{a=1}^{3} x^a \frac{\partial}{\partial x^a}\right).$$

By using $\bar{g}$ from (1.4) in case of R_1^4 we deduce $\bar{g}(N, \xi) = 1$ and $\bar{g}(N, N) = 0$. Hence we may take $tr(TM) = Span\ \{N\}$ and obtain

$$S(TM) = \left\{X \in \Gamma(TM) :\ X = \sum_{a=1}^{3} X^a \frac{\partial}{\partial x^a},\quad \sum_{a=1}^{3} x^a X^a = 0\right\}.$$

In a similar way it follows that the null cone of R_q^{m+2} is a degenerate hypersurface. In this case we have

$$\xi = \sum_{A=0}^{m+1} x^A \frac{\partial}{\partial x^A},$$

$$N = \frac{1}{2\sum_{i=0}^{q-1}(x^i)^2}\left\{-\sum_{i=0}^{q-1} x^i \frac{\partial}{\partial x^i} + \sum_{a=q}^{m+1} x^a \frac{\partial}{\partial x^a}\right\},$$

$$S(TM) = \left\{X \in \Gamma(TM) : X = \sum_{A=0}^{m+1} x^A \frac{\partial}{\partial x^A}, \sum_{i=0}^{q-1} x^i X^i = 0, \sum_{a=q}^{m+1} x^a X^a = 0\right\}.$$

Example 2.2. In R_2^4 consider the hypersurface M given by the equation

$$x^3 = x^0 + \frac{1}{2}(x^1 + x^2)^2.$$

It is easy to verify that M is a degenerate hypersurface and

$$TM^{\perp} = Span\left\{\xi = \frac{\partial}{\partial x^0} + (x^1 + x^2)\frac{\partial}{\partial x^1} - (x^1 + x^2)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right\}.$$

Consider

$$S(TM) = Span\left\{W_1 = \frac{\partial}{\partial x^1} - (x^1 + x^2)\frac{\partial}{\partial x^0}, W_2 = \frac{\partial}{\partial x^2} + (x^1 + x^2)\frac{\partial}{\partial x^3}\right\}$$

and obtain

$$N = \frac{1}{2(1+(x^1+x^2)^2)}\left\{\frac{\partial}{\partial x^0} + (x^1+x^2)\frac{\partial}{\partial x^1} + (x^1+x^2)\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\right\}.$$

Example 2.3. We cut the unit pseudosphere $S_1^{2m+1}(1)$ by the hyperplane $x^0 - x^1 = 0$ and obtain a degenerate hypersurface M of $S_1^{2m+1}(1)$ with

$$TM^\perp = Span\left\{\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right\}.$$

Consider the screen distribution $S(TM)$ spanned by

$$\left\{W_{i-1} = x^{2m+1}\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^{2m+1}}\right\}, \quad i \in \{2, \ldots, 2m\}.$$

Then by using (2.6) we obtain the transversal vector bundle spanned by

$$N = -\frac{1}{2}\left\{(1+(x^0)^2)\frac{\partial}{\partial x^0} + ((x^0)^2 - 1)\frac{\partial}{\partial x^1} + 2x^0\sum_{a=2}^{2m+1} x^a\frac{\partial}{\partial x^a}\right\}.$$

Now, we come back to the general study of a degenerate hypersurface (M, g) of $(\bar{M}, \bar{g})$. Denote by $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then by using the second decomposition in (2.9) we obtain

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \text{and} \tag{2.10}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$. It follows that ∇ is a torsion-free linear connection on M, but in general g is not parallel with respect to ∇. More precisely, by using (2.10) and taking into account that $\bar{\nabla}$ is a metric connection, we obtain

$$(\nabla_X g)(Y, Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \quad \forall\, X, Y, Z \in \Gamma(TM), \tag{2.12}$$

where

$$\eta(X) = \bar{g}(X, N), \quad \forall\, X \in \Gamma(TM). \tag{2.13}$$

B is a symmetric bilinear form on $\Gamma(TM)$ which we call the *second fundamental form* of M. The following result is important for the study which follows in the paper.

Proposition 2.1. The second fundamental form of M does not depend on the screen distribution on M.

Proof. Suppose $S(TM)$ and $S(TM)'$ are two screen distributions on M and B and B' are the corresponding second fundamental forms. Then by using (2.10) and (2.8) for both distributions we obtain

$$(2.14) \qquad B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi) = B'(X,Y), \quad \forall X, Y \in \Gamma(TM),$$

which proves our assertion.

As a consequence of (2.14) we deduce

$$(2.15) \qquad B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM);$$

that is, B is degenerate.

A_N from (2.11) is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ which we call the *shape operator* of M. The main property of A_N is stated in the next proposition.

Proposition 2.2. The shape operator of a degenerate hypersurface has an eigenvalue $\lambda = 0$.

Proof. From (2.11), taking into account that N is a null vector field, we obtain

$$(2.16) \qquad \bar{g}(A_N X, N) = 0 \quad \forall\, X \in \Gamma(TM).$$

Thus $A_N X$ has no component in $TM^{\perp}$ which implies rank $A_N < m+1$, and we get the assertion.

Remark 2.1. In general, the 1-form τ from (2.11) is not identically zero as it is in case of non-degenerate hypersurfaces. For the null cone we have $\tau(\xi) = -1$ (see Example 4.3).

Remark 2.2. We note that both B and τ depend on the section ξ of $TM^{\perp}$. Indeed, in case we have $\xi^* = \alpha\xi$, it follows $N^* = (1/\alpha)N$ and from (2.10) and (2.11) we get $B^* = \alpha B$ and $\tau(X) = \tau^*(X) + X(\log \alpha)$ for any $X \in \Gamma(TM)$.

As in the case of non-degenerate hypersurfaces we call (2.10) and (2.11) the *Gauss* and *Weingarten formulas* for the degenerate hypersurface M.

Now, according to the decomposition (2.4), we set locally

$$\nabla_X Y = \nabla^*_X Y + C(X,Y)\xi, \tag{2.17}$$

and

$$\nabla_X \xi = -A^*_\xi X + \mathcal{E}(X)\xi, \tag{2.18}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$. We call C the *second fundamental form* of $S(TM)$. From (2.17) we deduce that C is symmetric on $\Gamma(S(TM))$ if and only if $S(TM)$ is integrable. A^*_ξ is a $\Gamma(S(TM))$-valued linear operator on $\Gamma(TM)$ and we call it the *shape operator* of $S(TM)$. By using (2.18), (2.10) and (2.15) we obtain $\mathcal{E} = -\tau$. Hence (2.18) becomes

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi. \tag{2.19}$$

We have to note that B and A_N are not related as in the case of non-degenerate hypersurfaces. More precisely, by using (2.10), (2.11), (2.17) and (2.19) we easy obtain

$$B(X,Y) = \bar{g}(A^*_\xi X, Y), \quad \forall\, X, Y \in \Gamma(TM), \tag{2.20}$$

and

$$C(X,Z) = \bar{g}(A_N X, Z), \quad \forall\, X \in \Gamma(TM), \quad Z \in \Gamma(S(TM)). \tag{2.21}$$

Take $X = \xi$ in (2.20) and by using (2.15) we derive

$$A^*_\xi \xi = 0. \tag{2.22}$$

We call (2.17) and (2.18) the *Gauss* and *Weingarten formulas* for the screen distribution $S(TM)$.

We show here how to construct a screen distribution on a degenerate hypersurface of a time-orientable Lorentz manifold. To this end we recall (see [19], p. 149) that on a time-orientable Lorentz manifold $(\bar{M}, \bar{g})$ there exists a unit timelike vector field which we denote by L. Denote by D the timelike distribution spanned by L on $\bar{M}$ and set

$$T\bar{M} = D \perp D^{\perp},$$

where $D^{\perp}$ is the spacelike distribution that is complementary orthogonal to D. Thus at any point of M, ξ has the unique decomposition

$$\xi = \xi^{-} + \xi^{+}, \tag{2.23}$$

where $\xi^{-} \in \Gamma(D)$ and $\xi^{+} \in \Gamma(D^{\perp})$. Suppose $\xi^{-} = aL$ and define

$$V = -aL, \tag{2.24}$$

where a is a differentiable function on $\mathcal{U} \subset M$. It is easy to check that V is nowhere tangent to M. Indeed $\bar{g}(V, \xi) = a^2 \neq 0$ at any point of M, otherwise $\xi^{-} = 0$ which together with (2.23) implies $\xi = 0$, and this is a contradiction. As L is globally defined on $\bar{M}$ we obtain a line bundle $Span\{V\}$ over M. We consider the vector bundle $K = TM^{\perp} \oplus Span\{V\}$ and claim that it is non-degenerate. In fact, suppose there exists a point $x \in M$ and a vector $X_x \in K_x$ such that

$$\bar{g}(X_x, \xi_x) = 0; \quad \bar{g}(X_x, V_x) = 0.$$

From the first equality it follows $X_x \in T_xM$. As $T_xM \cap K_x = T_xM^{\perp}$ we deduce that X_x is colinear with ξ_x, and hence $\bar{g}(V_x, X_x) \neq 0$, which contradicts the above second equality. Finally, denote by $S(TM)$ the orthogonal complementary vector bundle to K in $T\bar{M}_{|M}$. As $S(TM)$ is ortogonal to $TM^{\perp}$, $S(TM) \cap TM^{\perp} = \{0\}$ and it is of rank m, it follows that $S(TM)$ is a distribution on M and $TM = S(TM) \perp TM^{\perp}$. Hence $S(TM)$ is a screen distribution on M which from now on we call the *canonical screen distribution* on M. By using (2.6) and (2.24) we obtain

$$N = \frac{1}{a^2}(V + \frac{1}{2}\xi). \tag{2.25}$$

We call the vector bundle spanned by N the *canonical transversal bundle* of M.

Theorem 2.3. *Let M be a time-orientable Lorentz manifold such that the timelike distribution D is parallel with respect to $\bar{\nabla}$. Then the canonical screen distribution on M is integrable.*

Proof. Let $X, Y \in \Gamma(S(TM))$. Then by using (2.25) and taking into account that $\bar{\nabla}$ is a metric connection we derive

$$\begin{aligned}\bar{g}([X,Y],N) &= -\frac{1}{a}\bar{g}([X,Y],L) = -\frac{1}{a}\bar{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, L) \\ &= \frac{1}{a}\{\bar{g}(Y,\bar{\nabla}_X L) - \bar{g}(X,\bar{\nabla}_Y L)\} = 0.\end{aligned}$$

Due to (2.8) we conclude that $[X,Y]$ has no component with respect to ξ. Hence $[X,Y] \in \Gamma(S(TM))$, that is, $S(TM)$ is integrable.

The above canonical screen distribution has been constructed (see [3]) for any degenerate hypersurface M of R_q^{m+2}, $q > 1$, in the following way. Suppose M is locally given by the equations

$$x^A = f^A(u^0, \ldots, u^m), \quad A \in \{0, \ldots, m+1\}.$$

Then $TM^\perp$ is spanned by

$$\xi = \sum_{i=0}^{q-1}(-1)^i D^i \frac{\partial}{\partial x^i} + \sum_{a=q}^{m+1}(-1)^{a-1} D^a \frac{\partial}{\partial x^a},$$

where D^A are the determinants

$$D^A = \begin{vmatrix} \frac{\partial f^0}{\partial u^0} & \cdots & \frac{\partial f^{A-1}}{\partial u^0} & \frac{\partial f^{A+1}}{\partial u^0} & \cdots & \frac{\partial f^{m+1}}{\partial u^0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f^0}{\partial u^m} & \cdots & \frac{\partial f^{A-1}}{\partial u^m} & \frac{\partial f^{A+1}}{\partial u^m} & \cdots & \frac{\partial f^{m+1}}{\partial u^m} \end{vmatrix}.$$

Then locally on $\mathcal{U} \subset M$ we consider the vector field

$$V = \sum_{i=0}^{q-1}(-1)^{i-1} D^i \frac{\partial}{\partial x^i}, \tag{2.26}$$

which is nowhere tangent to M. Moreover, we show that all vector fields V span a line bundle H over M. Then it is proved that the complementary orthogonal vector bundle to the non-degenerate vector bundle $K = H \oplus TM^{\perp}$ is a screen distribution on M. This is the canonical distribution on M.

Remark 2.3. In general, the canonical screen distribution on a degenerate hypersurface of R_q^{m+2} with $q > 1$ is not integrable. In R_2^4 consider M from Example 2.2 and conclude that the canonical screen distribution is spanned by $\{W_1, W_2\}$. But $[W_1, W_2] = \partial/\partial x^0 + \partial/\partial x^3$, which does not lie in $\Gamma(S(TM))$.

3. THE GAUSS-CODAZZI EQUATIONS AND THE FUNDAMENTAL THEOREM FOR DEGENERATE HYPERSURFACES

Let M be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Denote by R and $\bar{R}$ the curvature tensor fields of ∇ and $\bar{\nabla}$ respectively. Then by using (2.10) and (2.11) we obtain

$$\begin{aligned}(3.1)\quad \bar{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_NY - B(Y,Z)A_NX + \\ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, where we set

$$(3.2)\quad (\nabla_X B)(Y,Z) = X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

Taking the components of $\bar{R}(X,Y)Z$ with respect to $S(TM)$, $TM^{\perp}$ and $tr(TM)$, by direct calculations we obtain the following result.

Theorem 3.1. (cf. [6]). *Let $(M, g, S(TM))$ be a degenerate hypersurface of the semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the Gauss- Codazzi equations of M are given by:*

$$\begin{aligned}(3.3)\quad \bar{g}(\bar{R}(X,Y)Z, W) &= g((R(X,Y)Z, W) + B(X,Z)C(Y,W) \\ &\quad -B(Y,Z)C(X,W),\end{aligned}$$

$$\begin{aligned}(3.4)\quad \bar{g}(\bar{R}(X,Y)Z, \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad +\tau(X)B(Y,Z) - \tau(Y)B(X,Z),\end{aligned}$$

$$(3.5)\qquad \begin{aligned}\bar{g}(\bar{R}(X,Y)W,N) &= \bar{g}(R(X,Y)W,N)\\ &= (\nabla_X C)(Y,W) - (\nabla_Y C)(X,W)\\ &\quad +\tau(Y)C(X,W) - \tau(X)C(Y,W),\end{aligned}$$

$$(3.6)\quad \begin{aligned}\bar{g}(\bar{R}(X,Y)\xi,N) &= \bar{g}(R(X,Y)\xi,N) = C(Y,A^*_\xi X) - C(X,A^*_\xi Y)\\ &\quad -2d\tau(X,Y),\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$ and $W \in \Gamma(S(TM))$, where we set

$$(3.7)\quad (\nabla_X C)(Y,W) = X(C(Y,W)) - C(\nabla_X Y,W) - C(Y,\nabla^*_X W),$$

and

$$(3.8)\qquad d\tau(X,Y) = \frac{1}{2}\{X(\tau(Y)) - Y(\tau(X)) - \tau([X,Y])\}.$$

As in the case of Riemannian manifolds we define the *Ricci tensor* of the induced connection ∇ on $(M, g, S(TM))$ by

$$Ric\,(X,Y) = \text{trace}\,\{Z \;\rightarrow\; R(X,Z)Y\},\quad \forall\, X,Y \in \Gamma(TM).$$

Consider the local frame field $\{W_i, \xi\}$ on $\mathcal{U} \subset M$, where $\{W_i\}$, $i \in \{1,\ldots,m\}$ is an orthonormal basis of $\Gamma(S(TM)_{|\mathcal{U}})$. Then we deduce

$$(3.9)\qquad Ric\,(X,Y) = \sum_{i=1}^{m} \varepsilon_i g(R(X,W_i)Y,W_i) + \bar{g}(R(x,\xi)Y,N),$$

where $\varepsilon_i = -1$ or $+1$ according as W_i is timelike or spacelike respectively. By using the Bianchi first identity with respect to ∇ and taking account of (3.3) in (3.9) we obtain

$$(3.10)\qquad \begin{aligned}Ric\,(X,Y) - Ric\,(Y,X) &= \sum_{i=1}^{m} \varepsilon_i\{C(X,W_i)B(Y,W_i)\\ &\quad -C(Y,W_i)B(X,W_i)\}\\ &\quad +\bar{g}(R(X,Y)\xi,N).\end{aligned}$$

On the other hand, by direct calculation, using (2.20) and (2.21), we deduce

$$(3.11)\qquad C(Y,A^*_\xi X) = \sum_{i=1}^{m} \varepsilon_i B(X,W_i)C(Y,W_i).$$

Finally, by using (3.11) and (3.6) in (3.10) we obtain

$$Ric\,(X,Y) - Ric(Y,X) = -2d\tau(X,Y), \quad \forall X,Y \in \Gamma(TM).$$

Hence we may state the following important result.

Theorem 3.2. *Let $(M, g, S(TM))$ be a degenerate hypersurface of $(\bar{M}, \bar{g})$. Then the Ricci tensor of the induced connection ∇ is symmetric if and only if on each $\mathcal{U} \subset M$ there exists a closed 1- form τ, i.e., $d\tau = 0$ on $\mathcal{U}$.*

In particular, for a degenerate hypersurface of a 4-dimensional Lorentz manifold, Theorem 3.2 was proved in [15].

Next, we want to present a Fundamental Theorem for degenerate hypersurfaces, that is, to find geometrical conditions for the existence of an immersion of a degenerate manifold in a semi-Euclidean space. To this end we start with a 1-degenerate $(m+1)$-dimensional manifold M of index $q-1$, $m > 0$, $q > 0$. Suppose there exists a line vector bundle F over M such that $E = TM \oplus F$ is a semi- Riemannian vector bundle with a semi-Riemannian metric $\bar{g}$ satisfying the condition

$$(\mathrm{C}_1) \quad \bar{g}(X,Y) = g(X,Y), \; \bar{g}(Z,V) = \bar{g}(V,V') = 0,$$

for any $X, Y \in \Gamma(TM)$, $Z \in \Gamma(S(TM))$ and $V, V' \in \Gamma(F)$. Since $\bar{g}$ is non-degenerate on E we have $\bar{g}(U,V) \neq 0$ for any non-zero vector fields $U \in \Gamma(Rad\,TM)$ and $V \in \Gamma(F)$.

Furthermore, we suppose there exists a torsion-free linear connection ∇' on M and a linear connection ∇'' on vector bundle F, satisfying

$$(\mathrm{C}_2) \quad \bar{g}(\nabla'_X U, V) + \bar{g}(U, \nabla''_X V) = X(\bar{g}(U,V)),$$

for any $X \in \Gamma(TM)$, $U \in \Gamma(Rad\,TM)$ and $V \in \Gamma(F)$. Consider a screen distribution $S(TM)$ on M, i.e., we have

$$TM = S(TM) \perp Rad\,TM. \tag{3.12}$$

Hence we may set

$$\nabla'_X W = \overset{*}{\nabla}{}'_X W + \overset{*}{h}{}'(X,W), \; \forall X \in \Gamma(TM), \; W \in \Gamma(S(TM)), \tag{3.13}$$

and

$$(3.14)\quad \nabla'_X U = -\overset{*}{A'}(U,X) + \overset{*}{\nabla}{}''_X U,\ \forall\, X \in \Gamma(TM),\ U \in \Gamma(Rad\, TM),$$

where $\overset{*}{\nabla}{}'_X W$ and $\overset{*}{A'}(U,X)$ belong to $\Gamma(S(TM))$ while $\overset{*}{h'}(X,W)$ and $\overset{*}{\nabla}{}''_X U$ belong to $\Gamma(Rad\ TM)$. It is easy to verify that $\overset{*}{\nabla}{}'$ and $\overset{*}{\nabla}{}''$ are linear connections on vector bundles $S(TM)$ and $Rad\,TM$ respectively, and $\overset{*}{h'}$ and $\overset{*}{A'}$ are $\mathcal{F}(M)$-bilinear forms on $\Gamma(TM) \times \Gamma(S(TM))$ and on $\Gamma(Rad\ TM) \times \Gamma(TM)$, respectively. With respect to these geometric objects we suppose the following conditions are satisfied:

(C_3) $\overset{*}{A'}(U,U) = 0,\ g(\overset{*}{A'}(U,X),Y) = g(\overset{*}{A'}(U,Y),X),$

(C_4) $(\nabla'_X g)(W,W') = (\nabla'_X g)(U,U) = 0,\ (\nabla'_X g)(W,U) = g(\overset{*}{A'}(U,X),W)$

(C_5) $(\nabla'_X \overset{*}{A'})(U,Y) = (\nabla'_Y \overset{*}{A'})(U,X),$

for any $X, Y \in \Gamma(TM),\ W, W' \in \Gamma(S(TM))$ and $U \in \Gamma(Rad\,TM)$, where we put

$$(\nabla'_X \overset{*}{A'})(U,Y) = \overset{*}{\nabla}{}'_X (\overset{*}{A'}(U,Y)) - \overset{*}{A'}(\overset{*}{\nabla}{}''_X U, Y) - \overset{*}{A'}(U, \nabla'_X Y).$$

Finally, denote by R' the curvature tensor of ∇' and suppose the following conditions are fulfilled:

(C_6) $g(R'(X,Y)Z,W) = g(\overset{*}{A'}(\overset{*}{h'}(X,W),Y) - \overset{*}{A'}(\overset{*}{h'}(Y,W),X),Z),$

(C_7) $\bar{g}(R'(X,Y)Z,V) = 0,$

for any $X, Y, Z \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $V \in \Gamma(F)$. In what follows we denote by $\hat{g}$ the semi-Euclidean metric on R^{m+2}_q given by (1.4).

Theorem 3.3 (Fundamental Theorem for Degenerate Hypersurfaces). *Let $(M, g, S(TM))$ be a 1-degenerate simply connected $(m+1)$-dimensional manifold of index $q-1$, endowed with the vector bundle F and geometric objects $\bar{g}, \nabla', \nabla'', \overset{*}{h}{}'$ and, $\overset{*}{A'}$ satisfying conditions (C_1) -*

(C_7). Then there exists a degenerate isometric immersion

$$f : (M, g, S(TM)) \to (R_q^{m+2}, \hat{g}),$$

$$i.e., \ \hat{g}(f_*X, f_*Y) = g(X, Y), \ \forall \ X, Y \in \Gamma(TM)$$

and a vector bundle isomorphism $\bar{f} : F \to tr(TfM)$, such that

$$f_*(\nabla'_X Y) = \nabla_{f_*X} f_*Y, \ \bar{f}(\nabla''_X V) = \tau(X)\bar{f}(V),$$

$$f_*(\overset{*}{A'}(U, X)) = \overset{*}{A}_{f_*} U(f_*X), \ f_*(\overset{*}{h'}(X, W)) = C(f_*X, f_*W)f_*\xi,$$

for any $X, Y \in \Gamma(TM)$, $U \in \Gamma(Rad\ TM)$, $W \in \Gamma(S(TM))$ and $V \in \Gamma(F)$, where $tr(TfM)$ is the transversal vector bundle of M with respect to $f_*(S(TM))$, and $\nabla, \tau, \overset{*}{A}$ and C are geometric objects induced on fM by the Gauss and Weingarten formulas with respect to the immersion f.

Proof. First, by using $\overset{*}{A'}$ define the $\mathcal{F}(M)$-bilinear form

$$(3.15)\ h' : \Gamma(TM) \times \Gamma(TM) \to \Gamma(F); \bar{g}(h'(X, Y), U) = g(\overset{*}{A'}(U, X), Y),$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(Rad\ TM)$. Note that based on the last two equalities in condition (C_1), h' is well defined. Due to (C_3) we see that h' is symmetric and satisfies

$$(3.16) \qquad h'(X, U) = 0, \quad \forall X \in \Gamma(TM).$$

Next, by means of $\overset{*}{h}'$, and using (3.12), we define

$$A' \quad : \Gamma(F) \times \Gamma(TM) \to \Gamma(TM) \quad \text{by}$$

$$(3.17) \qquad \bar{g}(A'(V, X), W) = \bar{g}(\overset{*}{h'}(X, W), V); \quad \bar{g}(A'(V, X), V') = 0,$$

for any $X \in \Gamma(TM)$, $W \in \Gamma(S(TM))$ and $V, V' \in \Gamma(F)$. These two geometric objects enable us to define the differential operator $\bar{\nabla}$ by

$$(3.18) \qquad \bar{\nabla}_X Y = \nabla'_X Y + h'(X, Y), \quad \forall \ X, Y \in \Gamma(TM),$$

and

$$(3.19) \quad \bar{\nabla}_X V = -A'(V, X) + \nabla''_X V, \quad \forall \ X \in \Gamma(TM), \quad V \in \Gamma(F).$$

It is easy to verify that $\bar{\nabla}$ is a linear connection on E. Moreover, by using (3.18) we deduce

$$(3.20)\quad (\bar{\nabla}_X \bar{g})(Y,Z) = (\nabla'_X g)(Y,Z) - \bar{g}(h'(X,Y),Z) - \bar{g}(h'(X,Z),Y),$$

for any $X, Y, Z \in \Gamma(TM)$. By using (C_4), (C_1), (3.15) and (3.16) in the right hand side of (3.20) we infer

$$(\bar{\nabla}_X \bar{g})(Y,Z) = 0.$$

On the other hand, by using (C_1), (3.13) and (23.17) we obtain

$$(\bar{\nabla}_X \bar{g})(W,V) = 0, \quad \forall\, W \in \Gamma(S(TM)), \quad V \in \Gamma(F).$$

Conditions (C_1) and (C_2) yield

$$(\bar{\nabla}_X \bar{g})(U,V) = 0, \quad \forall\, U \in \Gamma(Rad\, TM), \quad V \in \Gamma(F).$$

Finally, by using (C_1), (3.19) and the second relation in (3.17) we obtain

$$(\bar{\nabla}_X \bar{g})(V,V') = 0, \quad \forall\, V, V' \in \Gamma(F).$$

Summing up, due to the above equalities, we conclude that $\bar{\nabla}$ is a metric connection on E.

Next, by using (3.15), we deduce

$$(3.21)\quad h'(X, \overset{*}{A'}(U,Y)) = h'(Y, \overset{*}{A'}(U,X)), \quad \forall\, X, Y \in \Gamma(TM), \quad U \in \Gamma(Rad\, TM).$$

Then, taking into account that $\bar{\nabla}$ is a metric connection and using (3.13) (3.15), (3.18), (319) and (3.21), we see that (C_5) is equivalent to

$$(3.22)\qquad (\nabla'_X h')(Y,Z) = (\nabla'_Y h')(X,Z), \quad \forall X, Y, Z \in \Gamma(TM),$$

where we set

$$(\nabla'_X h')(Y,Z) = \nabla''_X(h'(Y,Z)) - h'(\nabla'_X Y, Z) - h'(Y, \nabla'_X Z).$$

Moreover, by using (3.15), we see that (C_6) is equivalent to

$$
\begin{aligned}
(3.23) \qquad g(R'(X,Y)Z,W) &= \bar{g}(h'(Y,Z), \overset{*}{h'}(X,W)) \\
&\quad -\bar{g}(h'(X,Z), \overset{*}{h'}(Y,W)),
\end{aligned}
$$

By using (3.18), (3.19) , (C_7), (3.22) and (3.23) and performing some calculations similar to those for the Gauss-Codazzi equations we conclude that the curvature tensor $\bar{R}$ of $\bar{\nabla}$ vanishes identically. Now, consider a point $u \in M$ and orthonormal vectors $\{\overset{\circ}{W}_0, \ldots, \overset{\circ}{W}_{m+1}\}$ from the fibre E_u such that $\{\overset{\circ}{W}_0, \ldots, \overset{\circ}{W}_{q-1}\}$ and $\{\overset{\circ}{W}_q, \ldots, \overset{\circ}{W}_{m+1}\}$ are timelike and space-like, respectively. Since M is simply connected and $\bar{R}$ vanishes, there exist unique global extensions $\{\overset{\circ}{W}_0, \ldots, \overset{\circ}{W}_{m+1}\}$ parallel with respect to $\bar{\nabla}$. These global sections are pointwise orthonormal and have the same causal character as $\{\overset{\circ}{W}_0, \ldots, \overset{\circ}{W}_{m+1}\}$ since $\bar{g}$ is parallel with respect to $\bar{\nabla}$.

In the present proof we use the range of indices:$A, B, \ldots \in \{0, \ldots, m+1\}$; $\alpha, \beta, \ldots \in \{0, \ldots, m\}$; $i, j, \ldots \in \{0, \ldots, q-1\}$; $a, b \ldots \in \{q, \ldots, m+1\}$.

Now we consider a coordinate system $(\mathcal{U}; u^0, \ldots, u^m)$ around $u \in M$ and set $\partial/\partial u^\alpha = S^A_\alpha W_A$. Hence the local components of the degenerate metric g on M are given by

$$
(3.24) \qquad g_{\alpha\beta} = g\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) = -\sum_{i=0}^{q-1} S^i_\alpha S^i_\beta + \sum_{a=q}^{m+1} S^a_\alpha S^a_\beta .
$$

Taking into account that $\{W_A\}$ are parallel with respect to $\bar{\nabla}$ we obtain

$$
(3.25) \qquad \bar{\nabla}_{\frac{\partial}{\partial u^\beta}} \frac{\partial}{\partial u^\alpha} = \frac{\partial S^A_\alpha}{\partial u^\beta} W_A .
$$

By using (3.18) and (3.25) we derive

$$
(3.26) \qquad \frac{\partial S^A_\alpha}{\partial u^\beta} = \frac{\partial S^A_\beta}{\partial u^\alpha} .
$$

Thus the 1-forms $\omega^A = S^A_\alpha du^\alpha$ are closed and therefore they are exact on $\mathcal{U}$, that is, there exist smooth functions f^A such that $\omega^A = df^A$ or, equivalently,

$$\frac{\partial f^A}{\partial u^\alpha} = S^A_\alpha .$$

Define $f : \mathcal{U} \to R^{m+2}_q$ by $f = (f^0, \ldots, f^{m+1})$ and note that

$$f_*\left(\frac{\partial}{\partial u^\alpha}\right) = (S^0_\alpha, \ldots, S^{m+1}_\alpha).$$

Then, by using (3.24), we deduce

$$\hat{g}\left(f_*\left(\frac{\partial}{\partial u^\alpha}\right), f_*\left(\frac{\partial}{\partial u^\beta}\right)\right) = g_{\alpha\beta}.$$

That is, f is a degenerate immersion of $\mathcal{U}$ in R^{m+2}_q. As a consequence, $\bar{\mathcal{U}} = f(\mathcal{U})$ becomes a degenerate hypersurface of R^{m+2}_q.

Next, define the isomorphism of vector bundles (linear isometry between fibres)

$$\Phi : T\mathcal{U} \oplus F_{|\mathcal{U}} \to TR^{m+2}_q{}_{|\bar{\mathcal{U}}}, \quad \Phi(W_A) = E_A,$$

where $\{E_A\}$ is the canonical orthonormal frame field on $\bar{\mathcal{U}}$. Note that Φ carries isometrically $T\mathcal{U}$ onto $T\bar{\mathcal{U}}$. Indeed

$$\Phi\left(\frac{\partial}{\partial u^\alpha}\right) = S^A_\alpha \Phi(W_A) = S^A_\alpha E_A = f_*\left(\frac{\partial}{\partial u^\alpha}\right).$$

Moreover, the radical distribution and the screen distribution are preserved by Φ.

The Levi-Civita connection on R^{m+2}_q with respect to $\hat{g}$ is denoted by $\hat{\nabla}$. Then taking into account that $\{W_A\}$ and $\{E_A\}$ are parallel with respect to $\bar{\nabla}$ and $\hat{\nabla}$ respectively, we infer

$$\Phi(\bar{\nabla}_X Y) = \hat{\nabla}_{f_*X} f_*Y \quad \text{and} \quad \Phi(\bar{\nabla}_X V) = \hat{\nabla}_{f_*X} \Phi V,$$

for any $X, Y \in \Gamma(T\mathcal{U})$ and $V \in \Gamma(F_{|\mathcal{U}})$. By using (3.18), (3.19) and the Gauss and Weingarten formulas for the degenerate hypersurface $\mathcal{U}$ of R^{m+2}_q, we deduce

$$f_*(\nabla'_X Y) = \nabla_{f_*X} f_*Y, \Phi(h'(X,Y)) = B(f_*X, f_*Y)\Phi V,$$

and

$$f_*(A'(V,X)) = A_{\Phi V} f_* A, \Phi(\nabla''_X V) = \tau(X)\Phi V.$$

Moreover, from (3.13) and (3.14), we obtain

$$f_*(\overset{*}{A'}(U,X)) = \overset{*}{A}_{f_*U} f_*X, \; f_*(\overset{*}{h'}(X,W)) = C(f_*X, f_*W) f_*\xi.$$

It is easy to check that all these local immersions are determined up to an isometry of R_q^{m+2}.

Now, denote by $Tr(T\bar{\mathcal{U}})$ the transversal vector bundle of $\bar{\mathcal{U}}$ with respect to the screen distribution $S(T\bar{\mathcal{U}}) = f_*(s(T\mathcal{U}))$. It follows that $tr(T\bar{\mathcal{U}}) = \Phi(F_{|\mathcal{U}})$. Thus we have a vector bundle isomorphism $\bar{f}_{|\mathcal{U}} : F_{|\mathcal{U}} \to tr(T\bar{\mathcal{U}})$ which is the restriction of Φ to $F_{|\mathcal{U}}$.

Therefore we constructed both f and $\bar{f}$ on $\mathcal{U} \subset M$ satisfying the relations in the theorem.

Finally, since M is simply connected, the local degenerate immersions will be glued together as in the case of non-degenerate hypersurfaces, and give us the global degenerate isometric immersion $f : M \to R_q^{m+2}$. Moreover, $S(T\mathcal{U})$ and $S(T\mathcal{U}^*)$ coincide on $\mathcal{U} \cap \mathcal{U}^*$ since they are restrictions of $S(TM)$ to coordinate neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ respectively. Hence $S(T\bar{\mathcal{U}})$ and $S(T\bar{\mathcal{U}}^*)$ coincide on $\bar{\mathcal{U}} \cap \bar{\mathcal{U}}^*$ and thus we obtain a screen distribution $S(TfM)$ of fM. Since $tr(TfM)$ is unique and coincides locally with $tr(T\bar{\mathcal{U}})$, there exists a global isomorphism $\bar{f} : F \to tr(TfM)$ which, together with f, satisfy the relations of the theorem. This completes the proof of the theorem.

4. SPECIAL CLASSES OF DEGENERATE HYPERSURFACES

It is the purpose of this section to introduce and study some important classes of degenerate hypersurfaces and give examples.

Let $(M, g, S(TM))$ be a degenerate hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If any geodesic of M with respect to an induced connection ∇ is a geodesic of $\bar{M}$ with respect to the Levi-Civita connection

$\bar{\nabla}$, we say that M is *totally geodesic*. The theorem which follows shows that the definition does not depend on the screen distribution.

Theorem 4.1. *Let $(M, g, S(TM))$ be a degenerate hypersurface of $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

(i) M is totally geodesic.

(ii) The second fundamental form of M vanishes identically on M.

(iii) The shape operator of $S(TM)$ vanishes identically on M.

(iv) There exists a unique torsion-free metric connection ∇ induced by $\bar{\nabla}$.

(v) $TM^{\perp}$ is a parallel distribution with respect to ∇.

(vi) $TM^{\perp}$ is a Killing distribution on M.

(vii) M is a Reinhart degenerate manifold.

The equivalence of conditions (i) through (vi) is proved in [6], and due to Theorem 1.2 we see that (vii) is equivalent with (vi).

Example 4.1. For the sake of simplicity of calculations we consider $m = 1$ in Example 2.3. Hence the degenerate surface is obtained by cuting $S_1^3(1)$ with the plane $x^0 - x^1 = 0$. Denote by ∇' and $\bar{\nabla}$ the Levi-Civita connections on R_1^4 and $S_1^3(1)$ respectively. Then we have

$$\nabla'_X Y = \bar{\nabla}_X Y + h(X,Y)N' = \nabla_X Y + B(X,Y)N + h(X,Y)N',$$

for any $X, Y \in \Gamma(TM)$, where h is the second fundamental form of $S_1^3(1)$ in R_1^4 and N' is the position vector field on $S_1^3(1)$. Since $B(X,\xi) = 0$ for any $X \in \Gamma(TM)$ we only need to calculate $B(W_1, W_1)$, where $W_1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}$. Thus we set

$$\begin{aligned} \nabla'_{W_1} W_1 &= \alpha \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right) + \beta \left(x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} \right) \\ &\quad - \frac{1}{2} B(W_1, W_1) \left\{ (1 + (x^0)^2) \frac{\partial}{\partial x^0} + ((x^0)^2 - 1) \frac{\partial}{\partial x^1} \right. \end{aligned}$$

$$+2x^0x^2\frac{\partial}{\partial x^2}+2x^0x^3\frac{\partial}{\partial x^3}\Big\}+h(W_1,W_1)\sum_{A=0}^{3}x^A\frac{\partial}{\partial x^A}.$$

Using (1.5) for ∇' we obtain $\alpha = x^0$, $\beta = 0$, $B(W_1, W_1) = 0$ and $h(W_1, W_1) = -1$. Thus M is a totally geodesic degenerate surface of $S_1^3(1)$ and the unique induced metric connection on M is given by

$$\nabla_{W_1}W_1 = x^0\xi, \quad \nabla_{W_1}\xi = \nabla_\xi W_1 = 0.$$

In a similar way can be proved that any degenerate great hypersphere of $S_1^m(r)$ is totally geodesic.

Example 4.2. A hyperplane $M : x^0 = c + c_1x^1 + \ldots + c_{m+1}x^{m+1}$ is degenerate in R_q^{m+2} if and only if

$$1+\sum_{i=1}^{q-1}(c_i)^2=\sum_{a=q}^{m+1}(c_a)^2.$$

In this case $TM^\perp$ is spanned by

$$\xi=\frac{\partial}{\partial x^0}-\sum_{i=1}^{q-1}c_i\frac{\partial}{\partial x^i}+\sum_{a=q}^{m+1}c_a\frac{\partial}{\partial x^a},$$

and therefore $\bar{\nabla}_X\xi = 0$ for any $X \in \Gamma(TM)$, where $\bar{\nabla}$ is the Levi-Civita connection on R_q^{m+2}. Hence M is totally geodesic in R_q^{m+2}.

Next, we say that a degenerate hypersurface M of $(\bar{M}, \bar{g})$ is *totally umbilical* if there exists locally on each $\mathcal{U} \subset M$ a function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall\, X,Y \in \Gamma(TM).$$

According to Proposition 2.1 the definition does not depend on the screen distribution. Therefore, due to (2.15), M is totally umbilical if and only if there exists a function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall\, X,Y \in \Gamma(S(TM)), \tag{4.1}$$

where $S(TM)$ is a screen distribution on M. By using (2.20) and (4.1) it follows that M is totally umbilical if and only if the shape operator of a screen distribution $S(TM)$ satisfies

$$\overset{*}{A}_\xi X = \rho X, \quad ,\forall\, X \in \Gamma(S(TM)). \tag{4.2}$$

Theorem 4.2 ([6]). *Let $(M, g, S(TM))$ be a totally umbilical degenerate hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold of constant curvature $(\bar{M}(c), \bar{g})$. then ρ satisfies the partial differential equation*

$$\xi(\rho) + \rho\tau(\xi) - \rho^2 = 0,$$

and the curvature tensor of M is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} + \rho\{g(Y, Z)A_N X - g(X, Z)A_N Y\},$$

for any $X, Y, Z \in \Gamma(TM_{|\mathcal{U}})$. Moreover, if $m > 1$ then ρ satisfies the partial differential equations

$$X(\rho) + \rho\tau(X) = 0, \quad \forall\, X \in \Gamma(S(T\mathcal{U})_{|\mathcal{U}}).$$

The above necessary conditions for M to be totally umbilical seems to be very strong conditions on the geometry of M. That is why we need to prove the existence of totally umbilical submanifolds.

Example 4.3. Suppose M is the null cone of R_q^{m+2}. Then, based on Example 2.1, ξ is the position vector field, and it is globally defined on M. Thus, by using (2.10), (2.15) and (1.5), we obtain

$$\nabla_X \xi = \bar{\nabla}_X \xi = X, \quad \forall X \in \Gamma(TM). \tag{4.3}$$

Next, by using (2.19) in (4.3), we deduce

$$A^*_\xi X + \tau(X)\xi + X = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.4}$$

Hence for any $X \in \Gamma(S(TM))$ from (4.4) we derive

$$\tau(X) = 0, \tag{4.5}$$

and

$$\overset{*}{A}_\xi X = -X,$$

that is, M is totally umbilical and $\rho = -1$ with respect to the above ξ. Moreover, from (4.4) and (2.22) we obtain $\tau(\xi) = -1$. Hence Theorem 3.2 implies the following important result for the geometry of the null cone.

Theorem 4.3. *The Ricci tensor of the induced connection* ∇ *on the null cone of* R_q^{m+2} *is symmetric.*

By using the general theory we developed in the previous sections for degenerate hypersurfaces, we may obtain new results on the geometry of the null cone. The next theorems support this assertion.

Theorem 4.4. *On the null cone* M *of* R_q^{m+2} *there exists a foliation of codimension 1.*

Proof. We prove that the screen distribution presented in Example 2.1 is integrable. First, by using (1.5) and N from Example 2.1, we obtain

$$\bar{g}(Y, \bar{\nabla}_X N) = \frac{1}{2\sum_{i=0}^{q-1}(x^i)^2} \sum_{A=0}^{m+1} X^A Y^A, \tag{4.6}$$

where $X = X^A(\partial/\partial x^A)$ and $Y = Y^A(\partial/\partial x^A)$ belong to $\Gamma(S(TM))$. As $\bar{\nabla}$ is a torsion - free metric connection we deduce

$$\bar{g}([X,Y], N) = \bar{g}(X, \bar{\nabla}_Y N) - \bar{g}(Y, \bar{\nabla}_X N) = 0,$$

for any $X, Y \in \Gamma(S(TM))$. Hence $[X,Y] \in \Gamma(S(TM))$, and this completes the proof.

Theorem 4.5. *Let* M *be the null cone of* R_q^{m+2}, $m > 0$. *Then* $A_N\xi = 0$ *and any other eigenvector field* $Y \neq \alpha\xi$ *is either spacelike or timelike according as the corresponding eigenfunction is negative or positive, respectively.*

Proof. Take $X \in \Gamma(S(TM))$. By using (1.5), (2.10) and (2.15) we obtain

$$\nabla_\xi X = \bar{\nabla}_\xi X = x^A \frac{\partial X^B}{\partial x^A} \frac{\partial}{\partial x^B}. \tag{4.7}$$

Differentiating

$$\sum_{i=0}^{q-1} x^i X^i = 0 \quad \text{and} \sum_{a=q}^{m+1} x^a X^a = 0,$$

with respect to x^j and x^b, respectively, we easily obtain

$$\sum_{i=0}^{q-1} x^i x^A \frac{\partial X^i}{\partial x^A} = 0, \quad \sum_{a=q}^{m+1} x^a x^A \frac{\partial X^a}{\partial x^A} = 0.$$

Thus, taking account of (4.7), we deduce that $\nabla_\xi X \in \Gamma(S(TM))$ and (2.17) implies

$$C(\xi, X) = 0. \tag{4.8}$$

This, combined with (2.16) and (2.21), yields $A_N\xi = 0$. Now, denote by P the projection morphism of TM on $S(TM)$ and use (2.4) to obtain

$$Y = PY + \eta(Y)\xi, \quad \forall Y \in \Gamma(TM). \tag{4.9}$$

Then by using (4.9), (2.21), (4.8), (2.11) and (4.6) we infer

$$\begin{aligned} g(A_N Y, Y) &= g(A_N PY + \eta(Y) A_N \xi, PY) = g(A_N PY, PY) \\ &= -\frac{1}{2\sum_{i=0}^{q-1}(x^i)^2} \sum_{A=0}^{m+1} \{(PX)^A\}^2 < 0, \end{aligned} \tag{4.10}$$

for any $Y \neq \alpha\xi$. If Y is an eigenvector field and λ is the eigenfunction, i.e., $A_N Y = \lambda Y$, we can use (4.10) to deduce

$$\lambda g(Y, Y) < 0,$$

which proves the theorem.

Furthermore, as $c = 0$ and $\rho = -1$ for the null cone M of R_q^{m+2}, , we derive from Theorem 4.2 that

$$R(X, Y)Z = g(X, Z) A_N Y - g(Y, Z) A_N X, \ \forall\, X, Y, Z \in \Gamma(TM). \tag{4.11}$$

In particular, we obtain the following result.

Theorem 4.6. *Let M be the null cone of the Lorentz space R_1^{m+2}. Then we have*

(i) $A_N\xi = 0, \quad A_N = -\dfrac{1}{2(x^0)^2} X,$ *for any* $X \in \Gamma(S(TM))$.

(ii) The curvature tensor of the induced connection on M is given by

$$R(X, Y)Z = \frac{1}{2(x^0)^2}\{g(Y, Z) PX - g(X, Z) PY\}, \quad \forall X, Y, Z \in \Gamma(TM).$$

Proof. Consider $X, Y \in \Gamma(S(TM))$ and, by using (2.11) and (4.6), obtain

$$g(Y, A_N X) = -\bar{g}(Y, \bar{\nabla}_X N) = -\frac{1}{2(x^0)^2}\sum_{a=1}^{m+1} X^a Y^a = -\frac{1}{2(x^0)^2} g(X, Y),$$

since $X^0 = Y^0 = 0$. Thus by Theorem 4.5 we have the assertion (i). Finally, taking into account of (4.11) and assertion (i), we derive the formula in assertion (ii).

Theorem 4.7. *a degenerate surface M of a 3-dimensional Lorentz manifold $\bar{M}$ is either totally umbilical or totally geodesic.*

Proof. Let $\mathcal{U}$ be a coordinate neighborhood of M and $S(TM)$ be a screen distribution spanned by W on $\mathcal{U}$. If M is not totally geodesic, define $\rho = B(W, W)/g(W, W)$ and (4.1) is satisfied. Hence M is totally umbilical.

It is interesting to investigate the existence of some other totally umbilical degenerate hypersurfaces of semi-Euclidean spaces. In this respect the author succeeded in determining all totally umbilical degenerate hypersurfaces of R_2^4 (cf. [5]).

5. DEGENERATE HYPERSURFACES OF LORENTZ SPACES

Let M be a degenerate hypersurface of the Lorentz space R_1^{m+2} given by the equation

$$F(x^0, \dots, x^{m+1}) = 0, \tag{5.1}$$

where F is differentiable on a domain $D \subset R^{m+2}$ and rank $[F'_{x^0} \dots F'_{x^{m+1}}] = 1$ on M. Moreover, according to (2.3), the partial derivatives of first order of F satisfy

$$(F'_{x^0})^2 = \sum_{a=1}^{m+1} (F'_{x^a})^2. \tag{5.2}$$

Thus $F'_{x^0} \neq 0$ on M, which enables us to consider $TM^\perp$ spanned by

$$\xi = -\frac{\partial}{\partial x^0} + \frac{1}{F'_{x^0}} \sum_{a=1}^{m+1} F'_{x^a} \frac{\partial}{\partial x^a}. \tag{5.3}$$

Consider the transversal vector bundle spanned by

$$N = \frac{\partial}{\partial x^0} + \frac{1}{2}\xi. \tag{5.4}$$

$X = X^A(\partial/\partial x^A)$ belongs to $\Gamma(S(TM))$ if and only if

$$X^0 = 0 \quad \text{and} \quad \sum_{a=1}^{m+1} X^a F'_{x^a} = 0. \tag{5.5}$$

Remark 5.1. As R_1^{m+2} is a time-orientable Lorentz manifolds with $L = \partial/\partial x^0$, comparing (2.25) with (5.4) we see that the above distribution is just the canonical screen distribution on M.

By using (1.5) and (5.4) we obtain

$$\bar{g}(Y, \bar{\nabla}_X N) = \frac{1}{2}\bar{g}(Y, \bar{\nabla}_X \xi) = -\frac{1}{2}g(Y, A^*_\xi X) = -\frac{1}{2}B(X, Y), \tag{5.6}$$

for any $X, Y \in \Gamma(TM)$. Since B is symmetric, by using (2.6) we derive

$$\begin{aligned} \bar{g}([X,Y], N) &= \bar{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, N) \\ &= \bar{g}(X, \bar{\nabla}_Y N) - \bar{g}(Y, \bar{\nabla}_X N) = 0 \end{aligned} \tag{5.7}$$

and

$$\bar{g}(\nabla_\xi X, N) = \bar{g}(\bar{\nabla}_\xi X, N) = -\bar{g}(X, \bar{\nabla}_\xi N) = \frac{1}{2}B(\xi, X) = 0, \tag{5.8}$$

for any $X, Y \in \Gamma(S(TM))$. In view of (5.7) we may state the following result.

Theorem 5.1. *The canonical screen distribution of a degenerate hypersurface M of R_1^{M+2} is integrable.*

Taking account of (2.11) and (2.21) in (5.6) we see that the fundamental forms of M and $S(TM)$ are related by

$$B(X, Y) = 2C(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{5.9}$$

In a similar way, from (5.8) we deduce

$$(5.10) \qquad C(\xi, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

Next, using (2.11) and (5.6), and taking account of (2.15), we derive

$$(5.11) \qquad \tau(X) = \bar{g}(\bar{\nabla}_X N, \xi) = -\frac{1}{2}B(\xi, X) = 0.$$

Hence, due to Theorem 3.2 and (5.11) we state the following result.

Theorem 5.2. *The Ricci tensor of the induced connection on any degenerate hypersurface M of R_1^{m+2} is symmetric.*

Taking account of (5.9) - (5.11) we see that the Gauss-Codazzi equations of (3.3) - (3.6) become

$$\begin{aligned} (5.12) \qquad g(R(X,Y)Z, PW) &= 2\{C(PY, PZ)C(PX, PY) \\ &\quad -C(PX, PZ)C(PY, PW)\}, \\ (5.13) \qquad (\nabla_X B)(Y, Z) &= (\nabla_Y B)(X, Z), \\ (5.14) \qquad \bar{g}(R(X,Y)Z, N) &= 0, \end{aligned}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where P is the projection morphism of TM on the canonical screen distribution. Thus, by (5.12) and (5.14), we deduce that the curvature tensor of a degenerate hypersurface of R_1^{m+2} is given by

$$(5.15) \qquad R(X,Y)Z = 2\{C(PY, PZ)PX - C(PX, PZ)PY\}.$$

Now, suppose M is a totally umbilical degenerate hypersurface of R_1^{m+2}. Then by using (4.1), (5.9) and (5.15) we deduce the following result.

Theorem 5.3. *The curvature tensor of a totally umbilical degenerate hypersurface of R_1^{m+2} is given by*

$$R(X,Y)Z = \rho\{g(PY, PZ)PX - g(PX, PZ)PY\}.$$

Now, suppose M^* is a leaf of the canonical screen distribution and R^* is the curvature tensor field of the induced connection on M^* by $\bar{\nabla}$.

Then by direct calculations, using (2.17), (3.5), (5.11), (2.20) and (5.9), we obtain

$$(5.16)\quad g(R(X,Y)Z,W) = g(R^*(X,Y)Z,W) + 2\{C(X,Z)C(Y,W) - C(Y,Z)C(X,W)\},$$

for any $X, Y, Z, W \in \Gamma(TM^*)$. Comparing (5.16) with (5.15) we conclude

$$(5.17)\quad R(X,Y)Z = \frac{1}{2}R^*(PX,PY)PZ, \quad \forall\, X,Y,Z \in \Gamma(TM)$$

Finally, taking account of (2.10), (2.17) and (5.9) we deduce

$$(5.18)\quad B^*(X,Y) = B(X,Y)\left(\frac{1}{2}\xi + N\right),$$

for any $X, Y \in \Gamma(TM^*)$, where B^* is the second fundamental form of M^* as a non-degenerate submanifold of codimension two in R_1^{m+2}. Based on (5.17) and (5.18) we may state the following important result.

Theorem 5.4. *A degenerate hypersurface M of R_1^{m+2} is*

(i) flat

(ii) totally umbilical

(iii) totally geodesic

if and only if, any leaf of the canonical screen distribution is immersed as a non-degenerate submanifold of codimension two in R_1^{m+2}.

Thus, as a final conclusion we may say that the differential geometry of a degenerate hypersurface of a Lorentz space is intimately related to the geometry of an arbitrary leaf of the canonical screen distribution. In particular, if M is a degenerate Monge hypersurface of R_1^4, the results of this section are included in [7].

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FOUR-DIMENSIONAL ALMOST KÄHLER MANIFOLDS OF POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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ABSTRACT. In this paper we prove that the theorem of Schur holds for the class of 4-dimensional almost Kähler manifolds.

1. INTRODUCTION

Let $M = (M, J, g)$ be an almost Hermitian manifold and $U(M)$ the unit tangent bundle of M. Then the holomorphic sectional curvature $H = H(x)$ $(x \in U(M))$ can be regarded as a differentiable function on $U(M)$. If the function H is constant along each fiber, then M is called a space of pointwise constant holomorphic sectional curvature. Especially, if H is constant on the whole of $U(M)$, then M is called a space of constant holomorphic sectional curvature. An almost Hermitian manifold $M = (M, J, g)$ is called a Hermitian manifold if the almost complex structure J is integrable. On one hand, an almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form is closed (or equivalently, $\mathcal{S}_{X,Y,Z}g((\nabla_X J)Y, Z) = 0)$). A Hermitian manifold $M = (M, J, g)$ with the closed Kähler form is called a Kähler manifold. A Kähler manifold is characterized by an almost Hermitian manifold with the parallel almost complex structure with respect to the Levi-Civita connection (cf. [6]). It is well-known that the theorem of Schur holds for the class of Kähler manifolds, namely that a real $2n(\geq 4)$-dimensional Kähler manifold of pointwise constant holomorphic sectional curvature is of constant holomorphic sectional curvature. However, A. Gray and L. Vanhecke ([2]) have proved that the theorem of Schur does

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not hold anymore for the class of Hermitian manifolds by constructing examples of Hermitian manifolds of pointwise constant holomorphic sectional curvature which are not of constant holomorphic sectional curvature. It is a natural question to consider whether the theorem of Schur holds for a given class of almost Hermitian manifolds.

In the present paper, we show that the theorem of Schur holds for the class of four- dimensional almost Kähler manifolds. More precisely, we shall prove the following

Main Theorem. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is a Kähler manifold of constant holomorphic sectional curvature.*

The above Theorem is an improvement of the result by T.Sato ([4], Corollary 3.5).

2. PRELIMINARIES

Let $M = (M, J, g)$ be an almost Hermitian manifold. We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X,Y) = g(X, JY)$ and $N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY]$ for $X, Y \in \mathfrak{X}(M)$, ($\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M). We note the the Nihenhuis tensor satisfies the following equalities.

$$N(X,Y) = -N(Y,X), \quad N(JX,Y) = -JN(X,Y). \tag{2.1}$$

for $X, Y \in \mathfrak{X}(M)$. Further, we denote by ∇, R, ρ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. Here the curvature tensor, R, is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

for $X, Y, Z \in \mathfrak{X}(M)$. Further, we denote by ρ^* and τ^* the Ricci $*$-tensor and the $*$-scalar curvature defined respectively by

$$\rho^*(x,y) = g(Q^*x, y) = \text{trace } (z \mapsto R(x, Jz)Jy), \tag{2.2}$$

$$(2.3) \qquad \tau^* = \text{trace } Q^*,$$

for $x, y, z \in T_pM$, $p \in M$.

Now, we put

$$G(x,y,z,w) = R(x,y,z,w) - R(x,y,Jz,Jw),$$

where $R(x,y,z,w) = g(R(x,y)z,w)$, for $x,y,z,w \in T_pM(p \in M)$. T. Sato ([3]) has proved the following

Proposition 2.1. *Let $M = (M,J,g)$ be an almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c = c(p)$, $p \in M$. Then we have*

$$(2.4) \qquad R(x,y,z,w) = \frac{c(p)}{4} R_0(x,y,z,w) + P(x,y,z,w),$$

where

$$\begin{aligned} R_0(x,y,z,w) &= g(x,w)g(y,z) - g(x,z)g(y,w) \\ &\quad + g(x,Jw)g(y,Jz) - g(x,Jz)g(y,Jw) \\ &\quad - 2g(x,Jy)g(z,Jw), \end{aligned}$$

and

$$\begin{aligned} P(x,y,z,w) &= \frac{1}{96}[26\{G(x,y,z,w) + G(z,w,x,y)\} \\ &\quad -6\{G(Jx,Jy,Jz,Jw) + G(Jz,Jw,Jx,Jy)\} \\ &\quad +13\{G(x,z,y,w) + G(y,w,x,z)\} \\ &\quad -G(x,w,y,z) - G(y,z,x,w)\} \\ &\quad -3\{G(Jx,Jz,Jy,Jw) + G(Jy,Jw,Jx,Jz) \\ &\quad -G(Jx,Jw,Jy,Jz) - G(Jy,Jz,Jx,Jw)\} \\ &\quad +4\{G(x,Jy,z,Jw) + G(Jx,y,Jz,w)\} \\ &\quad +2\{G(x,Jz,y,Jw) + G(Jx,z,Jy,w) \\ &\quad -G(x,Jw,y,Jz) - G(Jx,w,Jy,z)\}], \end{aligned}$$

for $x,y,z,w \in T_pM$, $p \in M$.

It is also well-known that the following identities hold for an almost Kähler manifold $M = (M, J, g)$ (cf. [5]):

$$2g((\nabla_x J)y, z) = g(Jx, N(y, z)), \tag{2.5}$$

for $x, y, z, w \in T_pM (p \in M)$, and

$$\|\nabla J\|^2 = \frac{1}{4}\|N\|^2 = 2(\tau^* - \tau). \tag{2.6}$$

By (2.5) and the Ricci identity, we have further

$$\begin{aligned} G(x, y, z, w) &= \frac{1}{4}\{g(x, N(N(z, w), y)) - g(y, N(N(z, w), x))\} \\ &\quad -\frac{1}{2}\{g(x, J(\nabla_y N)(Jz, w)) - g(y, J(\nabla_x N)(Jz, w))\} \end{aligned} \tag{2.7}$$

for $x, y, z, w \in T_pM,\ p \in M$ (cf. [5]). Taking account of (2.7), Proposition 2.1 reduces to the following

Proposition 2.2. *Let $M = (M, J, g)$ be an almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$. Then we have*

$$R(x, y, z, w) = \frac{c(p)}{4} R_0(x, y, z, w) + P(x, y, z, w),$$

where

$$\begin{aligned} R_0(x, y, z, w) &= g(x, w)g(y, z) - g(x, z)g(y, w) \\ &\quad + g(x, Jw)g(y, Jz) - g(x, Jz)g(y, Jw) \\ &\quad - 2g(x, Jy)g(z, Jw), \end{aligned}$$

and

$$\begin{aligned} P(x, y, z, w) &= \frac{1}{4 \cdot 96}[28\{g(x, N(N(z, w), y)) - g(y, N(N(z, w), x))\} \\ &\quad + 20\{g(z, N(N(x, y), w)) - g(w, N(N(x, y), z))\} \\ &\quad + 14\{g(x, N(N(y, w), z)) - g(z, N(N(y, w), x))\} \\ &\quad + 10\{g(y, N(N(x, z), w)) - g(w, N(N(x, z), y))\} \end{aligned}$$

$$
\begin{aligned}
&+14\{g(w,N(N(y,z),x))-g(x,N(N(y,z),w))\}\\
&+10\{g(z,N(N(x,w),y))-g(y,N(N(x,w),z))\}]\\
&+\frac{1}{2\cdot 96}[26\{g(Jy,(\nabla_x N)(Jz,w))-g(Jx,(\nabla_y N)(Jz,w))\\
&+g(Jw,(\nabla_z N)(Jx,y))-g(Jz,(\nabla_w N)(Jx,y))\}\\
&-6\{g(y,(\nabla_{Jx}N)(z,Jw))-g(x,(\nabla_{Jy}N)(z,Jw))\\
&+g(w,(\nabla_{Jz}N)(x,Jy))-g(z,(\nabla_{Jw}N)(x,Jy))\}\\
&+13\{g(Jz,(\nabla_x N)(Jy,w))-g(Jx,(\nabla_z N)(Jy,w))\\
&+g(Jw,(\nabla_y N)(Jx,z))-g(Jy,(\nabla_w N)(Jx,z))\\
&+g(Jx,(\nabla_w N)(Jy,z))-g(Jw,(\nabla_x N)(Jy,z))\\
&+g(Jy,(\nabla_z N)(Jx,w))-g(Jz,(\nabla_y N)(Jx,w))\}\\
&-3\{g(z,(\nabla_{Jx}N)(y,Jw))-g(x,(\nabla_{Jz}N)(y,Jw))\\
&+g(w,(\nabla_{Jy}N)(x,Jz))-g(y,(\nabla_{Jw}N)(x,Jz))\\
&+g(x,(\nabla_{Jw}N)(y,Jz))-g(w,(\nabla_{Jx}N)(y,Jz))\\
&+g(y,(\nabla_{Jz}N)(x,Jw))-g(z,(\nabla_{Jy}N)(x,Jw))\}\\
&+4\{-g(y,(\nabla_x N)(Jz,Jw))-g(Jx,(\nabla_{Jy}N)(Jz,Jw))\\
&-g(x,(\nabla_y N)(Jw,Jz))-g(Jy,(\nabla_{Jx}N)(Jw,Jz))\}\\
&+2\{g(w,(\nabla_x N)(Jy,Jz))+g(Jx,(\nabla_{Jw}N)(Jy,Jz))\\
&+g(x,(\nabla_w N)(Jz,Jy))+g(Jw,(\nabla_{Jx}N)(Jz,Jy))\\
&-g(z,(\nabla_x N)(Jy,Jw))-g(Jx,(\nabla_{Jz}N)(Jy,Jw))\\
&-g(x,(\nabla_z N)(Jw,Jy))-g(Jz,(\nabla_{Jx}N)(Jw,Jy))\}],
\end{aligned}
$$

for $x,y,z,w\in T_pM, p\in M$.

From (2.1), we have also

$$
(2.8)\qquad (\nabla_Z N)(JX,Y) = -(\nabla_Z J)N(X,Y)-J(\nabla_X N)(X,Y) -N((\nabla_Z J)X,Y)
$$

for $X,Y,Z\in\mathfrak{X}(M)$.

In the sequel, we shall consider the case dim $M=4$. Then, from Proposition 2.2, taking account of (2.1)-(2.8) by long and tedius calculations, we have

Proposition 2.3. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$. Then we have*

$$
\begin{aligned}
R_{1212} &= R_{3434} = -c(p),\\
R_{1234} &= -\frac{c(p)}{2} - \frac{1}{16}(\tau^* - \tau),\\
R_{1324} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{1}{8}(A_{31} - A_{13} + A_{24} - A_{42}),\\
R_{1432} &= -\frac{c(p)}{4} - \frac{3}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} - A_{13} + A_{24} - A_{42}),\\
R_{1313} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{3}{8}(A_{13} - A_{31}) - \frac{1}{8}(A_{24} - A_{42}),\\
R_{1414} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) - \frac{1}{8}(A_{13} + A_{42}) - \frac{3}{8}(A_{31} + A_{24}),\\
R_{2323} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} + A_{24}) + \frac{3}{8}(A_{42} + A_{13}),\\
R_{2424} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{3}{8}(A_{24} - A_{42}) + \frac{1}{8}(A_{13} - A_{31}),\\
R_{1334} &= -R_{2434} = -\frac{1}{4}(A_{34} - A_{43}),\\
R_{1213} &= -R_{1224} = -\frac{1}{4}(A_{12} - A_{21}),\\
R_{1434} &= R_{2334} = -\frac{1}{4}(A_{33} + A_{44}),\\
R_{1214} &= R_{1223} = -\frac{1}{4}(A_{11} + A_{22}),\\
R_{1323} &= \frac{1}{8}(A_{41} + A_{14} + A_{32} - 3A_{23}),\\
R_{2324} &= \frac{1}{8}(A_{41} + A_{14} + A_{22} - 3A_{32}),\\
R_{1314} &= -\frac{1}{8}(A_{23} + A_{32} + A_{34} - 3A_{41}),\\
R_{1424} &= -\frac{1}{8}(A_{41} + A_{32} + A_{23} - 3A_{14}),
\end{aligned}
$$

where we set $A_{ij} = g(e_i, (\nabla_{e_j} N)(e_1, e_3)), R_{ijkl} = R(e_i, e_j, e_k, e_l)$ for a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of $T_pM,\ p \in M$.

From the above Proposition 2.3, we have easily

$$\tau + 3\tau^* = 24c. \tag{2.9}$$

3. PROOF OF THE MAIN THEOREM

Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$.

First, we shall calculate the square norm $\|R\|^2$ of curvature tensor at an arbitrary point p of M. Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any unitary basis of T_pM. By direct calculation, we have the following general formula

$$\begin{aligned} \|R\|^2 &= 4\sum_{a,b}(R^2_{1a1b} + R^2_{2a2b} + R^2_{3a3b} + R^2_{4a4b}) \\ &\quad -4(R^2_{1212} + R^2_{1313} + R^2_{1414} + R^2_{2323} + R^2_{2424} + R^2_{3434}) \\ &\quad +8(R^2_{1234} + R^2_{1324} + R^2_{1432}). \end{aligned} \tag{3.1}$$

Taking account of Proposition 2.3, we have

$$\begin{aligned} \sum_{a,b} R^2_{1a1b} &= \frac{9}{8}c^2 + \frac{1}{8}(A_{12} - A_{21})^2 + \frac{1}{8}(A_{11} + A_{22})^2 + \frac{26}{32^2}(\tau^* - \tau)^2 \\ &\quad + \frac{9}{64}(A_{13} - A_{31})^2 + \frac{1}{64}(A_{24} - A_{42})^2 \\ &\quad + \frac{1}{64}(A_{13} + A_{42})^2 + \frac{9}{64}(A_{31} - A_{24})^2 \\ &\quad - \frac{3c}{32}(\tau^* - \tau) + \frac{c}{4}(A_{13} + A_{24}) \\ &\quad - \frac{1}{32}(\tau^* - \tau)(A_{42} + 2A_{13} + 3A_{31} + 4A_{24}) \\ &\quad + \frac{3}{32}(A_{13} - A_{31})(A_{24} - A_{42}) + \frac{3}{32}(A_{13} + A_{42})(A_{31} + A_{24}) \\ &\quad + \frac{1}{32}(A_{23} + A_{32} + A_{14} - 3A_{41})^2. \end{aligned} \tag{3.2}$$

Similarly, we calculate R^2_{2a2b}, R^2_{3a3b} and R^2_{4a4b} and have

$$4\sum_{a,b}(R^2_{1a1b} + R^2_{2a2b} + R^2_{3a3b} + R^2_{4a4b}) \tag{3.3}$$

$$
\begin{aligned}
= \quad & 18c^2 + \frac{13}{32}(\tau^* - \tau)^2 - \frac{3c}{2}(\tau^* - \tau) + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42}) \\
& +(A_{12} - A_{21})^2 + (A_{11} + A_{22})^2 + (A_{34} - A_{43})^2 + (A_{33} + A_{44})^2 \\
& +\frac{5}{4}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\} \\
& +\frac{3}{2}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\} \\
& +\frac{1}{8}\{(A_{23} + A_{32} + A_{14} - 3A_{41})^2 + (A_{14}) + A_{41} + A_{23} - 3A_{32})^2 \\
& +(A_{14} + A_{41} + A_{32} - 3A_{23})^2 + (A_{23} + A_{32} + A_{41} - 3A_{14})^2\}
\end{aligned}
$$

Next, we calculate the second term of (3.1). Taking account of Proposition 2.3, we have

$$
\begin{aligned}
(3.4) \qquad & -4(R^2_{1212} + R^2_{1313} + R^2_{1414} + R^2_{2323} + R^2_{2424} + R^2_{3434}) \\
= \quad & -9c^2 - \frac{4 \cdot 52^2}{32}(\tau^* - \tau)^2 + \frac{3c}{4}(\tau^* - \tau) \\
& -\frac{\tau^* - \tau}{4}(A_{13} - A_{31} - A_{24} + A_{42}) \\
& -\frac{5}{8}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\} \\
& -\frac{3}{4}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\}.
\end{aligned}
$$

Lastly, we calculate the last term of (3.1). Taking account of Proposition 2.3, we have

$$
\begin{aligned}
(3.5) \qquad & 8(R^2_{1234} + R^2_{1324} + R^2_{1423}) \\
& = 3c^2 + \frac{1}{16}(\tau^* - \tau)^2 + \frac{3c}{8}(\tau^* - \tau) + \frac{\tau^* - \tau}{4}(A_{13} - A_{31} - A_{24} + A_{42}).
\end{aligned}
$$

Therefore by (3.1) and (3.3)-(3.5), we have

Lemma 3.1. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then we have*

$$
\|R\|^2 \quad = \quad 12c^2 + \frac{17}{64}(\tau^* - *)^2 + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42})
$$

$$
\begin{aligned}
&+(A_{12}-A_{21})^2+(A_{11}+A_{22})^2+(A_{34}-A_{43})^2+(A_{33}+A_{44})^2\\
(3.6)\quad &+\frac{5}{8}\{(A_{13}-A_{31})^2+(A_{24}-A_{42})^2+(A_{13}+A_{42})^2+(A_{31}+A_{24})^2\}\\
&+\frac{3}{4}\{(A_{13}-A_{31})(A_{24}-A_{42})+(A_{13}+A_{42})(A_{31}+A_{24})\}\\
&+\frac{1}{8}\{(A_{23}+A_{32}+(A_{14}-3A_{41})^2+(A_{14}+A_{41}+A_{23}-3A_{32})^2\\
&+(A_{14}-A_{41}+A_{32}-3A_{23})^2+(A_{23}+A_{32}+A_{41}-3A_{14})^2\}
\end{aligned}
$$

for any unitary basis $\{e_i\}=\{e_1,e_2=Je_1,e_3,e_4=Je_3\}$ *of* $T_pM, p\in M$.

Now we replace $\{e_3,e_4\}$ by $\{e_4,-e_3\}$. Then by virtue of (2.8), we see that A_{13}, A_{31}, A_{24} and A_{42} is changed respectively into

$$
\begin{aligned}
(3.7)\quad A_{13} &\longmapsto g(e_1,(\nabla_{e_4}N)(e_1,e_4))\\
&= A_{24}-\frac{1}{2}\{g(N(e_1,e_3),e_3)^2+g(N(e_1,e_3),e_4)^2\}\\
A_{31} &\longmapsto g(e_4,(\nabla_{e_1}N)(e_1,e_4))\\
&= -A_{31}+\frac{1}{2}\{g(N(e_1,e_3),e_1)^2+g(N(e_1,e_3),e_2)^2\}\\
A_{24} &\longmapsto -g(e_2,(\nabla_{e_3}N)(e_1,e_4))\\
&= -A_{13}+\frac{1}{2}\{g(N(e_1,e_3),e_3)^2+g(N(e_1,e_3),e_4)^2\}\\
A_{42} &\longmapsto -g(e_3,(\nabla_{e_2}N)(e_1,e_4))\\
&= -A_{42}-\frac{1}{2}\{g(N(e_1,e_3),e_1)^2+g(N(e_1,e_3),e_2)^2\}
\end{aligned}
$$

Thus from (2.6) and (3.7) we have

$$
\begin{aligned}
(3.8)\quad A_{13}-A_{31} &\longmapsto A_{24}+A_{31}-\frac{1}{2}(\tau^*-\tau),\\
A_{24}-A_{42} &\longmapsto A_{13}+A_{42}+\frac{1}{2}(\tau^*-\tau),\\
A_{13}+A_{42} &\longmapsto A_{24}-A_{42}-\frac{1}{2}(\tau^*-\tau),\\
A_{31}+A_{24} &\longmapsto A_{13}-A_{31}+\frac{1}{2}(\tau^*-\tau).
\end{aligned}
$$

Replacing $\{e_3,e_4\}$ by $\{e_4,-e_3\}$, we have further

$$
(3.9)\quad A_{12}-A_{21}\longmapsto A_{11}+A_{22},
$$

$$A_{11}+A_{22}\longmapsto -A_{12}+A_{21},$$
$$A_{34}-A_{43}\longmapsto A_{33}+A_{44},$$
$$A_{33}+A_{44}\longmapsto -A_{34}+A_{43},$$

(3.10)
$$A_{23}\longmapsto -A_{14},$$
$$A_{32}\longmapsto -A_{32},$$
$$A_{14}\longmapsto -A_{23},$$
$$A_{41}\longmapsto -A_{41}.$$

Taking account of (3.9) and (3.10), we see that
$\frac{1}{8}\{(A_{23}+A_{32}+A_{14}-3A_{41})^2+(A_{14}+A_{41}+A_{23}-3A_{32})^2+(A_{14}+A_{41}+A_{32}-3A_{23})^2+(A_{23}+A_{32}+A_{41}-3A_{14})^2\}$

and
$(A_{12}-A_{21})^2+(A_{11}+A_{22})^2+(A_{34}-A_{43})^2+(A_{33}+A_{44})^2$ are both invariant by the change $\{e_3,e_4\}$ into $\{e_4,-e_3\}$. On the other hand, from (3.8) we see that

$$A_{13}-A_{31}-A_{24}+A_{42}\longmapsto -(A_{13}-A_{31}-A_{24}+A_{42})(\tau^*-\tau), \tag{3.11}$$

$$\begin{aligned}&(A_{13}-A_{31})^2+(A_{24}-A_{42})^2+(A_{13}+A_{42})^2+(A_{31}+A_{24})^2\\ \longmapsto\ &(A_{13}-A_{31})^2+(A_{24}-A_{42})^2+(A_{13}+A_{42})^2+(A_{31}+A_{24})^2\\ &+2(\tau^*-\tau)(A_{13}-A_{31}-A_{24}+A_{42})+(\tau^*-\tau)^2,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&(A_{13}-A_{31})(A_{24}-A_{42})+(A_{13}+A_{42})(A_{31}+A_{24})\\ \longmapsto\ &(A_{13}-A_{31})(A_{24}-A_{42})+(A_{13}+A_{42})(A_{31}+A_{24})\\ &-(\tau^*-\tau)(A_{13}-A_{31}-A_{24}+A_{42})-\frac{1}{2}(\tau^*-\tau)^2.\end{aligned} \tag{3.13}$$

Since $\|R\|^2$ is invariant by the change $\{e_3,e_4\}$ into $\{e_4,-e_3\}$, taking account of Lemma 3.1 and (3.11)-(3.13), we have

$$(\tau^*-\tau)\{(A_{13}-A_{31}-A_{24}+A_{42})+\frac{1}{2}(\tau^*-\tau)\}=0. \tag{3.14}$$

Suppose that M is not Kählerian. Then there exists a point $p \in M$ such that $\tau^* - \tau > 0$ at p. From (3.14) it follows that

$$A_{13} - A_{31} - A_{24} + A_{42} = -\frac{1}{2}(\tau^* - \tau) \tag{3.15}$$

at p. From (3.2) and (3.15) we calculate at p

$$\begin{aligned} \sum_{a,b} R^2_{1a1b} = {} & \frac{9}{8}c^2 + \frac{1}{8}(A_{12} - A_{21})^2 + \frac{1}{8}(A_{11} + A_{22})^2 \\ & + \frac{26}{32^2}(\tau^* - \tau)^2 - \frac{3c}{32}(\tau^* - \tau) + \frac{c}{4}(A_{13} + A_{24}) \\ & + \frac{1}{4}(A_{13} - A_{31})^2 + \frac{1}{4}(A_{13} + A_{42})^2 \\ & + \frac{1}{16}(\tau^* - \tau)(A_{13} - A_{31}) + \frac{3}{16}(\tau^* - \tau)(A_{13} + A_{42}) \\ & + \frac{5}{128}(\tau^* - \tau)^2 + \frac{1}{32}(A_{23} + A_{32} + A_{14} - 3A_{41})^2. \end{aligned} \tag{3.16}$$

Since $\sum_{a,b} R^2_{1a1b}$ depends only on e_1 (invariant by the change $\{e_3, e_4\}$ into $\{e_4, -e_3\}$), taking account of (3.7) - (3.10) and (3.15) we have

$$(\tau^* - \tau)(A_{42} + A_{31}) = 0 \tag{3.17}$$

at p. Thus from (3.17) we have

$$A_{42} + A_{31} = 0 \tag{3.18}$$

at p. But by changing $\{e_1, e_2, e_3, e_4\}$ into $\{e_3, e_4, e_1, e_2\}$, we see that

$$A_{42} \longmapsto -A_{24}, \quad A_{31} \longmapsto -A_{13}. \tag{3.19}$$

So, (3.18) and (3.19) yield

$$A_{13} + A_{24} = 0 \tag{3.20}$$

at p. On the other hand, Proposition 2.3 yields

$$\begin{aligned} \rho_{11} &= \frac{\tau}{4} + \frac{1}{2}(A_{13} + A_{24}), \\ \rho_{22} &= \frac{\tau}{4} - \frac{1}{2}(A_{13} + A_{24}), \\ \rho_{33} &= \frac{\tau}{4} - \frac{1}{2}(A_{31} + A_{42}), \\ \rho_{44} &= \frac{\tau}{4} + \frac{1}{2}(A_{31} + A_{42}), \end{aligned} \tag{3.21}$$

where we set $\rho_{ij} = \rho(e_i, e_j)$. From (3.18), (3.20) and (3.21) we see that

$$\rho_{ii} = \frac{\tau}{4} \quad (1 \leq i \leq 4). \tag{3.22}$$

Thus we have

Lemma 3.2. *Under the same hypothesis as in Lemma 3.1, if M is not Kählerian, then the open subspace $M_0 = \{p \in M | \tau^* - \tau > 0$ at $p\}$ is an Einstein manifold.*

We put $M_1 = \{p \in M | \tau^* - \tau = 0 \text{ at } p\}$. Then $M = M_0 \cup M_1$. If the interior M_1' of M_1 is non-empty, then M_0' is a Kählerian and has constant holomorphic sectional curvature, and is Einsteinian. Thus we have

Proposition 3.3. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature, then M is an Einstein manifold.*

Now, we suppose that M_0 is non-empty, and we discuss on M_0. Then (3.15), (3.18) and (3.20) yield

$$\begin{aligned} A_{13} - A_{31} &= -\frac{1}{4}(\tau^* - \tau), \\ A_{24} - A_{42} &= \frac{1}{4}(\tau^* - \tau). \end{aligned} \tag{3.23}$$

From Proposition 2.3 and the Einstein condition we have

$$\begin{aligned} A_{11} + A_{22} &= A_{33} + A_{44}, \\ A_{12} - A_{21} &= A_{34} - A_{43}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} A_{14} &= A_{23}, \\ A_{41} &= A_{32}. \end{aligned} \tag{3.25}$$

Also, from Proposition 2.3, (3.23) and the definition of ρ^* we have

$$\begin{aligned} \rho^*_{14} &= \rho^*_{23} = -\rho^*_{41} = -\frac{1}{2}(A_{12} - A_{21}), \\ \rho^*_{13} &= \rho^*_{24} = -\rho^*_{31} = -\frac{1}{2}(A_{11} + A_{22}). \end{aligned} \tag{3.26}$$

Here, we recall the following general formula which holds in a four-dimensional almost Hermitian manifold (cf. [1]);

$$\rho(x,y)+\rho(Jx,Jy)-\rho^*(x,y)-\rho^*(Jx,Jy)=\frac{\tau^*-\tau}{2}g(x,y),$$

for any tangent vector x, y on the manifold. So, since M_0 is an Einstein manifold, we have

$$\rho^*(x,y)+\rho^*(y,x)=\frac{\tau^*}{2}g(x,y), \tag{3.27}$$

for any $x, y \in T_pM_0$ and any $p \in M_0$. From (3.16), taking account of (3.15), (3.18), (3.20), (3.23), (3.24), (3.25), (3.26) and (3.27) we have

$$\begin{aligned} \sum_{a,b} R_{1a1b}^2 &= \frac{9}{8}c^2+\frac{1}{8}\|\rho^*\|^2-\frac{1}{32}\tau^{*2}-\frac{3c}{32}(\tau^*-\tau) \\ &\quad +\frac{18}{32^2}(\tau^*-\tau)^2+\frac{1}{8}(A_{14}-A_{41})^2. \end{aligned} \tag{3.28}$$

Also, from Lemma 3.1 together with (3.15), (3.18), (3.20), (3.23), (3.24), (3.25), (3.26) and (3.27) we have

$$\|R\|^2=12c^2+\frac{5}{64}(\tau^*-\tau)^2+2(A_{14}-A_{41})^2+2(\|\rho^*\|^2-\frac{1}{4}\tau^{*2}). \tag{3.29}$$

From (3.29) we see that $(A_{14}-A_{41})^2$ is a function on M_0, and hence from (3.28), $\sum_{a,b} R_{1a1b}^2$ is independent with the choice of e_1. Thus we see that M_0 is a 2-stein space. Further, since $(A_{14}-A_{41})^2$ is a function on M_0, $(A_{14}-A_{41})^2$ is independent with the choice of a unitary basis $\{e_i\}=\{e_1, e_2=Je_1, e_3, e_4=Je_3\}$. Thus, from (2.8), (3.18), (3.20) and (3.23) we have

$$\begin{aligned} (A_{14}-A_{41})^2 &= \{g(\cos te_1+\sin te_2, (\nabla_{e_4}N)(\cos te_1+\sin te_2, e_3)) \\ &\quad -g(e_4, (\nabla_{\cos te_1+\sin te_2}N)(\cos te_1+\sin te_2, e_3))\}^2 \\ &= (A_{14}-A_{41})^2\cos^2 2t \end{aligned}$$

for all real number t. So, we see that

$$A_{14}-A_{41}=0,$$

and hence from (3.28) and (3.29) respectively we have

$$(3.30)\qquad \sum_{a,b} R^2_{1a1b} = \frac{9}{8}c^2 + \frac{1}{8}\|\rho^*\|^2 - \frac{1}{32}\tau^{*2} - \frac{3c}{32}(\tau^* - \tau) + \frac{18}{32^2}(\tau^* - \tau)^2$$

and

$$(3.31)\qquad \|R\|^2 = 12c^2 + \frac{5}{64}(\tau^* - \tau)^2 + 2(\|\rho^*\|^2 - \frac{1}{4}\tau^{*2}).$$

Now, let f denote the smooth function on M_0 defined by

$$f = \sum_{a,b} R^2_{1a1b}.$$

Then for any tangent vector $x \in T_pM_0$ $(p \in M_0)$ we have

$$(3.32)\qquad \sum_{a,b} R^2_{xaxb} = f(p)g(x,x)g(x,x),$$

where we set $R_{xaxb} = g(R(x,e_a)x,e_b)$. By taking twice Euclidean Laplacian in T_pM in both members of (3.32), we have

$$(3.33)\qquad f(p) = \frac{1}{96}\tau^2 + \frac{1}{16}\|R\|^2.$$

Thus from (3.30), (3.31) and (3.33) we have

$$(3.34)\qquad \frac{3}{8}c^2 - \frac{1}{96}\tau^2 + \frac{13}{32^2}(\tau^* - \tau)^2 - \frac{3c}{32}(\tau^* - \tau) = 0.$$

At last, from (2.9) and (3.34) we have

$$\frac{7}{32^2}(\tau^* - \tau)^2 = 0,$$

and hence $\tau^* - \tau = 0$ on M_0. This is a contradiction. Therefore we conclude that M_0 is empty, and we have proved our Main Theorem.

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HYPERSURFACES IN THE EUCLIDEAN SPACE R^4

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ABSTRACT. For a compact and connected hypersurface M in the Euclidean space R^4, it is proved that, if the mean curvature is nowhere zero and the scalar curvature S satisfies $||\psi||^2 S = 6$, where ψ is the position vector field of M in R^4, then M is isometric to a sphere. It is also proved that if the Ricci curvature of the hypersurface M satisfies $0 < Ric \leq \frac{2}{3}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on M, then M is isometric to a sphere.

1. INTRODUCTION

In the geometry of compact hypersurfaces of a Euclidean space R^n, one of the interesting questions is to obtain conditions under which the hypersurface is isometric to a sphere in R^n. This problem becomes more interesting for the hypersurfaces of even-dimensional Euclidean spaces R^{2n} owing to the complex geometry of R^{2n} and, in particular, of R^4 which has quaternion structure. The motivation for the present paper comes from the following considerations:

Let $S^n(c)$ be the n-sphere of constant curvature c in R^{n+1} centered at the origin and $\psi : S^n(c) \to R^{n+1}$ be the inclusion map. Then the mean curvature α and the scalar curvature S of S^n satisfy:

$$\alpha = -n\sqrt{c} < 0 \quad \text{and} \quad ||\psi||^2 S = n(n-1).$$

This raises the question: Is a compact and connected immersed hypersurface $\psi : M \to R^{n+1}$ whose mean curvature α is nowhere zero and

1991 Mathematics Subject Classification: Primary 53C40, Secondary 53C42, 53C20 whose scalar curvature S satisfies $\|\psi\|^2 S = n(n-1)$, necessarily isometric to a sphere in R^{n+1}?

Also the Ricci curvature Ric of $\psi : S^n(c) \to R^{n+1}$ satisfies $0 < \text{Ric} = \frac{(n-1)}{n}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on $S^n(c)$. This raises another question: Is a compact and connected immersed hypersurface $\psi : M \to R^{n+1}$ whose Ricci curvature Ric satisfies $0 < \text{Ric} = \frac{(n-1)}{n}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on M, necessarily isometric to a sphere in R^{n+1}?

In this paper we answer these questions in the affirmative for compact and connected hypersurfaces of R^4. Indeed, we prove the following theorems:

Theorem 1. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface. If the mean curvature α of M is nowhere zero and the scalar curvature S of M satisfies $\|\psi\|^2 S = 6$, then M is isometric to a sphere.

Theorem 2. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface. If the Ricci curvature of M satisfies $0 < \text{Ric} \le \frac{2}{3}\lambda_1$, where λ_1 is the first nonzero eigenvalue of the Laplacian on M with respect to the induced metric, then M is isometric to a sphere.

The proof of Theorem 2 depends heavily on the quaternion structure of R^4. However, it is surprising to note that we use no quaternion structure of R^4 in the proof of Theorem 1 though the proof works only for this dimension and cannot be extended to $\psi : M \to R^{n+1}$.

2. PRELIMINARIES

Let $< \cdot, \cdot >$ be the inner product on R^4 and $\bar{\nabla}$ be the Euclidean connection on R^4. Let $\psi : M \to R^4$ be an orientable hypersurface with unit normal vector field N. We denote by g and ∇ the induced metric and the Riemannian connection on M, respectively. Then we have

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M),$$

where A is the shape operator of M and $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M. The shape operator A satisfies

$$(\nabla_X A)(Y) = (\nabla_Y A)(X), \tag{2.2}$$

$$g((\nabla_X A)(Y), Z) = g((\nabla_X A)(Z), Y), \quad X, Y, Z \in \mathfrak{X}(M). \tag{2.3}$$

For a local orthonormal frame $\{e_1, e_2, e_3\}$ on M, the mean curvature α is given by $\alpha = \frac{1}{3}\sum_{i=1}^{3} g(Ae_i, e_i)$ and it satisfies

$$X(\alpha) = \frac{1}{3}\sum_{i=1}^{3} g((\nabla_{e_i} A)(e_i), X), \quad X \in \mathfrak{X}(M). \tag{2.4}$$

The Ricci curvature tensor Ric and the scalar curvature S of M are given by

$$\text{Ric}\,(X, Y) = 3\alpha g(AX, Y) - g(AX, AY), \;\; S = 9\alpha^2 - \|A\|^2, \quad X, Y \in \mathfrak{X}(M), \tag{2.5}$$

where $\|A\|^2 = trA^2$.

For the orientable hypersurface $\psi : M \to R^4$, we can treat ψ as the position vector field of M in R^4 and therefore it can be expressed as

$$\psi = t + \rho N, \tag{2.6}$$

where $t \in \mathfrak{X}(M)$ and $\rho = <\psi, N>$ is called the *support function* of M. Using (2.1) and (2.6) one immediately obtains

$$\nabla_X t = X + \rho AX, \qquad X(\rho) = -g(AX, t), \quad X \in \mathfrak{X}(M). \tag{2.7}$$

If M is compact, we have Minkowski's formula

$$\int_M (1 + \rho\alpha) dv = 0. \tag{2.8}$$

We denote by J_1, J_2 and J_3 the complex structures on R^4 which define the quaternion structure of R^4. Then we have

$$J_1 J_2 = -J_2 J_1 = J_3, \quad J_2 J_3 = -J_3 J_2 = J_1, \quad J_3 J_1 = -J_1 J_3 = J_2; \tag{2.9}$$

$$(2.10)\qquad \bar{\nabla} J_i = 0, \qquad < J_i, J_i > = <,>, \quad i = 1,2,3.$$

Define the vector fields ξ_1, ξ_2, ξ_3 on M by $\xi_i = -J_i N$, $i = 1,2,3$ and set $J_i X = \phi_i X + \eta_i(X)N$, where ϕ_i is a (1,1)-tensor field on M and η_i is a 1-form dual to ξ_i on $M, i = 1,2,3$. It is easy to verify that each triplet (ϕ_i, ξ_i, η_i) satisfy

$$(2.11)\qquad \phi_i^2 = -I + \eta_i \otimes \xi_i, \quad \phi_i(\xi_i) = 0, \quad \eta_i \circ \phi_i = 0,$$

$$(2.12)\qquad g(\phi_i X, \phi_i Y) = g(X,Y) - \eta_i(X)\eta_i(Y), \quad X, Y \in \mathfrak{X}(M).$$

For a local unit vector field e on M satisfying $g(e, \xi_i) = 0$ for a fixed i, $\{e, \phi_i(e), \xi_i\}$ is a local orthonormal frame and such a local frame will be referred to as an adapted frame. Using (2.1) and (2.10) with $\xi_i = -J_i N$, we obtain $\nabla_X \xi_i = \phi_i AX$, $X \in \mathfrak{X}(M)$. From this equation, using an adapted frame, we get

$$(2.13)\qquad div\ \xi_i = 0, \quad i = 1,2,3.$$

Using the complex structures J_i and the position vector field ψ of M in R^4, we define the smooth functions $\rho_i : M \to R$ by $\rho_i = < J_i\psi, N >$ and the vector fields $t_i \in \mathfrak{X}(M)$ by setting $J_i\psi = t_i + \rho_i N$. Using (2.1), (2.10) and $J_i\psi = t_i + \rho_i N$, we now obtain

$$\nabla_X t_i = \phi_i X + \rho_i AX, \quad X(\rho_i) = -g(At_i, X) + \eta_i(X), \quad X \in \mathfrak{X}(M).$$

(2.14)

We also have

$$g(t_i, \xi_i) = < J_i\psi - \rho_i N, \xi_i > = - < \psi, J\xi_i > = - < \psi, N > = -\rho.$$

(2.15)

3. PROOF OF THEOREM 1

Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface of R^4. We define $F : M \to R$ by $F = \frac{1}{2}\|\psi\|^2$. Then, using (2.6) and (2.7), we obtain

$$(3.1)\qquad \Delta F = 3(1 + \rho\alpha),$$

where Δ is the Laplacian operator on M. Using (2.6) and (2.7), we compute $\Delta\rho$ to obtain

$$\Delta\rho = -3t(\alpha) - 3\alpha - \rho\|A\|^2. \tag{3.2}$$

We have $X(F) = g(t, X)$, $X \in \mathfrak{X}(M)$, and consequently grad $F = t$. Also from the second equation in (2.7), we get grad $\rho = -At$. Thus, using (3.1) and (3.2), we obtain

$$\begin{aligned}\Delta(\rho F) &= F\Delta\rho + \rho\Delta F + 2g(\text{grad } F, \text{grad}\rho)\\ &= -3Ft(\alpha) - 3\alpha F - \rho F\|A\|^2\\ &\quad +3\rho(1+\rho\alpha) - 2g(At, t)\end{aligned}$$

Since $div\,(fX) = X(f) + f\,div\,X$, $X \in (M), f \in c^\infty(M)$ and $t(\rho) = -g(At, t)$, we have

$$\begin{aligned}\Delta(\rho F) &= -3[div\,(F\alpha t) - \alpha\, div(Ft)]\\ &\quad -3\alpha F - F\rho\|A\|^2 + 3\rho + 3\alpha\rho^2 + 2[div(\rho t) - \rho div t]\\ &= -3div(F\alpha t) + 2div(\rho t) + 3\alpha[t(F) + F div\, t]\\ &\quad -3\alpha F - F\rho\|A\|^2 + 3\rho + 3\alpha\rho^2 - 2\rho div t\\ &= -3div(F\alpha t) + 2div(\rho t) + 3\alpha\|t\|^2 + F\rho[g\alpha^2 - \|A\|^2]\\ &\quad +6F\alpha - 3\rho - 3\rho^2\alpha,\end{aligned}$$

where we have used $t(F) = \|t\|^2$ and $div t = 3(1+\rho\alpha)$. Consequently, with $\|\psi\|^2 = \|t\|^2 + \rho^2$, we have

$$\begin{aligned}\Delta(\rho F) + 3div(F\alpha t) - 2div(\rho t) &= 3\alpha(\|t\|^2 - \rho^2 + 2F) + F\rho S - 3\rho\\ &= 6\alpha\|t\|^2 + \frac{\rho}{2}[\|\psi\|^2 - 6].\end{aligned}$$

Integrating this last equation over M and using the hypothesis of the theorem we get

$$\int_M \alpha\|t\|^2 dv = 0.$$

Since α is nowhere zero and M is connected we must have $\alpha < 0$ (as there is a point where all eigenvalues of A are negative; indeed this point is where the height function of M attains its maximum). Hence the above integral gives $t = 0$ on M. Then, the equations in (2.7) yield $\rho AX = -X$

and $\rho =$ constant. That $\rho \neq 0$ follows from $\psi = \rho N$ (as $t = 0$). Hence $A = \frac{-1}{\rho} I$, that is, M is a totally umbilical hypersurface of R^4 and, as such, it is isometric to a sphere.

Remark: If we take M to be a hypersurface of R^{n+1} and proceed with the computation as in the above proof, we arrive at

$$\Delta(\rho F)+3div(F\alpha t)-2div(\rho t) = n\alpha(\|t\|^2(\frac{n+1}{2})+\rho^2\frac{(n-3)}{2})+\rho(FS-n),$$

which suggests that the conclusion of the theorem works only for $n = 3$, and therefore this cannot give information beyond dimension 3, unless some additional assumptions are made. It will be an interesting problem to generalise the theorem for a compact and connected hypersurface of R^{n+1}.

4. PROOF OF THEOREM 2.

First we prove the following

Proposition. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface of R^4 with non-negative Ricci curvature. If $\rho_i = 0$, $i = 1,2,3$, then M is isometric to a sphere in R^4.

Proof. Note that for an adapted frame $\{e, \phi_i(e), \xi_i\}$, for a fixed i, we can use (2.11) to obtain $\phi_i^2(e) = -e$ as $e \perp \xi_i$. Thus if $\{e_1, e_2, e_3\}$ is such a frame, we have

$$(4.1) \quad \sum_{j=1}^{3} g(\phi_i(e_j), A(e_j) = g(\phi_i(e), Ae) + g(\phi_i(\phi_i(e)), A(\phi_i(e)) = 0,$$

Since $\rho_1 = \rho_2 = \rho_3 = 0$, the second equation in (2.14) gives $At_i = \xi_i$, $i = 1,2,3$. As ξ_i are globally defined unit vector fields on M and A is a linear operator, it follows from $At_i = \xi_i$, that t_i are nowhere zero on M. Moreover, using (2.9), we have

$$g(t_1, t_2) =< J_1\psi, J_2\psi >= - < \psi, J_1J_2\psi >= - < \psi, J_3\psi >= 0,$$

and similarly $g(t_2, t_3) = 0$, $g(t_3, t_1) = 0$, that is, the vector fields t_1, t_2, t_3 are mutually orthogonal. Thus the unit vector fields $\hat{t}_i$ along t_i give the orthonormal frame $\{\hat{t}_1, \hat{t}_2, \hat{t}_3\}$. Using (2.13), and $At_i = \xi_i$, we get

$$0 = div(At_i) = \sum_{j=1}^{3} g(\nabla_{e_j} At_i, e_j) = \sum_{j=1}^{3} [g((\nabla_{e_j} A)(t_i), e_j) + g(\nabla_{e_j} t_i, Ae_j)].$$

Using (2.4), (2.14) and (4.1) in the above equation we get $3t_i(\alpha) = 0$ or $\hat{t}_i(\alpha) = 0$, $i = 1, 2, 3$. This proves that the mean curvature α is a constant.

Then the equation (3.2), after integration, gives

$$\int_M (3\alpha + \rho tr A^2) dv = 0. \tag{4.2}$$

The integral formula (2.8) with α a constant can be restated as

$$\int_M (3\alpha + 3\rho\alpha^2) dv = 0 \tag{4.3}$$

The integrals (4.2) and (4.3) give

$$\int_M \rho(3\alpha^2 - tr A^2) dv = 0. \tag{4.4}$$

We use (2.5) and $At_i = \xi_i$, to arrive at Ric $(t_1, t_1) = 3\alpha g(t_i, \xi_i) - \|\xi_i\|^2$, or Ric $(t_1, t_1) = -3\alpha\rho - 1 \geq 0$, as the Ricci curvature is non-negative from the hypothesis. This last inequality suggests that there is no point $p \in M$ such that $\rho(p) = 0$. Thus, M being connected, we have either $\rho > 0$ or $\rho < 0$. Moreover the Schwarz inequality states that $3\alpha^2 \leq tr A^2$, with equality holding at a point if and only if it is an umbilic point. The integral (4.4) gives $3\alpha^2 = tr A^2$, proving that M is an umbilical hypersurface of R^4, and this proves the proposition.

Now we proceed to prove Theorem 2. Let $\psi : M \to R^4$ be a compact and connected immersed hypersurface of R^4. We assume that the center of mass of M is at the origin of R^4 (for otherwise an isometry $\phi : R^4 \to R^4$ can be chosen which maps the center of mass of M to the origin of R^4, and then $\psi' = \phi \circ \psi$ will be the desired immersion). Thus, using the minimal principle with $\int_M \psi dv = 0$, we get

$$\lambda_1 \leq 3vol(M) / \int_M \|\psi\|^2 dv,$$

where λ_1 is the first nonzero eigenvalue of the Laplacian operator on M. Consequently we have

$$\int_M \|\psi\|^2 dv \leq \frac{3vol(M)}{\lambda_1}. \tag{4.5}$$

We use (2.3), (2.4), (2.14) and (4.1) to compute $div(At_i)$:

$$div(At_i) = \sum_{j=1}^{3} g(\nabla_{e_j} At_i, e_j) = 3t_i(\alpha) + \rho_i \|A\|^2 = 3[div(\alpha t_i) - \alpha div t_i] + \rho_i \|A\|^2.$$

This gives

$$div(At_i) = 3div(\alpha t_i) + \rho_i[\|A\|^2 - 9\alpha^2], \tag{4.6}$$

where we have used $div(t_i) = 3\rho_i\alpha$ as a result of (2.14). Now the second equations in (2.14) and (2.15), yield

$$\begin{aligned} div(\rho_i \alpha t_i) &= \rho_i div(\alpha t_i) + \alpha t_i(\rho_i) \\ &= \rho_i div(\alpha t_i) - \alpha g(At_i, t_i) - \alpha\rho, \end{aligned}$$

that is,

$$3\rho_i div(\alpha t_i) = 3div(\rho_i \alpha t_i) + 3\alpha g(At_i, t_i) + 3a\rho. \tag{4.7}$$

Finally we use (2.14), (4.6) and (4.7) to compute $div(\rho_i At_i)$ and obtain

$$\begin{aligned} div(\rho_i At_i) &= At_i(\rho_i) + \rho_i div(At_i) \\ &= -g(At_i, At_i) + \eta_i(At_i) + 3\rho_i div(\alpha t_i) + \rho_i^2[\|A\|^2 - 9\alpha^2] \\ &= -\|At_i\|^2 + g(At_i, \xi_i) + 3div(\rho_i \alpha t_i) + 3\alpha g(At_i, t_i) \\ &\quad +3\alpha\rho - \rho_i^2 S. \end{aligned}$$

Since $g(At_i, \xi_i) = -[-g(At_i, \xi_i) + \eta_i(\xi_i) - 1] = -[\xi_i(\rho_i) - 1] = -div(\rho_i\xi_i) + 1$, (where we have used (2.13)), the above equation becomes

$$div[\rho_i At_i - 3\rho_i \alpha t_i + \rho_i \xi_i] = \text{Ric}\ (t_i, t_i) - \rho_i^2 S + 3\rho\alpha + 1. \tag{4.8}$$

Let $\hat{t}_i$ be the unit vector field defined on the open subset of M where $t_i \neq 0$. Using $\|\psi\|^2 = \|t_i\|^2 + \rho_i^2$, which follows from $J_i\psi = t_i + \rho_i N$, in equation (4.8), we arrive at

$$div[\rho_i At_i - 3\rho_i \alpha t_i + \rho_i \xi_i] = -\rho_i^2[\text{Ric}(\hat{t}_i, \hat{t}_i) + S] + \|\psi\|^2\ \text{Ric}(\hat{t}_i, \hat{t}_i) + 3\alpha\rho + 1.$$

From the hypothesis of the theorem that $0 < \mathrm{Ric} \leq \frac{2}{3}\lambda_1$, we obtain

$$div[\rho_i A t_i - 3\rho_i \alpha t_i + \rho_i \xi_i] \leq -\rho_i^2[\mathrm{Ric}(\hat{t}_i, \hat{t}_i) + S] + \frac{2}{3}\lambda_1 \|\psi\|^2 + 3\alpha\rho + 1.$$

Integrating the above inequality and using formula (2.8), we get

$$\int_M \rho_i^2[\mathrm{Ric}(\hat{t}_i, \hat{t}_i) + S]dv \leq -2vol(M) + \frac{2}{3}\lambda_1 \int_M \|\psi\|^2 dv.$$

This, together with (4.5), give

$$\int_M \rho_i^2[\mathrm{Ric}(\hat{t}_i, \hat{t}_i) + S]dv \leq 0.$$

Since $\mathrm{Ric} > 0$, this integral inequality gives $\rho_i = 0$ $i = 1, 2, 3$, and the above proposition completes the proof.

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ASYMPTOTIC TEST FOR MONOTONE VARIANCE RESIDUAL LIFE

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ABSTRACT. In this paper we give an outline of the main properties of decreasing (increasing) variance remaining life distributions, DVRL (IVRL). The connections of this class with DMRL and equilibrium distributions are displayed. An asymptotic test for exponentiality against DVRL (IVRL) class is developed. It can be used to test exponentiality against DMRL (IMRL) property of the associated class of renewal distributions.

1. INTRODUCTION

Let T, denote the life time of an equipment with distribution function $F(t) = P(T \leq t)$ and survival function $\bar{F}(t) = P(T > t)$, where $\bar{F}(O) = 1$. We assume that the density $f(t)$ can be obtained by differentiating $F(t)$. The failure rate $r(t) = f(t)/\bar{F}(t)$. The mean life $\mu = \int_0^\infty \bar{F}(t)dt$ and the variance life $\sigma^2 = \text{var } (T)$, will be assumed finite. It is well known that $\mu(t) = E(T - t|T > t) = \int_t^\infty \bar{F}(x)dx/\bar{F}(t)$, and $\sigma^2(t)$ will denote var $(T - t|T > t) = \text{var } (T|T > t)$.

A distribution function F is said to have decreasing (increasing) variance residual life DVRL, (IVRL) if $\sigma^2(t)$ is nonincreasing (nondecreasing) function of t on $(0,\infty)$. Launer [9]; Gupta [4]; Gupta, Kirmani and Launer (5); studied characterization of this class and used it to find better bounds on moments and survival functions. In section 2 below we outline some known results for the convenience of reader, although some of them are derived in a new different way. In section 3, some preliminary work is presented and in section 4 an asymptotic test for exponentiality against DVRL (IVRL) class is derived and an example is given.

2. CHARACTERIZATION OF DVRL(IVRL) AND SOME USEFUL RESULTS

(a) Consider $E(u^2|t) = -\int_0^\infty u^2 d\bar{F}(u|t)$. Integrating by parts we have,

$$\sigma^2(t) + \mu^2(t) = E(u^2|t) = 2\int_0^\infty u[\bar{F}(u+t)/\bar{F}(t)]du.$$

Let $u + t = x$, then,

$$\begin{aligned} \sigma^2(t) + \mu^2(t) &= [2/\bar{F}(t)]\int_t^\infty (x-t)\bar{F}(x)dx \\ &= [2/\bar{F}(t)]\int_t^\infty \int_t^x \bar{F}(x)dydx \\ &= [2/\bar{F}(t)]\int_t^\infty \int_y^\infty \bar{F}(x)dxdy. \qquad (2.1) \end{aligned}$$

(b) As in Hall and Wellner [6], let

$$\bar{F}^{(r)}(t) = \int_t^\infty \bar{F}^{(r-1)}(x)dx, \quad r = 1, 2, \ldots; \quad \bar{F}^{(0)} = \bar{F} = 1 - F. \qquad (2.2)$$

and let $\phi_r(t) = r!\bar{F}^{(r)}(t)/\bar{F}(t)$. Then from (2.1) and (2.2) we have

$$\begin{aligned} \phi_1(t) = \mu(t); \phi_2(t) &= [2/\bar{F}(t)]\int_t^\infty \int_y^\infty \bar{F}(x)dxdy \\ &= \sigma^2(t) + \mu^2(t). \qquad (2.3) \end{aligned}$$

$$\begin{aligned} \sigma^2(t) &= [2/\bar{F}(t)]\int_t^\infty \int_y^\infty \bar{F}(x)dxdy - \mu^2(t) \\ &= \phi_2(t) - \mu^2(t) \qquad (2.4) \end{aligned}$$

or, using the definition of $\mu(y)$:

$$\begin{aligned} \sigma^2(t) &= [2/\bar{F}(t)]\int_t^\infty \bar{F}(y)\mu(y)dy - \mu^2(t) \qquad (2.5) \\ \frac{d\sigma^2(t)}{dt} &= [2f(t)/\bar{F}^2(t)]\int_t^\infty \bar{F}(y)\mu(y)dy \\ &\quad -[2/\bar{F}(t)]\bar{F}(t)\mu(t) - 2\mu(t)\ \mu'(t) \\ &= r(t)[\sigma^2(t) + \mu^2(t)] - 2\mu(t)[1 + \mu'(t)] \\ &= r(t)[\sigma^2(t) - \mu^2(t)] \\ &= r(t)\mu^2(t)[\gamma^2(t) - 1], \quad \text{where} \gamma^2(t) = \sigma^2(t)/\mu^2(t). \qquad (2.6) \end{aligned}$$

Hence F $\in$ DVRL (IVRL) iff

$$\gamma^2(t) \leq (\geq) 1. \tag{2.7}$$

(c) It can be easily shown that $\frac{d}{dt}[\phi_2(t)/\phi_1(t)] = \gamma^2(t) - 1$; so that

$$[\phi_2(t)/\phi_1(t)] \downarrow \Leftrightarrow F \in \text{ DVRL} \tag{2.8}$$

$$[\phi_2(t)/\phi_1(t)] \uparrow \Leftrightarrow F \in \text{ IVRL} \tag{2.9}$$

(d) To see the connection with the equilibrium (renewal) distribution, let $g(x) = \bar{F}(x)/\mu$, $x > 0$, be the density of the equilibrium (or renewal) distribution corresponding to F, then $\bar{G}(t) = [1/\mu]\int_t^\infty \bar{F}(x)dx$, where μ, σ^2, r without subscript are μ_F; σ_F^2, and r_F respectively.

Now,

$$\begin{aligned} \phi_2(t)/\phi_1(t) &= 2\int_t^\infty \int_y^\infty \bar{F}(x)dxdy / \int_t^\infty \bar{F}(y)dy \\ &= 2\int_t^\infty \mu\bar{G}(y)dy/\mu\bar{G}(t) = 2\mu_G(t). \end{aligned}$$

or, using (2.3),

$$\mu_G(t) = [\sigma_F^2(t) + \mu_F^2(t)]/2\mu_F(t) \tag{2.10}$$

From (2.8), (2.9) and (2.10) we have immediately:

$$\begin{aligned} F \in \text{ DVRL} &\Leftrightarrow G \in \text{ DMRL} \\ F \in \text{ IVRL} &\Leftrightarrow G \in \text{ IMRL}. \end{aligned} \tag{2.11}$$

(e) From (2.5) we have:

$$\begin{aligned} [2/\bar{F}(t)]\int_t^\infty \bar{F}(y)[\mu(y) - \mu(t)]dy &= \sigma^2(t) + \mu^2(t) - 2\mu^2(t) \\ &= \sigma^2(t) - \mu^2(t), \end{aligned} \tag{2.12}$$

thus, $\mu(y) - \mu(t) \leq 0 \; \forall \; y > t \Rightarrow \gamma^2(t) \leq 1 \Leftrightarrow F \in$ DVRL; so that

$$F \in \text{DMRL} \Rightarrow F \in \text{ DVRL}. \tag{2.13}$$

i.e. the class DMRL of a subset of the class DVRL. From (2.12) and (2.13) we have:

$$F \in \text{ NBUE } \Leftarrow F \in \text{DMRL } \rightarrow F \in \text{ DVRL } \Leftrightarrow G \in \text{ DMRL}$$

$$(2.14) \qquad \rightarrow G \in \text{ NBUE.}$$

(f) Solving (2.6) as a differential equation in $\sigma^2(t)$ we get:

$$\sigma^2(t) = \frac{1}{\bar{F}(t)}[\sigma^2(0) - \int_0^t r(x)\mu^2(x)\bar{F}(x)dx].$$

$$(2.15) \qquad \frac{\sigma^2(t)}{\sigma^2(0)} = 1 + \frac{F(t)}{\bar{F}(t)} - \frac{1}{\sigma^2(0)\bar{F}(t)}\int_0^t \mu^2(x)f(x)dx$$

If $\mu(x)$ is $\downarrow$ we can write (2.15), using the mean value theorem for integrals as

$$\frac{\sigma^2(t)}{\sigma^2(0)} = 1 + \frac{F(t)}{\bar{F}(t)} - \frac{\mu^2(0)}{\sigma^2(0)\bar{F}(t)}\int_0^{\theta t} f(x)dx$$

$$(2.16) \qquad = 1 + \frac{F(t)}{\bar{F}(t)} - \frac{F(\theta t)}{\gamma^2(0)\bar{F}(t)}, \quad 0 < \theta < 1$$

but $F \in$ DMRL $\Rightarrow F \in$ DVRL $\Rightarrow \frac{\sigma^2(t)}{\sigma^2(0)} < 1, \quad \forall\, t > 0$, this leads in turn to,

$$\frac{F(t)}{\bar{F}(t)} - \frac{F(\theta t)}{\gamma^2(0)\bar{F}(t)} < 0, \quad \forall\, t > 0.$$

or $\gamma^2(0) < \frac{F(\theta t)}{F(t)} \leq 1$. Thus we have the result

$$(2.17) \qquad F \in \text{ DMRL } \Rightarrow F \in \text{ DVRL } \Rightarrow \gamma^2(0) < 1.$$

(g) Further we have from (2.10),

$$(2.18) \qquad \mu_G(t)/\mu_F(t) = \frac{1}{2}[\gamma_F^2(t) + 1].$$

so that

$$(2.19) \qquad \mu_G(t) < \mu_F(t) \quad \text{iff} \quad \gamma_F^2(t) < 1$$

or

(2.20) $$\mu_G(t) < \mu_F(t) \Leftrightarrow F \in \text{ DVRL}.$$

From the definition of the equilibrium distribution $G(t)$ we can see easily that:

(2.21) $$F \in \text{ NBUE } \Leftrightarrow \bar{G}(t) < \bar{F}(t), \quad \forall\, t > 0.$$

But $\bar{G}(t) < \bar{F}(t) \Rightarrow \mu_G(0) < \mu_F(0)$. Using (2.18) again we have the result:

(2.22) $$F \in \text{ NBUE } \Rightarrow \gamma^2(0) < 1.$$

3. PRELIMINARIES

Let $x_1, x_2, \ldots, x_n$ be a random sample from

(3.1) $$f_1(x;\theta) = \theta e^{-\theta x}; \quad x > 0, \;\; \theta > 0.$$

Let $z_i = \frac{X_i}{T}, \quad i = 1, 2, \ldots, n$; where $T = \sum_{i=1}^n X_i$. It is well known (see [3]) that the joint distribution of any $r(< n)$ of these variables is:

$$f(z_1, \ldots, z_r) = \frac{(n-1)!}{(n-r-1)!}(1-z_1-z_2-\ldots-z_r)^{n-r-1}, \quad r = 1, 2, \ldots, n-1$$

This is a Dirichlet distribution $D(1, \ldots, 1, n-r)$, (see [10]) from which $E(z_i) = 1/n$; $E(z_i^2) = 2/n(n+1)$; $E(z_i^2 z_j^2) = 4/p(n)$; $E(z_i^4) = 24/p(n)$; where

(3.2) $$p(n) = n(n+1)(n+2)(n+3) \quad i, j = 1, \ldots, r.$$

We notice that $\sum_{i=1}^n z_i^2 - \frac{1}{n} = \frac{ns^2}{T^2}$ where $s^2 = \frac{1}{n}\sum_1^n (x_i - \bar{x})^2$ so that

(3.3) $$\hat{\gamma}^2 = \frac{s^2}{\bar{X}^2} = n\sum_1^n z_i^2 - 1$$

4. TEST FOR DVRL (IVRL) CLASS

We notice from (2.10) that

$$\mu_G(t) = \frac{\sigma_F^2(t) + \mu_F^2(t)}{2\mu_F(t)} \tag{4.1}$$

It can be shown that, (see [6])

$$F(t) = \frac{\mu}{\mu(t)} \exp[-\int_0^t \frac{du}{\mu(u)}] \tag{4.2}$$

Thus $\mu(t)$ determines uniquely, a continuous survival distribution with finite mean. From (4.1) it is clear that $\gamma^2(t) = 1 \Leftrightarrow \mu_G(t) = \mu_F(t)$ and this leads to $F(t) = G(t)$ leading to $F(t)$ is exponential. Thus $\gamma^2(t) = 1$ is a characteristic property of exponential distribution . Consider now the hypothesis:

$H_0 : F(x)$ is exponential; against $H_1 : F(x)$ is DVRL (IVRL). Since $\gamma^2(t) = 1$ characterizes exponentiality, while $\gamma^2(t) < 1$ (> 1) characterizes DVRL (IVRL) class of survival functions, it is reasonable to adopt as a measure of departure from H_0, the parameter

$$\int_0^\infty (\gamma^2(t) - 1))dF(t) = \int_0^\infty \gamma^2(t)dF(t) - 1.$$

In other words, small (large) values of

$$\Delta = \int_0^\infty \gamma^2(t)dF(t) \tag{4.3}$$

will indicate the rejection of H_0 leading to the claim that $F(t) \in$ DVRL (IVRL). In terms of Δ, the hypothesis can be formulated as:

$$H_0 : \Delta = 1 \quad \text{ag.} \quad H_1 : \Delta < 1 \ (> 1).$$

Consider now, $\hat{\gamma}_k^2 = s_K^2 / \bar{x}_k^2, \quad k = 0, 1, \ldots, n_1 \leq n - 2.$

Where $\bar{x}_k = \sum_{i=k+1}^n x_i/(n-k)$, and $s_k^2 = \sum_{i=k+1}^n (x_i - \bar{x}_k)^2/(n-k)$;

$$\bar{x}_0 = \bar{x} = \sum_{i=1}^n x_i/n; s_0^2 = s^2 = \sum_{i=1}^n (x_i - \bar{x}_n)^2/n. \tag{4.4}$$

Notice that $\hat{\gamma}_k^2$ is based on the last $(n-k)$ observations, of the sample, $k = 0, 1, \ldots, n_1$, (Remember that life time data is naturally ordered) . Consider the test statistic:

$$Q = \frac{1}{n_1} \sum_{k=0}^{n_1-1} \hat{\gamma}_k^2 \tag{4.5}$$

which is the sample analogue of Δ in (4.3).

Lemma 4.1. *Let $\bar{x}$ and s^2 be the mean amd variance of a random sample of size n from a distribution with mean μ and variance σ^2 then:*

$$\sqrt{n}\left[\frac{s^2}{\bar{x}^2} - \gamma^2\right] \xrightarrow{D} N[0, \gamma^2(\alpha_4 - 1)] \tag{4.6}$$

where γ^2 and α_4 are square coefficient of variation and the kurtosis μ_4/σ^4, respectively. In case of exponential

$$\sqrt{n}\left[\frac{s^2}{\bar{x}^2} - \gamma^2\right] \xrightarrow{D} N(0, 8) \tag{4.7}$$

Proof. It is well known that $\sqrt{n}(s^2 - \sigma^2) \xrightarrow{D} N(0, \mu_4 - \sigma^4)$; see [8]; from which

$$\sqrt{n}\left(\frac{s^2}{\bar{x}^2} - \frac{\sigma^2}{\mu^2}\right) \xrightarrow{D} N\left[0, \frac{\mu_4 - \sigma^4}{\mu^4}\right] \equiv N[0, \gamma^4(\alpha_4 - 1)],$$

using Slutsky's theorem. In exponential $\gamma^2 = 1$, $\alpha_4 = 9$, hence we have (4.8). □

Now, by lemma 4.1 we have, under H_0:

$$\sqrt{(n-k)}[\hat{\gamma}_k^2 - 1] \xrightarrow{D} N(0, 8) \tag{4.8}$$

Thus the test statistic $Q = \frac{1}{n_1} \sum_{k=0}^{n_1-1} \hat{\gamma}_k^2$ has a normal limiting distribution for every value of $n_1 \geq 1$. Since $\hat{\gamma}_k$'s are not independent for different values of k, it is not easy to compute the variance of Q. This difficulty will be overcomed by considering another statistic which is asymptotically equivalent, and whose variances and covariances can be computed more easily.

Lemma 4.2. $\hat{\gamma}_k^2+1$ *has the same limiting distribution as* $\frac{n^2}{n-k}\sum_{i=k+1}^{n} z_i^2$

Proof,

$$\hat{\gamma}_k^2+1 = (n-k)\frac{x_{k+1}^2+\cdots+x_n^2}{T_k^2}; \quad \text{where} \quad T_k=\sum_{i=k+1}^{n} x_i$$

$$(4.9) \qquad = (n-k)\frac{T^2}{T_k^2}\frac{x_{k+1}^2+\cdots+x_n^2}{T^2} \equiv \frac{(n-k)}{U_k^2}\sum_{i=k+1}^{n} z_i^2,$$

$$\text{where} U_k = T_k/T.$$

Thus $\frac{n^2}{n-k}\sum_{i=k+1}^{n} z_i^2 = \frac{n^2}{(n-k)^2}U_k^2(\hat{\gamma}_k^2+1)$. It is clear that
$\frac{n^2}{(n-k)^2}U_k^2 = \frac{n^2}{(n-k)^2}\cdot\frac{T_k^2}{T^2} = \frac{\bar{x}_k^2}{\bar{x}^2} \xrightarrow{P} 1.$

Since both $\bar{x}_k^2$ & $\bar{x}^2$ converge in probability to the same μ^2. The required result follows by Slutsky's theorem. □

For n large enough, one can write,

$$(4.10) \qquad \operatorname{cov}(\hat{\gamma}_k^2,\hat{\gamma}_j^2) \simeq \operatorname{cov}\left(\frac{n^2}{n-k}y_k, \frac{n^2}{n-j}y_j\right)$$

where $y_k=\sum_{i=k+1}^{n} z_i^2$.

The product $y_j y_k$, $j>k$, contains $(n-j)(n-k-1)$ pairs like $z_r^2 z_t^2 (r\neq t)$, and $(n-j)$ pairs like z_t^4. From (3.2), we have therefore,

$$(4.11) \qquad \operatorname{cov}(Y_k,Y_j) = \frac{4(n-j)(n^2+4nk-n+6k)}{n(n+1)p(n)}.$$

Asymptotically, one has,

$$\operatorname{cov}(\hat{\gamma}_k^2,\hat{\gamma}_j^2) = \frac{n^4}{(n-j)(n-k)}\operatorname{cov}(y_k,y_j) =$$

$$\frac{4n^4(n-j)(n^2+4nk-n+6k)}{(n-j)(n-k)n(n+1)p(n)} = \frac{4n^2(n^2+4nk-n+6k)}{(n-k)(n+1)^2(n+2)(n+3)} \tag{4.12}$$

which is of the order $\frac{4}{n-k}$ as $n \to \infty$, and k fixed.

$$\begin{aligned} V(Q) &= V\left[\frac{1}{n_1}\sum_{k=0}^{n_1-1}\hat{\gamma}_k^2\right] \\ &= \frac{1}{n_1^2}\left\{\sum_{k=0}^{n_1-1}V(\hat{\gamma}_k^2)+2\sum_{k=0}^{n_1-1}\sum_{j=k+1}^{n_1-1}\operatorname{cov}(\hat{\gamma}_k^2,\hat{\gamma}_j^2)\right\} \\ &= \frac{1}{n_1^2}\left\{\sum_{k=0}^{n_1-1}\frac{8}{n-k}+2\sum_{k=0}^{n_1-1}\sum_{j=k+1}^{n_1-1}\frac{4}{n-k}\right\} \\ &= \frac{8}{n_1^2}\sum_{k=0}^{n_1-1}\frac{n_1-k}{n-k}. \end{aligned} \tag{4.13}$$

For $n-n_1$ large enough, the statistic $Q = \frac{1}{n_1}\sum_{k=0}^{n_1-1}\hat{\gamma}_k^2$ could be used to test H_0 vs H_1, using the standardized normal table. Small (large) values of $D = (Q-1)/\sigma_Q$ indicate that $F(t)$ belongs to DVRL (IVRL). From (2.11) we can say that this test is at the same time a test of $H_0' : G(t)$ is exponential vs $H_1' : G(t)$ belongs to DMRL (IMRL), where $G(t)$ is the renewal distribution corresponding to $F(t)$.

Notice that Q is the average of n_1 asymptotically normal variables, so that it approaches normality for every $n_1 \geq 1$. For this test $n - n_1$ should exceed 30, and n_1 should be fairly large.

Example: Bryson & Siddiqui (1969) have analysed data which are survival times, in days from diagnosis of patients suffering from chronic granalocytic leukemia. The order statistics $X_1 < \ldots < X_{43}$ $(n = 43)$ are: 7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455, 468, 495, 497, 532, 571, 579, 581, 650, 702, 715, 779, 881, 900, 930, 968, 1077, 1109, 1314, 1334, 1367, 1534, 1712, 1784, 1877, 1886, 2045, 2056, 2260, 2429, 2509.

We shall test here whether this data is exponential against, the data belongs to DVRL (IVRL) class.

Using $n_1 = 13$, so that we leave $n - n_1 = 30$, (i.e. $n - k \geq 30$), allowing the use of asymptotic results stated above.

$$\begin{aligned} Q &= \frac{1}{n_1}\sum_{k=0}^{n_1-1}\hat{\gamma}_k^2 = 0.400255727; \quad \sigma_Q = 0.333506121 \\ D &= -1.798, \quad p\text{-value} = 0.036. \end{aligned}$$

A significant result indicating that $F(t) \in$ DVRL class and that the corresponding renewal distribution belongs to DMRL class.

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CHARACTERIZATION OF FUZZY T_0 AND R_0 TOPOLOGICAL SPACES

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ABSTRACT. It is the purpose of this note to suggest new definitions of fuzzy T_0 and fuzzy R_0-spaces using the Wong defintion of fuzzy points. It will also be shown that these new definitions are equivalent to those introduced by Srivastava. Moreover, the properties of T_0-ness and R_0-ness are shown to be both productive and hereditary and that a topologically generated fuzzy topological space is T_0 or R_0 if the original topological space is T_0 or R_0, respectively.

1. INTRODUCTION

The fundamental concept of a fuzzy set was introduced by Zadeh in 1965 [9]. Since then, intensive studies of fuzzy sets have been developed. In particular, the definition of a fuzzy point was first given in 1974 by Wong [8]. It is notable that, with this definition, an ordinary point of a set is not a special case of a fuzzy point. In 1980, Pu and Liu [5] remedied this drawback by redefining fuzzy point in a way that can be used to develop the theory of fuzzy topology in a satisfactory way. In 1984, R. Srivastava, S.N. Lal, and A.K. Srivastava [5] studied the concept of fuzzy T_1-topological space using the Wong fuzzy point [8]. Latter in 1988, they introduced an equivalent definition (depending upon the ordinary points of a set) of a fuzzy T_1-space [7]. On the other hand, a fuzzy T_0-topological space has been defined and studied by Hutton and Reilly [2], Pu and Liu [4], R. Srivastava, S.N. Lal, and A.K. Srivastava [6]. Hutton [2] and Srivastava [6] studied, in addition, the concept of a fuzzy R_0 topological spaces in order to establish a satisfactory relationship between fuzzy T_1, T_0, and R_0-spaces. It can be seen that in papers [2,3,4] the authors investigated fuzzy T_0 and fuzzy R_0 -spaces depending upon the ordinary points of a set and not the fuzzy points. In 1990, Ali, Wuyts, and

Srivastava introduced and intensively studied fuzzy R_0-spaces [1]. It is the purpose of this note to suggest new definitions of fuzzy T_0 and fuzzy R_0-spaces using the Wong definition of fuzzy points [8]. These new definitions can be extended in a straightforward manner to the case of fuzzy T_1-spaces studied by Srivastava in [5]. It will be also shown that the new definition of fuzzy T_0-spaces is equivalent to that introduced by Srivastava [6]. The new definitions of fuzzy R_0-spaces are shown to be equivalent to that introduced by Ali in [1], but they are more general than that introduced by Srivastava in [6]. Moreover, the properties of T_0-ness and R_0-ness are shown to be both productive and hereditary and that a topologically generated fuzzy topological space is T_0 or R_0 iff the original topological space is T_0 or R_0, respectively.

2. BASIC DEFINITIONS AND PROPERTIES

A function U from a non empty set X to the unit interval [0,1] is called a fuzzy set in X. The fuzzy set that takes the value 0 at all points $x \in X$ is denoted by ϕ and that which takes the value 1 at all points of X is denoted by X itself. A fuzzy topology τ on X (in Lowen's sense [3]) is that contain, in addition to the above properties, all constant fuzzy sets. The term "fuzzy topological space" will be abbreviated as fts. A fuzzy point p in X is a fuzzy set in X such that $p(x_p) = t$ for $x = x_p$, and $p(x) = 0$, otherwise. The point x_p is called the support of p and t its value, $t \in (0,1)$. A fuzzy point p is said to belong to a fuzzy set U in $X(p \in U)$ iff $p(x_p) < U(x_p)$. Two fuzzy points are said to be distinct iff they have different supports. For a fuzzy point p in X and a fuzzy set U in X, we say that $p \cap U = 0$ iff $U(x_p) = 0$. We denote the characteristic function of a singleton set $\{x\}$ by 1_x and its closure by $\bar{1}_x$.

3. FUZZY T_0-TOPOLOGICAL SPACES

Definition 3.1. (Pu and Liu [4]). A fts (X, τ) is said to be fuzzy T_0 iff for any $s, t \in [0,1)$ and $x, y \in X$, $x \neq y$, $\exists U \in \tau$ such that $U(x) = s$ and $U(y) > t$, or $U(x) > s$ and $U(y) = t$.

Definition 3.2. (Hutton and Reilly [2]). A fts (X,τ) is said to be fuzzy T_0 iff each fuzzy set in X can be written as $\sup_i \inf_j U_{ij}$, where $i \in I$, $j \in J$, and each U_{ij} is fuzzy open or fuzzy closed in (X,τ).

Definition 3.3. (R. Srivastava, S.N. Lal, and A.K. Srivastava [9]). A fts (X,τ) is said to be fuzzy T_0 iff for all $x, y \in X$, $x \neq y$, $\exists U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$, or $U(x) = 0$ and $U(y) = 1$.

Now we introduce our new definition of a fuzzy T_0 topological space

Definition 3.4. A fts (X,τ) is said to be fuzzy T_0 iff for any two distinct fuzzy points p and q in X, $\exists U \in \tau$ such that either $p \in U$ and $q \cap U = 0$ or $q \in U$ and $p \cap U = 0$.

We now compare the above four definitions of fuzzy T_0-ness in the following theorem:

Theorem 3.1. *Consider the following statements for the fts (X,τ):*

(I) For any distinct fuzzy points p, q in X, $\exists U \in \tau$ such that $p \in U$ and $q \cap U = 0$, or $q \in U$ and $p \cap U = 0$.

(II) $\forall\, x, y \in X$, $x \neq y$, $\exists U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$, or $U(y) = 1$ and $U(x) = 0$.

(III) Each fuzzy set in X can be written in the form $\sup_i \inf_j U_{ij}$ where each U_{ij}, $i \in I, j \in J$, is a fuzzy open or a fuzzy closed set.

(IV) For any two distinct points $x, y \in X$ and for all $s, t \in [0,1)$, there exists $U \in \tau$ such that either $U(x) = s$ and $U(y) > t$, or $U(x) > s$ and $U(y) = t$.

We have the following implications:

(I) $\Leftrightarrow$ (II)

(I) $\Rightarrow$ (III)

(III) $\not\Rightarrow$ (I)

(I) $\Rightarrow$ *(IV)*

(IV) $\not\Rightarrow$ *(I)*

Proof. It suffices to prove that (I) $\Leftrightarrow$ (II). The remaining implications follow directly using [6, Theorem 2.1].

(I) $\Rightarrow$ (II). Let $x, y \in X$, $x \neq y$ and let p_n, q_n be fuzzy points in X with supports x, y, respectively, and such that $p_n(x) = q_n(y) = 1 - \frac{1}{2n}$, $n \in N$. Since $x \neq y$ then $p_n \neq q_n$ for every $n \in N$ and by (I) $\exists U_n \in \tau$ such that either $p_n \in U_n$ and $q_n \cap U_n = 0$, or $q_n \in U_n$ and $p_n \cap U_n = 0$. Assume that there is an infinite subset J of N such that $p_n \in U_n$ and $q_n \cap U_n = 0$ for all $n \in J$ (the other case can be treated similarly) then $U_n(x) > 1 - \frac{1}{2n}$, $U_n(y) = 0$ for every $n \in J$. Define $U = \bigcup_{n \in J} U_n$ then, $U \in \tau$ and $U(x) = 1$, $U(y) = \bigcup_{n \in J} U_n(y) = 0$. So we have (II).

(II) $\Rightarrow$ (I). Suppose that p, q are two distinct fuzzy points in X with supports x, y, and values $r, s \in (0,1)$, respectively, then $x \neq y$ and by (II) $\exists U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$, or $U(x) = 0$ and $U(y) = 1$. Assume that $U(x) = 1$ and $U(y) = 0$ (the other case can be treated similarly). Since $p(x) = r < 1$, and $q(y) = s > 0$, it follows that $p \in U$ and $q \cap U = 0$. So we have (I).

Remark 3.1. Definition 3.4 can be replaced by an equivalent definition where we replace the fuzzy open set U by a fuzzy closed set V. In this case all the implications of theorem 3.1 remain valid.

The following theorem shows that the property of T_0-ness of a fuzzy topological space is productive.

Theorem 3.2. *Let $\{(X_i, \tau_i) : i \in I\}$ be a family of fuzzy topological spaces, then the product space $(X, \tau) = \Pi_i(X_i, \tau_i)$ is fuzzy T_0 iff each coordinate fts is fuzzy T_0 (in the sense of Definition 3.4).*

Proof. Let (X_j, τ_j) be fuzzy T_0, for $j \in I$ and let p, q be two distinct fuzzy points in X, $p = < p_j >$, $q = < q_j >$. Then $p_i \neq q_i$ for at least one $i \in I$. Then $\exists\ U_i \in \tau_i$ such that $p_i \in U_i$ and $q_i \cap U_i = 0$ or $q_i \in U_i$ and $p_i \cap U_i = 0$. Suppose that $p_i \in U_i$ and $q_i \cap U_i = 0$ (the other case can be treated similarly). Let $U = \Pi_j U'_j$, where $U'_j = X_j$, for $j \neq i$, $U'_j = U_j$ for $j = i$. It is clear that $U \in \tau$ and $p \in U$, $q \cap U = 0$. Hence (X, τ) is fuzzy T_0. Conversely, let (X, τ) be fuzzy T_0 and consider any (X_i, τ_i), $i \in I$. Let p_i, q_i be two distinct fuzzy points in X_i and construct the two distinct fuzzy points $p = < p'_j >$, $q = < q'_j >$ in X where $p'_j = q'_j$ for $j \neq i$ and $p'_i = p_i$, $q'_i = q_i$. Then $\exists\, U \in \tau$ such that either $p \in U$ and $q \cap U = 0$, or $q \in U$ and $p \cap U = 0$. Suppose that $p \in U$ and $q \cap U = 0$ (the other case can be treated similarly). Then we can find a basic fuzzy open set $\Pi_j U_j$ such that $p \in \Pi_j U_j \subset U$. It follows that $p_i \in U_i$, and since $q \cap U = 0$ then $q \cap \Pi U_j = 0$ and hence $\Pi_j q_j \cap \Pi_j U_j = 0$. Since $q_j = p_j$ for $j \neq i$ and $p_j \in U_j$ then $q_j \cap U_j \neq 0$, for $j \neq i$. Hence, we must have that $q_i \cap U_i = 0$. This proves that (X_i, τ_i) is fuzzy T_0.

Using the definitions of a fuzzy subspace introduced by Pu and Liu [4, Definition 8.1] and the topologically generated fuzzy topological space (introduced by Lowen [3]) together with Definition 3.4 we can easily prove the following theorems.

Theorem 3.3. *Every fuzzy subspace of a fuzzy T_0-space is also a fuzzy T_0-space.*

Theorem 3.4. *Let (X, T) be a topological space. Then (X, T) is T_0 iff $(X, w(T))$ is fuzzy T_0, where $w(T)$ is the topologically generated fuzzy topology generated by the topology T [5, Definition 2.8].*

4. FUZZY R_0-TOPOLOGICAL SPACES

Fuzzy R_0-spaces have been defined by Hutton and Reilly [2], R. Srivastava, S.N. Lal, A.K. Srivastava [6], and D.M. Ali, P. Wuyts, and A.K. Srivastava [1] as follows:

Definition 4.1. (Hutton and Reilly [2]). An fts (X, τ) is said to be fuzzy R_0 iff each fuzzy open set can be written as a supremum of fuzzy closed sets.

Definition 4.2. (R. Srivastava, S.N. Lal, A.K. Srivastava [6]). A fts (X, τ) is fuzzy R_0 iff $\forall x, y\ X$, $x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

Definition 4.3. (D.M. Ali, P. Wuyts, and A.K. Srivastava [1]). A fts (X, τ) is said to be fuzzy R_0^7 iff $\forall (x, y) \in X^{(2)}$ it follows that $\bar{1}_x(y) = \bar{1}_y(x) \in \{0, 1\}$.

It has been shown in [6] that Definitions 4.1, 4.2 are totally independent and that the latter definition is a good extension of the concept of an R_0 topological space. In 1990 Ali, Wuyts, and A.K. Srivastava [1] introduced and studied carefully many fuzzy R_0 topological spaces. We propose here more general definitions of fuzzy R_0 topological space and show that these new definitions are not implied by that introduced by Srivastava in [6], but are equivalent to R_0^7 introduced in [1].

Definition 4.4. A fts (X, τ) is said to be fuzzy R_0^a iff $\forall x, y \in X, x \neq y$ if $\bar{1}_y(x) < 1$ then $\bar{1}_x(y) = 0$.

Definition 4.5. A fts (X, τ) is fuzzy R_0 iff $\forall x, yX, x \neq y$, whenever there is a $U \in \tau$ such that $U(x) \neq 0$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

Definition 4.6. A fts (X, τ) is said to be fuzzy R_0 iff for every two distinct fuzzy points p, q in X, whenever there is a $U \in \tau$ such that $p \in U$ and $q \cap U = 0$ there is also $V \in \tau$ such that $q \in V$ and $p \cap V = 0$.

Theorem 4.1. *For a fts (X, τ) consider the following statements:*

(1) $\forall x, y \in X, x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

(2) $\forall (x, y) \in X^{(2)}$ it follows that $\bar{1}_x(y) = \bar{1}_y(x) \in \{0, 1\}$.

(3) $\forall x, y \in X, x \neq y$ if $\bar{1}_y(x) < 1$ then $\bar{1}_x(y) = 0$.

(4) $\forall x, y \in X, x \neq y$, whenever there is a $U \in \tau$ such that $U(x) \neq 0$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

(5) for every two distinct points p, q in X, whenever there is a $U \in \tau$

such that $p \in U$ *and* $q \cap U = 0$ *there is also* $V \in \tau$ *such that* $q \in V$ *and* $p \cap V = 0$. *Then the following implications hold:*

(i) (2) $\Leftrightarrow$ *(3)* $\Leftrightarrow$ *(5)*

(ii) (5) $\Rightarrow$ *(1) and (1)* $\not\Rightarrow$ *(5)*

(iii) (1) $\Leftrightarrow$ *(4).*

Proof. (i) (2) $\Rightarrow$ (3). Let $x, y \in X, x \neq y$ are such that $\bar{1}_y(x) < 1$, then by (2) it follows that $\bar{1}_y(x) = 0$, this implies that $\bar{1}_x(y) = 0$. Hence, (X, τ) satisfies (3).

(3) $\Rightarrow$ (2). Suppose that $x, y \in X, x \neq y$ are such that $\bar{1}_x(y) = 0$. Since $\bar{1}_x(y) = 0 < 1$ then, again by (3) we must have that $\bar{1}_y(x) = 0$. Therefore, $\bar{1}_y(x) = \bar{1}_x(y) = 0$. Now, suppose that $\bar{1}_y(x) = 1$. If $\bar{1}_x(y) < 1$ then by (3) it follows that $\bar{1}_y(x) = 0$ which is not true. So $\bar{1}_x(y)$ must be 1. Hence, $\bar{1}_y(x) = \bar{1}_x(y) = 1$.

This proves that (X, τ) satisfies (2).

(3) $\Rightarrow$ (5). Let (X, τ) be fuzzy R_0. Suppose that $x, y \in X, x \neq y$ are such that $\bar{1}_y(x) = \gamma < 1$. Choose the real number t such that $t + \gamma < 1$. Let p be a fuzzy point in X supported at x and with value t. $\forall \alpha < 1$, let q_α be a fuzzy point supported at y and with value α. Let $U = \text{co}\ (\bar{1}_y)$, then $U(y) = 1 - \bar{1}_y(y) = 0$, $U(x) = 1 - \bar{1}_y(x) = 1 - \gamma > t$. Therefore, $p \in U$, $q_\alpha \cap U = 0$, hence $\exists V_\alpha \in \tau$ such that $q_\alpha \in V_\alpha$, $p \cap V_\alpha = 0$. Take $V = \bigcup_\alpha V_\alpha$. It follows that $V(y) = 1$, $V(x) = 0$, Hence, $1_x \subseteq \text{co}\ (V)$, $\bar{1}_x \subseteq \overline{\text{co}(V)} = \text{co}(V)$. Therefore, $\bar{1}_x(y) \leq \text{co}(V)(y) = 0$, this implies that $\bar{1}_x(y) = 0$.

(5) $\Rightarrow$ (3). Let p, q be two distinct fuzzy points in X with supports $x, y \in X$ and values $r, s \in (0, 1)$, respectively. Let U be such that $p \in U$ and $q \cap U = 0$. Therefore, $\text{co}(U(y)) = 1$, $\bar{1}_y \subseteq \overline{\text{co}(U)} = \text{co}(U)$. Hence, $\bar{1}_y(x) \leq j\text{co}(U(x)) = 1 - U(x) < 1 - \gamma < 1$. So, by R_0^a we get $\bar{1}_x(y) = 0$. Take $V = \text{co}(\bar{1}_x)$. Hence, $V(x) = 1 - \bar{1}_x(x) = 0$, $V(y) = 1 - \bar{1}_x(y) = 1$. This implies that $q \in V$ and $p \cap V = 0$.

(ii) (5) $\Rightarrow$(1). Let $x, y \in X, x \neq y$, and suppose that there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$. Let p_n and q_n be fuzzy points in X with supports x and y, respectively, and such that $p_n(x) = q_n(y) = 1 - \frac{1}{2n}$, $n \in N$. It is clear that $p_n \in U$ and $q_n \cap U = 0$ for all $n \in N$. Hence, by (1) $\exists V_n \in \tau$ such that $q_n \in V_n$ and $p_n \cap V_n = 0$, for all $n \in N$. Let $V = \bigcup_n V_n$. Then, $V(y) = 1$ and $V(x) = \bigcup_n V_n(x) = 0$. so we have (1).

(1) $\not\Rightarrow$ (5). Consider the following counterexample: Let $X = \{x, y\}$ be a set of two points x and y, and let $\tau = \{U : U$ is a fuzzy set such that $U(x) = U(y)$, or $\frac{1}{2} \geq U(x) > U(y)\}$. Clearly, (X, τ) is a fuzzy topology on X, in both Chang's sense and Lowen's sense. The fts (X, τ) satisfies (1) because, the premise $U(x) = 1$ and $U(y) = 0$ and the premise $U(y) = 1$ and $U(x) = 0$, are both impossible. On the other hand (X, τ) does not satisfy (5). Take the two distinct fuzzy points p and q in X such that $p(x) = \frac{1}{4}$, $p(y) = 0$ and $q(y) = \frac{1}{4}, q(x) = 0$. Then, there exists $U \in \tau$ such that $U(x) = \frac{3}{8} > p(x), U(y) = 0$. Clearly, $p \in U$ and $q \cap U = 0$. But, for all $V \in \tau$, if $q \in V$, then $V(x) \geq V(y) > \frac{1}{4}$, and so $p \cap V = 0$.

(iii) (1) $\Leftrightarrow$ (4). This can be shown easily by applying the same technique used in (ii).

If the fuzzy open sets U and V in Definition 4.6 are replaced by fuzzy closed sets U' and V', respectively, then the statement of Theorem 4.1 remains valid.

Theorem 4.2. *Let $\{(X_i, \tau_i), i \in I\}$ be a family of fuzzy topological spaces. Then their product space $(X, \tau) = \Pi_i(X_i, \tau_i)$ is fuzzy R_0 iff each coordinate fts is fuzzy R_0*

Proof. Let (X_j, τ_j) be fuzzy R_0 for all $j \in I$ and let p, q be any two distinct fuzzy points in $X, p =< p_j >, q =< q_j >$ (see Srivastava [7]) and $U \in \tau$ such that $p \in U$, $q \cap U = 0$. Then p_i and q_i are distinct for at least one $i \in I$, and there exists a basic fuzzy open set $\Pi_j U_j$ such that $U = \Pi_j U_j$. Therefore, $p_j \in U_j$, $j \in I$. Also, $q_i \cap U_i = 0$, for at least one $i \in I$. Then $\exists V_i \in \tau_i$ such that $q_i \in V_i$ and $p_i \cap V_i = 0$. Construct $V = \Pi_j V_j'$, where $V_j' = X_j$, for $j \neq i$, and $V_i' = V_i$. Clearly, $q \in V$, and

$p \cap V = 0$. Hence, (X, τ) is fuzzy R_0.

Conversely, let (X, τ) be fuzzy R_0. Suppose that p_i and q_i are distinct fuzzy points of X_i, $U_i \in \tau_i$, $p_i \in U_i$ and $q_i \cap U_i = 0$. Construct $p =< p'_j >, q =< q'_j >, p'_j = q'_j, j \neq i$, $p'_i = p_i, q'_i = q_i$, and $U = \Pi_j U'_j, U'_j = X_j$, for $j \neq i$, and $U'_i = U_i$. Then p and q are distinct fuzzy points of X, and $p \in U$, $q \cap U = 0$. Then $\exists V = \Pi_j V_j \in \tau$ such that $q \in V$, $p \cap V = 0$. Then by the construction of p and q we must have that $q_j \in V_j$, $j \in I$, and $p_i \cap V_i = 0$. This proves that (X_i, τ_i) is fuzzy R_0.

Using Definition 4.6, Theorem 3.1 of [6] and Proposition 3.2 of [7], we can easily prove the following theorems:

Theorem 4.3. *A fuzzy subspace of a fuzzy R_0 space is also fuzzy R_0.*

Theorem 4.3. *A topological space (X, T) is R_0 iff the fts $(X, w(T))$ is fuzzy R_0.*

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ON NEW RENEWAL BETTER THAN USED CLASSES OF AGEING

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ABSTRACT. Suppose that a device is subject to shocks occurring randomly in time according to the counting process $N = \{N(t), t \geq 0\}$. Let $\bar{P}_k$, be the probability that the device survives the first k shocks, $k = 0, 1, 2, \ldots$, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \ldots$. The probability that the device survives beyond t, $\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\}\bar{P}_k$, is proved to have the new renewal better (worse) than used NRBU (NRWU) property under some conditions on $\{\bar{P}_k\}_{k=0}^{\infty}$ when N is homogeneous and nonhomogeneous Poisson processes. Laplace transform and generating function characterizations for these NRBU (NRWU) properties are given.

1. INTRODUCTION

Suppose that a device is subject to shocks occuring randomly in time according to the counting process $N = \{N(t), t \geq 0\}$. Let the device have the probability $\bar{P}_k$ of surviving k shocks $k = 0, 1, 2, \ldots$, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \ldots$ The probability $\bar{H}(t)$ that the device survives beyond t is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\}\bar{P}_k. \tag{1.1}$$

Such shock models have been studied by Esary et al. (1973) when N is a homogeneous Poisson process and by A-Hameed and Proschan (1973, 1975) when N is a nonhomogeneous Poisson process. In all these cases the authors prove that $\bar{H}(t)$ is IFR, IFRA, DMRL, NBU, or NBUE under suitable conditions on N if $\{\bar{P}_k\}_{k=0}^{\infty}$ has the corresponding discrete property. Klefsjo (1981) has considered (1.1) for the HNBUE class. Abouammoh et al. (1988) have studied some shock models for NBUFR and NBAFR classes. Abouammoh and Hendi (1991) considered shock models for the new better than used renewal failure rate (NBURFR) and the new better than average renewal failure rate (NBARFR). Recently Abouammoh and Ahmed (1993) studied NRBU (NRWU) closure properties under some reliability operations such as convolutions, mixtures and coherent systems. They examine also, the relationships between these classes and other classes.

The main theme of this paper is to establish different results of shock models for the class of new renewal better than used (NRBU) distributions and its dual class of new renewal worse than used (NRWU) distributions. In section 3 the survival function $\bar{H}(t)$ is studied when N is a homogeneous and nonhomogeneous Poisson processes. The Laplace transforms for these classes and characterization of generating functions of the renewal failure rate classes are established in section 4.

2. THE NEW RENEWAL BETTER THAN USED PROPERTIES

Let H be a life distribution, that is $H(0-) = 0$ and $\bar{H}(t) = 1 - H(t)$, $\forall t \geq 0$ be its corresponding survival function and $\mu_H = \int_0^\infty \bar{H}(u)du$. The new renewal better than used (NRBU) and the new renewal worse than used (NRWU) properties are given below.

Definition 2.1 : A life distribution H with $H(0-) = 0$ and finite mean $\mu_H = \int_0^\infty \bar{H}(u)du$ is said to have the NRBU(NRWU) property if

$$(2.1) \qquad \mu_H \bar{H}(t+x) \leq (\geq) \bar{H}(t) \int_x^\infty \bar{H}(u)du, \qquad \forall\, t, x \geq 0$$

Classes of life distributions with these properties are introduced by Abouammoh and Ahmed (1993). Their behaviour under some reliability operations such as convolution, mixing and formation of coherent systemes has been also studied. Moreover, they investigated the relationships between these classes with other classes.

Now consider a device with life distribution H which is replaced instantaneously upon failure by a sequence of mutually independent devices. These devices are independent of the first device and each has the same life distribution H. In the long run the residual life of a device under operation is given by

$$(2.2) \qquad W_H(t) = \mu_H^{-1} \int_0^t \bar{H}(u)du, \qquad t \geq 0$$

the stationary renewal distribution of H. Using (2.2) relation (2.1) can be expressed as

$$(2.3) \qquad \bar{H}(x|t) \leq \bar{W}_H(x); \qquad t, x \geq 0$$

where $\bar{W}_H(x) = 1 - W_H(x)$. The last inequality (2.3) account for the name NRBU.

Now we give the following definition of the discrete NRBU and NRWU properties.

Definition 2.2 : A life distribution or its survival $\bar{P}_k = 1 - P_k$, $k = 0, 1, 2, \ldots$ is called NRBU (NRWU) if

$$(2.4) \qquad \mu \bar{P}_{l+k} \leq (\geq) \bar{P}_l \sum_{j=k}^{\infty} \bar{P}_j \; ; \qquad k \geq 0, 1, \ldots$$

where $\mu = \sum_{k=0}^{\infty} \bar{P}_k$.

3. A POISSON SHOCK MODEL LEADING TO NRBU

In this section we consider the shock model given by (1.1) such that the arriving shocks to the device occur according to a counting process N which is a homogeneous and nonhomogeneous Poisson processes.

Assume first that a device is subjected to shocks occuring randomly in time according to a homogeneous Poisson process with constant intensity λ. Thus the shock model (1.1) is reduced to the form

$$(3.1) \qquad \bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \bar{P}_k.$$

Now we prove that the discrete NRBU property of $\bar{P}_k$, $k = 0, 1, 2, \ldots$ is preserved for $\bar{H}(t)$ under the model (3.1).

Theorem 3.1. *The survival $\bar{H}(t)$ in model (3.1) is NRBU if $\{\bar{P}_k\}_{k=0}^{\infty}$ is discrete NRBU.*

Proof. By (3.1)

$$\begin{aligned} \mu_H &= \int_0^\infty \bar{H}(u)du = \int_0^\infty \sum_{k=0}^{\infty} e^{-\lambda u}\frac{(\lambda u)^k}{k!}\bar{P}_k du \\ &= \frac{\mu}{\lambda}. \end{aligned} \tag{3.2}$$

Note that $\bar{H} \in$ NRBU if

$$\mu_H \bar{H}(t+x) \leq \bar{H}(t)\int_x^\infty \bar{H}(u)du \tag{3.3}$$

L. H. S. of (3.3) is

$$\begin{aligned} \mu_H \bar{H}(t+x) &= \frac{\mu}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda(t+x)}\frac{[\lambda(t+x)]^k}{k!}\bar{P}_k \\ &= \frac{\mu}{\lambda}e^{-\lambda(t+x)}\sum_{k=0}^{\infty}\bar{P}_k\sum_{j=0}^{k}\frac{\binom{k}{j}(\lambda t)^j(\lambda x)^{k-j}}{k!} \\ &= \frac{\mu}{\lambda}e^{-\lambda(t+x)}\sum_{k=0}^{\infty}\bar{P}_k\sum_{j=0}^{k}\frac{(\lambda t)^j}{j!}\frac{(\lambda x)^{k-j}}{(k-j)!} \\ &= \frac{\mu}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\sum_{k=j}^{\infty}\frac{(\lambda x)^{k-j}}{(k-j)!}\bar{P}_k, \\ &= \frac{\mu}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\sum_{l=0}^{\infty}\frac{(\lambda x)^l}{l!}\bar{P}_{j+l}. \end{aligned}$$

Since $\bar{P}_k$ is NRBU, i.e. $\mu\bar{P}_{j+l} \leq \bar{P}_j\sum_{i=l}^{\infty}\bar{P}_i$, then

$$\mu_H \bar{H}(t+x) \leq \frac{1}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\sum_{l=0}^{\infty}\frac{(\lambda x)^l}{l!}\bar{P}_j\sum_{i=l}^{\infty}\bar{P}_i$$

$$
\begin{aligned}
&= \frac{1}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\sum_{l=0}^{\infty}\frac{(\lambda x)^l}{l!}\sum_{i=l}^{\infty}\bar{P}_i \\
&= \frac{1}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\sum_{i=0}^{\infty}\bar{P}_i\sum_{l=0}^{i}\frac{(\lambda x)^l}{l!} \\
&= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\sum_{i=0}^{\infty}\bar{P}_i(\frac{1}{\lambda}\sum_{l=0}^{i}\frac{(\lambda x)^l}{l!}e^{-\lambda x}) \\
&= \sum_{j=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^j}{j!}\bar{P}_j\int_x^{\infty}\sum_{i=0}^{\infty}\frac{(\lambda u)^i}{i!}e^{-\lambda u}\bar{P}_i du.\square
\end{aligned}
$$

By reversing all inequalities we get a dual theorem in the NRWU case.

Next, let the shocks in (1.1) occur according to non-homogeneous Poisson process, with mean value function $\Lambda(t)$ and event rate $\Lambda'(t)$. Let $\bar{P}_k, k = 0, 1, \ldots,$ be the probability that the device survives the first k shocks, in this case model (1.1) is reduced to

$$
\bar{H}(t) = \sum_{k=0}^{\infty} e^{-\Lambda(t)}\frac{[\Lambda(t)]^k}{k!}\bar{P}_k, \quad t \geq 0. \tag{3.4}
$$

Now we investigate the preservation of discrete NRBU (NRWU) property of $\bar{P}_k, k = 0, 1, 2, \ldots$ for $\bar{H}(t)$ under the model (3.4).

Theorem 3.2. *$\bar{P}_k, k = 0, 1, 2, \ldots$ are preserved for $\bar{H}(t)$ under model (3.4) if $\{\bar{P}_k\}_{k=0}^{\infty}$ is discrete NRBU, $\Lambda'(0) \neq 0$ and $\Lambda'(\infty) = \infty$ if*

$$
\Lambda'(0)\bar{H}_k(t)\{\sum_{k=0}^{\infty}\bar{H}_k(x)\} \leq \Lambda'(x)\bar{H}(t+x)(\sum_{j=k}^{\infty}\bar{P}_j) \tag{3.5}
$$

Proof. The survival function $\bar{P}_k$ has the discrete NRBU properties if $\mu\bar{P}_{l+k} \leq \bar{P}_l\sum_{j=k}^{\infty}\bar{P}_j, l, k = 0, 1, 2, \ldots$ i.e.

$$
\bar{P}_l \geq \mu\bar{P}_{l+k}/(\sum_{j=k}^{\infty}\bar{P}_j), \qquad l, k = 0, 1, 2, \ldots
$$

Multiplying both sides of the above inequality by kernal $e^{-\Lambda(t)}\frac{\Lambda^l(t)}{l!}$ and taking summation over $l = 0, 1, \ldots$ we get

$$\bar{H}(t) \geq \mu \bar{H}_k(t) / (\sum_{j=k}^{\infty} \bar{P}_j) \quad k > 0, \tag{3.6}$$

where

$$\bar{H}_k(t) = \sum_{l=0}^{\infty} e^{-\Lambda(t)} \frac{\Lambda^l(t)}{l!} \bar{P}_{k+l}, \quad k = 1, 2, \ldots \ .$$

The survival function $\bar{H}(t)$ has continuous NRBU properties if

$$\bar{H}(t) \geq \mu_H \frac{\bar{H}(t+x)}{\int_x^{\infty} \bar{H}(u) du} \ . \tag{3.7}$$

Note that

$$\begin{aligned} \mu_H &= \int_0^{\infty} \bar{H}(u) du \\ &= \sum_{k=0}^{\infty} \bar{P}_k \int_0^{\infty} e^{-\Lambda(u)} \frac{\Lambda^k(u)}{k!} \frac{d\Lambda(u)}{\Lambda'(u)}, \end{aligned}$$

where $du = \dfrac{d\Lambda(u)}{\Lambda'(u)}$.

Applying 2^{nd} mean value theorem for the above equality and taking $\Lambda'(0) \neq 0$ and $\Lambda'(\infty) = \infty$, we get $\mu_H = \frac{\mu}{\Lambda'(0)}$ and (3.7) may be written as follows:

$$\bar{H}(t) \geq \frac{\mu \Lambda'(x) \bar{H}(t+x)}{\Lambda'(0) \{\sum_{k=0}^{\infty} \bar{H}_k(x)\}}; \qquad t, x \geq 0. \tag{3.8}$$

From (3.6) and (3.8) we conclude that $\bar{H}(t)$ satisfies the assumption of NRBU property if

$$\mu \bar{H}_k(t) / (\sum_{j=k}^{\infty} \bar{P}_j) \leq \frac{\mu \Lambda'(x) \bar{H}(t+x)}{\Lambda'(0) \{\sum_{k=0}^{\infty} \bar{H}_k(x)\}}.$$

The proof is complete.

The proof for the NRWU case can be obtained by reversing all the inequalities in the above proof.

4. LAPLACE TRANSFORMS AND GENERATING FUNCTIONS FOR NRBU

Here we establish the necessary and sufficient conditions for the life distribution to have the NRBU properties by using the laplace transforms. These conditions can be used to investigate the closure under convolution.

Now let F be a distribution function such that $F(0-) = 0$ and let $\phi(s) = \int_0^\infty e^{-su} dF(u)$; $s \geq 0$ be the Laplace transform of $F(x)$. Define

$$(4.1) \qquad a_n(s) = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left(\frac{1-\phi(s)}{s} \right); \qquad n \geq 0\ ,\ s \geq 0$$

Let $b_{n+1}(s) = s^{n+1} a_n(s)$ for $n \geq 0$ and $b_0(0) = 1$ for $s \geq 0$. The transforms $a_n(s)$ and $b_{n+1}(s)$ have the forms

$$(4.2) \qquad a_n(s) = \frac{1}{n!} \int_0^\infty u^n e^{-su} \bar{F}(u) du$$

$$(4.3) \qquad b_{n+1}(s) = \frac{1}{n!} \int_0^\infty s^{n+1} u^n e^{-su} \bar{F}(u) du, \qquad n \geq 0,\ s \geq 0$$

Venogradov (1973) characterized the IFR property in terms of $b_n(s)$. Block and Savits (1980) obtained similar characterization for the IFRA, DMRL, NBU and NBUE properties. Abouammoh et al. (1988) have characterized the NBAFR property by $b_n(s)$. Abouammoh and Hendi (1990) characterized the NBURFR and NBARFR properties by $b_n(s)$. In the following we establish the corresponding characterization for the new renewal better than used (NRBU).

Theorem 4.1. *Let F be a life distribution with $F(0-) = 0$, then F has the NRBU property if and only if (iff)*

$$(4.4) \qquad \bar{F}(t) \sum_{j=n+1}^{\infty} b_{j+1}(s) \geq \mu c_{n+1}(s,t) \qquad \text{for } n > 0, s > 0$$

where

$$c_{n+1}(s,t) = s \int_0^\infty e^{-su} \frac{(su)^n}{n!} \bar{F}(t+u) du, \qquad t > 0.$$

Proof. Assume that F is NRBU, then by using the form (4.3) for $n > 0$, $s > 0$,

$$\begin{aligned}\sum_{j=n+1}^{\infty} b_{j+1}(s) &= \sum_{j=n+1}^{\infty} s^{j+1} \int_0^{\infty} e^{-su} \frac{(u)^j}{j!} \bar{F}(u) du \\ &= s \int_0^{\infty} \bar{F}(u) (\sum_{j=n+1}^{\infty} \frac{(su)^j}{j!} e^{-su}) du \\ &= s \int_0^{\infty} \bar{F}(u) (\int_0^{u} e^{-sv} \frac{(sv)^n}{n!} dv) du,\end{aligned}$$

i.e.,

$$\sum_{j=n+1}^{\infty} b_{j+1}(s) = s \int_0^{\infty} e^{-sv} \frac{(sv)^n}{n!} (\int_v^{\infty} \bar{F}(u) du) dv.$$

Since F is NRBU i.e. $\mu \bar{F}(t+u) \leq \bar{F}(t) \int_u^{\infty} \bar{F}(v) dv$, then

$$\begin{aligned}\sum_{j=n+1}^{\infty} b_{j+1}(s) &\geq s \int_0^{\infty} e^{-su} \frac{(su)^n}{n!} \mu \frac{\bar{F}(t+u)}{\bar{F}(t)} du \\ &= \frac{\mu s}{\bar{F}(t)} \int_0^{\infty} e^{-su} \frac{(su)^n}{n!} \bar{F}(t+u) du,\end{aligned}$$

i.e.

$$\sum_{j=n+1}^{\infty} b_{j+1}(s) \geq \frac{\mu}{\bar{F}(t)} c_{n+1}(s,t).$$

This completes the proof of necessary part.

To prove that the condition (4.4) is sufficient, note that it may be written in the form

$$\int_0^{\infty} G_n(u) \bar{F}(u) du \geq \frac{\mu}{\bar{F}(t)} c_{n+1}(s,t), \tag{4.5}$$

where

$$G_n(u) = s \sum_{j=n+1}^{\infty} \frac{(su)^j}{j!} e^{-su}.$$

It is obvious that

$$G_n(u) = \int_0^u s \frac{(sv)^n}{\Gamma(n+1)} e^{-sv} dv = P\{\sum_{i=1}^{n+1} Y_i \leq u\},$$

where $Y_1, Y_2, \ldots, Y_{n+1}$ are mutually independent and exponential with rate s. Hence $G_n(u)$ represent a gamma distribution function with parameters $(n+1, s)$ and its characteristic function is given by $\phi_{n+1}(w) = (1 - \frac{iw}{s})^{-(n+1)}$.

Letting $\frac{(n+1)}{s} \longrightarrow x$, it can be shown that

$$\lim_{n\to\infty} \phi_{n+1}(w) = \exp(iwx),$$

that is

$$G_n(u) \longrightarrow I_x(u) = \begin{cases} 0 & \text{for } u < x, \\ 1 & \text{for } u \geq x. \end{cases} \tag{4.6}$$

On the other hand, by Lemma (2.3) due to Block and Savits (1980),

$$\lim_{n\to\infty} c_{n+1}(s,t) = \bar{F}(t+x). \tag{4.7}$$

The condition (4.5) implies that

$$\lim_{n\to\infty} \int_0^\infty G_n(u)\bar{F}(u)du \geq \lim_{n\to\infty} \frac{\mu}{\bar{F}(t)} c_{n+1}(s,t).$$

From (4.6) and (4.7), it follows that

$$\int_x^\infty \bar{F}(u)du \geq \frac{\mu}{\bar{F}(t)} \bar{F}(t+x).$$

This complete the proof. □

The corresponding Laplace transforms of the dual NRWU properties are satisfied with inequalities sign reversed.

In the following we translate NRBU (NRWU) properties in terms of generating functions.

Let $p_0 = 1 - \bar{P}_0$ and $p_i = \bar{P}_{i-1} - \bar{P}_i, i = 1, 2, \ldots$ be the probability mass function (pmf) of a nonnegative random variable X. The probability generating function of X is

$$\psi(\theta) = E[\theta^X] = 1 - \sum_{j=0}^{\infty}(1-\theta)\theta^j \bar{P}_j. \tag{4.8}$$

Relation (4.8) may be written in the form

$$\psi(\theta) = 1 - \sum_{j=0}^{\infty} P(Y=j)\bar{P}_j \tag{4.9}$$

where Y has the geometric distribution

$$g_k = P(Y=k) = (1-\theta)\theta^k, \quad k = 0, 1, 2, \dots \tag{4.10}$$

Let $Y_i, i = 1, 2, \dots, n$ be iid random variables with common pmf given by (4.10). The variable $V = \sum_{i=1}^{n} Y_i$ has the negative binomial distribution,

$$P(V = n+j) = \binom{n+j-1}{j}\theta^j(1-\theta)^n. \qquad j = 0, 1, \dots \tag{4.11}$$

Next define

$$B_n(\theta) = \begin{cases} \sum_{j=0}^{\infty}\binom{n+j-1}{j}\theta^j(1-\theta)^n\bar{P}_j, & \text{for } n = 1, 2, \dots \\ 1 & \text{for } n = 0. \end{cases} \tag{4.12}$$

The form (4.12) has the following interesting physical meaning. Suppose that a device is subjected to two different types of shocks W_1 and W_2 say. At every time unit a shock of type W_1 occurs with probability $1-\theta$. If Y_i denote the number of W_1 shocks between the $(i-1)^{\underline{th}}$ and $i\underline{th}$, $i \in N$ of W_2 shocks, then Y_i has geometric distribution with pmf given by (4.10) and V has a negative binomial distribution given by (4.11). Hence $B_n(\theta), n \in N$, represents the probabilities that the device survives n shocks of type W_1 type, where $\bar{P}_j$ represents the probability that the device survives the first j shocks of type W_2.

Using the form (4.9) Abouammoh and Hendi (1990, 1991) found conditions for discrete life distributions, namely IFR, IFRA, NBU, NBUFR, NBAFR, NBURFR and NBARFR in terms of $B_n(\theta)$. Next, we translate the NRBU (NRWU) properties in terms of $B_n(\theta)$. The proof of the following Theorem is direct and therefore omitted.

Theorem 4.2. *Let $B_n(\theta)$ be given by (4.12), and*

$$C_{n,m}(\theta) = \sum_{l=0}^{\infty} \binom{n+l-1}{l} \theta^l (1-\theta)^n \bar{P}_{l+m}, \qquad l,m = 1,2,\ldots, \ .$$

Then $\{\bar{P}_j\}_{j=0}^{\infty}$ is NRBU (NRWU) iff

$$B_n(\theta) \geq (\leq) C_{n,m}(\theta).$$

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