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ON THE DIOPHANTINE EQUATION $d_1x^2 + 4d_2 = y^n$

FADWA S. ABU MURIEFAH

ABSTRACT.The object of this paper is to prove the following Theorem:
The diophantine equation

$$d_1x^2 + 4d_2 = y^n,$$

where d_1, d_2, x, y, n are positive integers such that $(d_1, d_2) = (x, y) = (2, y) = 1, d_1, d_2$ are square -free integers, n is an odd integer > 3 and $(n, h) = 1$ where h is the class number of the field $K = Q\left(\sqrt{-d_1d_2}\right)$, has no solutions in (d_1, d_2, x, y, n).

1. INTRODUCTION

Let d_1, d_2, x, y, p be positive integers. Many special cases of the diophantine equation

$$(1) \qquad d_1x^2 + 4d_2 = y^p, (d_1x, d_2) = 1, \quad p \text{ prime}>3, (x, y) = 1, p \nmid h$$

where d_1, d_2 are square-free integers and h is the class number of the field $Q\left(\sqrt{-d_1d_2}\right)$, have been considered in the last few years. The first result regarding this equation is due to Nagell [9], who proved that when $d_1 = d_2 = 1$, then equation (1) has only the positive solutions $x = y = 2$, $p = 3$ and $x = 11$, $y = 5, p = 3$. Ljunggren [6] studied this equation in full generality and he has found interesting theorems concerning its solutions. Also Le Maohua [8] proved that if $p \geq 8.5.10^6$ then equation (1) has no solution with $2 \nmid y$. In [1] we proved that if d_1 is odd, $d_2 = 2^{2k}$ and $p \equiv 1 \pmod 4$, then equation (1) has no solution with x odd. Recently Luca [7] proved that when $d_1 = 1$, $d_2 = 3^a$, then (1) has no solution. In the interesting paper [3] Bugeaud and Shory

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studied equation (1) where y is a fixed odd integer k coprime with d_1d_2 and they gave a necessary and sufficient conditions on d_1, d_2 and k under which this equation has at most $2^{w(k)-1}$ solutions where $w(k)$ denotes the number of distinct prime divisors of k. In this paper we solve equation (1) completely.

2. Preliminaries

We start by giving some important definitions.

Definitions

A ***Lehmer pair*** is a pair (α, β) of algebraic integers such that $(\alpha+\beta)^2$ and $\alpha\beta$ are non-zero co-prime rational integers and α/β is not a root of unity.Given a Lehmer pair (α, β)one defines the corresponding sequence of Lehmer numbers by

$$u_n(\alpha, \beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } n \text{ is odd,} \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & \text{if } n \text{ is even.} \end{cases}$$

A prime number p is a ***primitive divisr*** of $u_n(\alpha, \beta)$ if p divides u_n, but does not divides $(\alpha^2 - \beta^2)^2 u_1 u_2 \dots u_{n-1}$.

A Lehmer pair (α, β) such that $u_n(\alpha, \beta)$ has no primitive divisors will be called $n-$***defective Lehmer pair***.

Now we reproduce the following results for future use.

Lemma 2.1 ([2]). *For $n > 30$, the nth term of any Lehmer sequences has a primitive divisor.*

Lemma 2.2 ([10]). *Let n satisfy $6 < n \leq 30$. Then up to equivalence all parameters of n-defective Lehmer pairs are given as follows:*

i. $n = 7$, $(a, b) = (1, -7), (1, -19), (3, -5), (5, -7), (13, -3), (14, -22)$.
ii. $n = 9, (a, b) = (5, -3), (7, -1), (7, -5)$,
iii. $n = 13, (a, b) = (1, -7)$,
iv. $n = 14, (a, b) = (3, -13), (7, -1), (7, -5), (19, -1), (22, -14)$,
v. $n = 15, (a, b) = (7, -1), (10, -2), (1, -7), (3, -5), (5, -7)$,
vi. $n = 18, (a, b) = (1, -7), (3, -50), (5, -7)$,

vii. $n = 24, (a, b) = (3, -5), (5, -3)$,
viii. $n = 26, (a, b) = (7, -1)$,
ix. $n = 30, (a, b) = (1, -7), (2, -10)$.

3. MAIN RESULTS

Theorem 3.1. *The diophantine equation*

$$d_1x^2 + 4d_2 = y^n, \tag{2}$$

where d_1, d_2, x, y, n *are positive integers such that* $(d_1, d_2) = (x, y) = (2, y) = 1$, d_1, d_2 *are square -free integers,* n *is an odd integer>3 and* $(n, h) = 1$ *where* h *is the class number of the field* $K = Q\left(\sqrt{-d_1d_2}\right)$, *has no solutions in* (d_1, d_2, x, y, n).

Proof. Let (d_1, d_2, x, y, n) be a solution of (2). The impossibility of the equations $3x^2 + 4 = y^n$ and $x^2 + 12 = y^n$, with x odd has been proved in [1, Theorem 1.8] and [4] respectively. We may therefore assume $d_1d_2 \neq 3$. Now factorize (2) in the field K,

$$\left(x\sqrt{d_1} + 2\sqrt{-d_2}\right)\left(x\sqrt{d_1} - 2\sqrt{-d_2}\right) = y^n$$

The principal ideal $[x\sqrt{d_1} + 2\sqrt{-d_2}]$ and its conjugate ideal are co-prime, so

$$\left[x\sqrt{d_1} + 2\sqrt{-d_2}\right] = \pi^n,$$

for some ideal π in K. It follows that π^n is principal and since $(n, h) = 1$, therefore π is principal ideal, say $\pi = [\xi]$ for some element ξ in K. So we get the equation

$$\left[x\sqrt{d_1} + 2\sqrt{-d_2}\right] = [\xi]^n,$$

and consequently

$$\left(x\sqrt{d_1} + 2\sqrt{-d_2}\right) = \varepsilon\xi^n,$$

for some ε in K. Therefore we have the following two cases:

$$x\sqrt{d_1} + 2\sqrt{-d_2} = \left(\frac{a\sqrt{d_1} + b\sqrt{-d_2}}{2}\right)^n, \quad a \equiv b \equiv 1 \pmod 2$$

$$x\sqrt{d_1} + 2\sqrt{-d_2} = \left(a\sqrt{d_1} + b\sqrt{-d_2}\right)^n,$$

for some rational integers a and b.

By equating the coefficients of $\sqrt{-d_2}$, we deduce that the first case is impossible.

Now we consider the second case. Again equating the coefficients of $\sqrt{-d_2}$, we get the relation

$$2 = \sum_{r=0}^{\frac{n-1}{2}} \binom{n}{2r+1} a^{n-1-2r} b^{2r+1} d_1^{\frac{n-1}{2}-r} \left(-d_2\right)^r, \tag{3}$$

such that $y = a^2 d_1 + b^2 d_2$, and $(ad_1, bd_2) = 1$. Since y is odd, therefore, a and b have the opposite parity. Let

$$\alpha = a\sqrt{d_1} + b\sqrt{-d_2},\ \beta = a\sqrt{d_1} - b\sqrt{-d_2}.$$

Then we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{2}{b}. \tag{4}$$

It is easy to verify that (α, β) is a Lehmer pair. Further, let

$$u_t(\alpha, \beta) = \frac{\alpha^t - \beta^t}{\alpha - \beta},\quad t \geq 0$$

be the corresponding sequence of Lehmer number.

By Waring's formula [5, Formula 1.76], we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \sum_{i=0}^{\frac{n-1}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \left(-4b^2 d_2\right)^{\frac{n-1-2i}{2}} y^i,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!n}{(n-2i)!i!},\quad i = 0, \ldots, \frac{n-1}{2},$$

are positive integers. This implies that $\left(\alpha^n - \beta^n\right) / \left(\alpha - \beta\right)$ is an odd integer.

So $b = \pm 2$, hence ad_1 is odd, and from (4) we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \pm 1. \tag{5}$$

It implies that $u_n(\alpha, \beta)$ has no primitive divisor. By Lemmas 2.1 and 2.2, we see that if (5) holds, then $n \leq 5$. So we have $n = 5$, then from (3), we get

$$\pm 2 = \sum_{r=0}^{2} \binom{5}{2r+1} a^{4-2r} 2^{2r+1} d_1^{2-r} \left(-d_2\right)^r,$$

dividing this equation by 2, we get

$$\pm 1 = 5a^4d_1^2 - 40a^2d_1d_2 + 16d_2^2 \qquad (6)$$

Considering equation (6) modulo 8, we get a contradiction. This completes the proof of our theorem. □

Now we consider the case $n = 3$.

Corollary 3.1. *The diophantine equation (1), may have a solution with x and n odd only when $n = 3$, d_1 odd and $3 \nmid d_2$. Furthermore the solution (if it exists) is given by $y = a^2d_1 + 4d_2$, where a satisfies $a^2 = (4d_2 \pm 1)/3d_1$.*

Proof. From the above theorem it sufficient to consider $n = 3$, then from (3) we get

$$\pm 1 = 3d_1a^2 - 4d_2,$$

From this relation we deduce that d_1 odd and $3 \nmid d_2$. □

Examples:

(1) Consider the diophantine equation $3x^2 + 28 = y^n$. Here $h = 4, (n, 4) = 1$ for all odd integers n, so from the Theorem, there is no solution for $n > 3$. If $n = 3$, then $a^2 = (28 \pm 1)/9$ which impossible. So the equation has no solution in odd integers x and n for all $n \geq 3$. This is first shown by Ljunggren in [6].

(2) Consider the diophantine equation $3x^2 + 8 = y^n$. Here $h = 2$, so from the Theorem, there is no solution for $n > 3$. If $n = 3$, then $a^2 = (8 \pm 1)/9 = 1$. Hence

$$y = a^2d_1 + 4d_2 = 1.3 + 4.2 = 11. \quad x = 21.$$

So the equation has a unique solution in odd integers x and n.

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INTERSECTION, FIXED POINTS AND MINIMAX INEQUALITIES WITH A GENERALIZED COERCIVITY IN H-SPACES

SOUHAIL CHEBBI[1]

ABSTRACT. We introduce a generalized coercivity type condition for correspondences defined on topological vector spaces endowed with a generalized convex structure. An extension of the Fan's matching theorem is obtained and used to prove results on fixed points and minimax inequalities with a weakned compactness condition.

1. INTRODUCTION

This paper is a study of a coercivity type condition for correspondences defined on topological spaces endowed with a generalized convex structure. We introduce the concept of coercing family in H-spaces and we propose the systematic development of the method based on the Fan's matching type theorem.

We firstly recall the structure of H-convexity defined by Horvath in [10] and H-KKM correspondence defined by Bardaro and Cepitelli in [4]. We then introduce the notion of H-coercing family for correspondences defined in H-spaces and give some examples from the literature. In section 3, we prove a Fan's type theorem on the intersection of correspondences defined in H-spaces and satisfying a weakened compactness condition. Theorem 1 and Theorem 2 of this section generalize recent results of Lassonde [[12], Theorem I], Horvath [[10], Theorem 1] and Bardaro and Cepitelli [[4], Theorem 1 and Theorem 2] as

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well as corresponding results obtained in Fan [8], Ben El-Mechaiekh, Deguire and Granas [3], Ben El-Mechaiekh, Chebbi and Florenzano [2] when the H-convexity is replaced by the usual convexity of a topological vector space.

In Section 4, we generalize the results on coincidence and fixed point obtained by Horvath in [11], Bardaro and Cepitelli in [5] and Lassonde in [12] to coercive correspondences defined on non-compact H-spaces and we prove minimax inequalities for functions defined on H-spaces and satisfying a generalized coercivity type condition. Our results generalize minimax inequalities obtained by Allen in [1], Granas in [9], Ben El-Mechaiekh, Deguire and Granas in [3], Lassonde in [12], Horvath in [11] and Ding and Tan in [6].

2. PRELIMINARIES

Let $\langle X\rangle$ denote the family of all non-empty finite subsets of X. In order to define the setting of this paper, we firstly recall some basic concepts:

Definition 2.1. (a) (X,Γ) is said to be an *H-space* if X is a topological space and $\Gamma : \langle X\rangle \to 2^X$ a map such that $\Gamma(A) \subset \Gamma(B)$ if $A \subset B$ and assumed to have non-empty C^∞ values

(b) A subset $C \subset X$ is said to be *H-convex* if for every $A \in \langle C\rangle$, $\Gamma(A) \subset C$.

(c) A subset $K \subset X$ is said to be *H-compact* if for every $A \in \langle X\rangle$, there is a compact H-convex set D such that $A \cup K \subset D$.

Note that the class of H-spaces, which was firstly defined by Horvath in [11], contains topological vector spaces as well as a number of spaces with abstract topological convexity (the pseudo-convexity of Horvath in [10] and the concept of convex space due to Lassonde in [12] for example). For More details about generalized convexity, refer to [12], [10], [11], [4], [5] and [13]. The notion of H-compactness generalizes the c-compactness in [12].

A subset X of a topological space is said to be C^∞ (or ∞-*connected*) if for each integer n, any continuous function $f : \partial\Delta_n \to X$ can be continuously extended to a continuous function $g : \Delta_n \to X$.

Definition 2.2. Let (X, Γ) be an H-space. A set-valued map (simply called correspondence) $F : X \to X$ is called *H-KKM* if and only if:

$$\forall A \in \langle X \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x)$$

Definition 2.3. As was defined in [12], we say that a subet A of a topological space X is *compactly closed* (*open*, respectively) in X if for every compact set $C \subset X$, the set $A \cap C$ is closed (open respectively) in X.

We now introduce the concept of a generalized coercivity condition for correspondences as follow:

Definition 2.4. Let (X, Γ) be an H-space and Y a topological space. A family $\{(C_i, K_i)\}_{i \in I}$ is said to be *H-coercing* for a correspondence $F : X \to Y$ if and only if:

(i) For each $i \in I$, C_i is an H-compact subset of X and K_i is a compact subset of Y;
(ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$;
(iii) For each $i \in I$, there exists $k \in I$ such that:

$$\bigcap_{x \in C_k} F(x) \subseteq K_i.$$

Example 2.1. If $F : X \to X$ is a correspondence satisfying the following condition given in [11]: For some $x_0 \in X$, $F(x_0)$ is compact. Then F admits a coercing family.

Proof. Take, for all $i \in I$, $C_i = \{x_0\}$ and $K_i = F(x_0)$. □

Example 2.2. If $F : X \to X$ is a correspondence satisfying the following condition given in [6]: There exists an H-compact subset X_0 of X such that

$$\cap_{x \in X_0} F(x)$$

is compact. Then F admits a coercing family.

Proof. Take, for all $i \in I$, $C_i = \{X_0\}$ and $K_i = \cap_{x \in X_0} F(x)$. □

Note that when X is a subset of a topological vector space, the notion of coercing family in this generality was used by Ben El-Mechaiekh, Chebbi and Florenzano in [2] and generalized the concept of coercivity (with two sets K and C) used in [1],[3] and [8]. For more details about coercing family in topological vector space, see [2].

3. INTERSECTION THEOREMS

The main result of this paper is the following extension of Theorem 4 in [8]:

Theorem 3.1. *Let (X, Γ) be a an H-space, Y any topological space and $F : X \to Y$ a correspondence such that:*

1. *For every $x \in X$, $F(x)$ is compactly closed in X.*
2. *For some continuous map $s : X \to Y$, the correspondence $G : X \to X$ given by :*
$$G(x) = s^{-1}(F(x))$$
is H-KKM.
3. *There exists an H-coercing family $\{(C_i, K_i)\}_{i \in I}$ for F. Then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Proof. For $j = (i, a) \in J = I \times \langle X \rangle$, let $\hat{C}_j = C_i \cup a$ and $\hat{K}_j = K_i \cup a$. Since C_i is an H-compact subset, the family $\{(\hat{C}_j, \hat{K}_j)\}_{j \in J}$ is also H-coercing for F and, furthermore, $X = \bigcup_{j \in J} \hat{C}_j$.

For every $j \in J$, let Z_j be the compact and H-convex set containing $\hat{C}_j$ and Let $Y_j = s(Z_j)$. We consider the correspondence $G_j : \hat{C}_j \to Z_j$ defined by :
$$G_j(x) = s_j^{-1}(F(x) \cap Y_j)$$
where s_j is the restriction of s to Z_j.

By (1), for each $x \in \hat{C}_j$, $G_j(x)$ is compact and it is easy to check that G_j is H-KKM since $G_j(x) = G(x) \cap Z_j$ and G is H-KKM. It follows from Corollary 1 in [10] that $\bigcap_{x \in \hat{C}_j} G_j(x)$ is not empty, so $\bigcap_{x \in \hat{C}_j} F(x)$ is also not empty . Using condition (ii) of Definition 2.4, we can see that the family $\{\bigcap_{x\in\hat{C}_j} F(x)\}_{j\in J}$ has the finite intersection property. Since for some $j \in J$, $\bigcap_{x\in\hat{C}_j} F(x)$ is contained in a compact set, we conclude that $\bigcap_{j\in J}\bigcap_{x\in\hat{C}_j} F(x)$ is not empty. Since $X = \bigcup_{j\in J}\hat{C}_j$, we just have to notice that $\bigcap_{j\in J}\bigcap_{x\in\hat{C}_j} F(x) = \bigcap_{x\in X} F(x)$, in order to complete the proof. □

Theorem 3.1 extends Theorem 1 in [4] which in turn generalizes Corollary 1 of Horvath in [10]. When I is a singleton and the H-convexity is replaced by the convexity of Lassonde, then Theorem 3.1 is reduced to Theorem I in [12].

For any correspondence $F : X \to Y$, let $F^* : Y \to X$ be the correspondence defined by:

$$F^*(y) = X \setminus F^{-1}(y)$$

The following result is more specially adapted to the study of minimax inequalities:

Theorem 3.2. *Let (X, Γ) be an H-space and $F, G : X \to X$ two correspondences such that:*

(a) *For every $x \in X$, $G(x)$ is compactly closed and $F(x) \subset G(x)$.*
(b) *For every $x \in X$, $x \in F(x)$*
(c) *$F^*(x)$ is H-convex .*
(d) *There exists an H-coercing family $\{(C_i, K_i)\}_{i\in I}$ for G. Then $\bigcap_{x\in X} G(x) \neq \emptyset$.*

Proof. By virtue of Theorem 3.1, it suffices to show that G is H-KKM. Suppose that for some finite subset $A \subset X$, there exists $y \in \Gamma(A)$ and $y \notin G(x)$ for every $x \in A$ and so $A \subset G^*(y)$. Since $G^*(y) \subset F^*(y)$ and by (c) $\Gamma(A) \subset F^*(y)$, hence $y \in F^*(y)$ which is equivalent to $y \notin F(y)$ and this contradicts (b). □

Note that Condition (d) of Theorem 3.2 extends the non-compactness condition of Theorem 2 in [10] and Theorem 1 in [11].

4. FIXED POINTS AND MINIMAX INEQUALITIES

As application of section 3, we prove a generalization of Ky Fan's fixed point theorem as presented by Ben-El Mechaiekh, Deguire and Granas in [3]:

Proposition 4.1. *Let (X,Γ) be an H-space, Y a topological space and $S : X \to Y$ a correspondence such that:*

(i) *For each $x \in X$, $S(x)$ is compactly open in Y.*
(ii) *For each $y \in X$, $S^{-1}(y)$ is non-empty and H-convex.*
(iii) *There exists an H-coercing family $\{(C_i, K_i)\}_{i\in I}$ for the correspondence $F : X \to Y$ defined by $F(x) = Y \backslash S(x)$, $\forall x \in X$.*

Then, for each continuous function s from X to Y, there exists an $x_0 \in X$ such that $s(x_0) \in S(x_0)$. In particular, S has a fixed point.

Proof. By (i), $F(x)$ is compactly closed for each $x \in X$. Let $s : X \to Y$ be any continuous map and $G : X \to X$ a correspondence defined, for all $x \in X$, by $G(x) = s^{-1}(F(x))$. G is not H-KKM, otherwise condition (ii) is not satisfied. Thus, there is a finite set $A \subset X$ and $x_0 \in \Gamma(A)$ such that $s(x_0) \in \cap_{x\in A} S(x)$ and so $s(x_0) \in S(x_0)$. □

Theorem 3.2 can be also used to prove results on minimax inequalities:

Proposition 4.2. *Let (X,Γ) be an H-space and let (E,C) be an order complete topological Riesz space, where C is the closed positive cone with a non-empty interior $int(C)$. Let $f,\ g : X \times X \to (E,C)$ be two functions satisfying the following conditions :*

(a) *For every $(x,y) \in X \times X$, $g(x,y) \le f(x,y)$.*
(b) *For every $y \in X$ and any $\lambda \in E$, the set $\{x \in X : f(x,y) \in \lambda + int(C)\}$ is H-convex.*
(c) *For every $x \in X$ and any $\lambda \in E$, the set $\{y \in X : g(x,y) \in \lambda + int(C)\}$ is compactly open.*
(d) *There exists a family $\{(C_i, K_i)\}_{i\in I}$ satisfying condition (i) and (ii) of Definition 2.4 and the following one:*

$\forall i \in I,\ \exists k \in I$ such that $\{\mathrm{y} \in \mathrm{Y} : \mathrm{g(x,y)} \notin \lambda + \mathrm{int(C)} \quad \forall \mathrm{x} \in \mathrm{C_k}\} \subset \mathrm{K_i}$

Then, for every $\lambda \in E$, the following alternative holds:
(1) *There exists $y_0 \in X$ such that for every $x \in X$, $g(x, y_0) \notin \lambda + int(C)$.*
(2) *There exists $x_0 \in X$, such that $f(x_0, x_0) \in \lambda + int(C)$.*

Proof. For fixed $\lambda \in E$, we define $F(x) = \{y \in X : f(x, y) \notin \lambda + int(C)\}$ and $G(x) = \{y \in X : g(x, y) \notin \lambda + int(C)\}$. Condition (a) implies $F(x) \subset G(x)$ for every $x \in X$; indeed if $y \notin G(x)$, then $g(x, y) \in \lambda + int(C)$ and there is a neighborhood V of $0 \in E$ such that $g(x, y) + V \subset \lambda + int(C)$. But $g(x, y) \leq f(x, y)$ implies $\lambda < g(x, y) + v \leq f(x, y) + v$, for every $v \in V$ thus $f(x, y) + V \subset \lambda + int(C)$, that is $y \notin F(x)$.

If there exists $x_0 \in X$ with $x_0 \notin F(x_0)$, then $f(x_0, x_0) \in int(C)$ so we have (2). Otherwise $x \in F(x)$ for each $x \in X$. Hence all assumptions of Theorem 3.2 are satisfied, then $\bigcap_{x \in X} G(x) \neq \emptyset$ which implies property (1) of the alternative □

Corollary 4.1. *Let (X, Γ) be an H-space, (E, C) a completely ordered topological Riesz space. Suppose that $f : X \times X \to (E, C)$ is a function satisfying the following properties:*

(a) *f is upper bounded on the set $\triangle = \{(x, x) : x \in X\}$.*
(b) *For every $y \in X$ and any $\lambda \in E$, the set $\{x \in X : f(x, y) > \lambda\}$ is H-convex.*
(c) *For every $x \in X$ and any $\lambda \in E$, the set $\{y \in X : f(x, y) \leq \lambda\}$ is compactly closed.*
(d) *There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying condition (i) and (ii) of Definition 4.2 and the following one:*

$$\forall i \in I,\ \exists k \in I \text{ such that } \{\mathrm{y} \in \mathrm{Y} : \mathrm{f(x,y)} \leq \lambda \ \ \forall \mathrm{x} \in \mathrm{C_k}\} \subset \mathrm{K_i} \quad \forall \lambda \in \mathrm{E}$$

Then: $\inf_{y \in X} \sup_{x \in X} f(x, y) \leq \sup f(x, x)$
(whenever the "inf" in the left-hand side exists).

Proof. Take $\lambda = \sup_{x \in X} f(x, x)$ which is well defined. By Proposition 4.2, there exists $y_0 \in X$ such that:

$$f(x, y_0) \leq \sup_{x \in X} f(x, x) \qquad \forall x \in X$$

Since (E, C) is completely ordered, it follows that $\sup_{x \in X} f(x, y_0)$ exists and the result follows. $\square$

Note that Proposition 4.2 generalizes Theorem 3 in [3] and Proposition 5.1 in [10] by relaxing the compactness condition. In case $E = \mathbb{R}$, Corollary 4.1 is reduced to minimax inequalities obtained in [1] and [7].

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APPLICATIONS OF BRIOT - BOUQUET DIFFERENTIAL SUBORDINATION TO SOME CLASSES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. The object of this paper is to define and study some classes of meromorphic functions using a new differential operator. Inclusion relations, integral operators, and other results are found by the application of Briot - Boq
uet differential subordination

1. INTRODUCTION

Let A be the set of all functions analytic in the unit disc $E = \{z : |z| < 1\}$. Let $g, G \in A$. We say that g is subordinate to G written $g \prec G$, if G is univalent, $g(0) = G(0)$ and $g(E) \subset G(E)$.

Let Σ be the set of all meromorphic functions f in E and having in $D = E| \{0\}$, the the Laurent expansion $f(z) = \frac{1}{z} + \sum\limits_{k=1}^{\infty} a_k z^k$. The convolution or Hadamard Product $f * g$ of two functions f and g in Σ is defined as follows

If $f(z) = \frac{1}{z} + \sum\limits_{k=1}^{\infty} a_k z^k$ and $g(z) = \frac{1}{z} + \sum\limits_{k=1}^{\infty} b_k z^k$, then

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k.$$

If follows from the definition of the Hadamard Product that

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$$z(f*g)'(z) = (zf'*g)(z) = (f*zg')(z).$$

In [1], the first author defined the operator $D_\lambda^n f$, which generalizes the well-known Salagean operator [7]. In this paper we modify the operator $D_\lambda^n f$ for meromorphic functions as follows

$$\begin{aligned} D_\lambda^0 f(z) &= f(z), \\ D_\lambda^1 f(z) &= (1-\lambda)f(z) + \lambda\frac{(z^2 f(z))'}{z}, \ \lambda \geq 0, \\ &= D_\lambda f(z), \\ D_\lambda^2 f(z) &= D_\lambda(D_\lambda^1 f(z)), \ldots \\ D_\lambda^n f(z) &= D_\lambda(D_\lambda^{n-1} f(z)) = (1-\lambda)D_\lambda^{n-1} f(z) \\ &\qquad + \lambda\frac{(z^2 D_\lambda^{n-1} f(z))'}{z}, \ n \in \mathbb{N} \end{aligned} \tag{1.1}$$

Using the operator $D_\lambda^n f$, we introduce three subclasses $\Sigma\,(m; n; \lambda; h)$, $Q\,(m; n; \lambda; h)$ and $T\,(m; n; \lambda; h)$ of meromorphic functions and investigate certain properties of functions belonging to these classes. We require the following lemmas to prove the results of this paper.

Lemma 1.1 ([3]). *Let $\beta, \sigma \in C$. Let $h \in A$ be convex univalent in E with* $\operatorname{Re}[\beta h(z) + \sigma] > 0$, $z \in E$, $h(0) = 1$ *and $P \in A$ with $P(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in E, then*

$$P(z) + \frac{zP'(z)}{\beta P(z) + \sigma} \prec h(z) \rightarrow P(z) \prec h(z).$$

Lemma 1.2 ([5]). *Let $\beta, \sigma \in C$. Let $h \in A$ be convex univalent in E with $h(0) = 1$ and* $\operatorname{Re}[\beta h(z) + \sigma] > 0$, *$z \in E$, and $q \in A$ with $q(0) = 1$ and $q(z) \prec h(z)$, $z \in E$. If $P(z) = 1 + p_1 z + +p_2 z^2 + \cdots$ is analytic in E, then*

$$P(z) + \frac{zP'(z)}{\beta q(z) + \sigma} \prec h(z) \rightarrow P(z) \prec h(z)$$

2. THE CLASS $\Sigma(m;n;\lambda;h)$

Definition 2.1. Let $f=\{f_1,f_2,...,f_m\}$, $f_i\in\Sigma$, $1\le i\le m$ be such that

$$\frac{-z[D_\lambda^n f_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)}\prec h(z),\ z\in E,\ i=\overline{1,m},$$

where $z\sum\limits_{j=1}^{m} D_\lambda^n f_j(z)\neq 0$ in E, h is convex univalent in E with $h(0)=1$.

We say that $f=\{f_1,f_2,...,f_m\}$ belongs to the class $\Sigma(m;n;\lambda;h)$.

Remark 2.1. If $m=1$ and $\lambda=0$ or $n=0$ and $h(z)=\frac{1-z}{1+z}$, then $D_\lambda^n f(z)=f(z)$ and $\Sigma(1;n;0;h)=\Sigma^*$, the class of meromorphic starlike functions which has been studied by Clunie [2], Pommerenke [6] and others.

Theorem 2.1. *Let* $f=\{f_1,f_2,...,f_m\}\in\sum(m;n;\lambda;h)$ *and* $F(z)=\frac{1}{m}\sum\limits_{i=1}^{m} f_i(z)$. *Then F satisfies the condition*

$$\frac{-z[D_\lambda^n F(z)]'}{D_\lambda^n F(z)}\prec h(z),\qquad z\in E \tag{2.1}$$

Proof. Let $f=\{f_1,f_2,...,f_m\}\in\sum(m;n;\lambda;h)$. Then for any $z_0\in E$,

$$\frac{-z_0[D_\lambda^n f_i(z_0)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n f_j(z_0)}\in h(E)$$

and hence equal $h(w_i)$ (say) for some $w_i\in E$, $i=\overline{1,m}$. Hence

$$-\frac{\sum\limits_{i=1}^{m} z_0[D_\lambda^n f_i(z_0)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n f_j(z_0)}=\sum_{i=1}^{m} h(w_i).$$

Let $f(z)=\frac{1}{z}+\sum\limits_{k=1}^{\infty} a_k z^k$. Then, from (1,1) we see that

$$\begin{aligned} D_\lambda^n f &= f(z)*\left\{\frac{1}{z}+\sum_{k=1}^{\infty}[1+\lambda(k+1)]^n z^k\right\}\\ &=(f*k_n)(z), \end{aligned}$$

where

$$k_n(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [1+\lambda(k+1)]^n z^k, \qquad n \in \mathbb{N}_0. \tag{2.2}$$

Hence

$$\frac{-z_0[D_\lambda^n F(z_o)]'}{D_\lambda^n F(z_0)} = \frac{-z_0[k_n * \sum\limits_{i=1}^{m} f_i(z_0)]'}{k_n * \sum\limits_{j=1}^{m} f_j(z_0)}.$$

Since

$$D_\lambda^n \sum_{i=1}^{m} f_i(z) = \sum_{i=1}^{m} D_\lambda^n f_i(z),$$

we have

$$\begin{aligned}\frac{-z_0[D_\lambda^n F(z_0)]'}{D_\lambda^n F(z_0)} &= \frac{1}{m}\left[\frac{-z_0 \sum\limits_{i=1}^{m} [D_\lambda^n f_i(z_0)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^n f_i(z_0)}\right] \\ &= \frac{1}{m}\sum_{i=1}^{m} h(w_i) = h(w_0).\end{aligned}$$

for some $w_0 \in E$, since h is convex in E. This completes the proof of the theorem. □

Remark 2.2. If $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$ and $h(z) = \frac{1-z}{1+z}$, then Theorem 2.1 shows that $D_\lambda^n F \in \Sigma^*$, where $F(z) = \frac{1}{m}\sum\limits_{i=1}^{m} f_i(z)$, and hence $D_\lambda^n f_i$ are close-to-convex meromorphic functions [4].

Theorem 2.2. *Let $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$. Define*

$$F_i(z) = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} f_i(t)dt, \qquad (\gamma \in C, \operatorname{Re}\gamma > 0), \qquad i = \overline{1, m}.$$

If $\operatorname{Re} h$ *is bounded in* E *and* $\operatorname{Re}(\gamma+2) > \max\limits_{z\in E} \operatorname{Re} h(z)$, *then* $F = \{F_1, F_2, \ldots, F_m\} \in \sum(m; n; \lambda; h)$.

Proof. From the definition of $F_i(z)$ it follows that

$$zF_i'(z) + (\gamma+2)F_i(z) = (\gamma+1)f_i(z)$$

and on taking convolution with k_n, given by (2.2), we obtain

$$z[D_\lambda^n F_i(z)]' + (\gamma+2)D_\lambda^n F_i(z) = (\gamma+1)D_\lambda^n f_i(z), \qquad i=\overline{1,m}. \tag{2.3}$$

Let

$$P_i(z) = \frac{mz[D_\lambda^n F_i(z)]'}{\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}. \tag{2.4}$$

From (2.3) we get

$$\frac{P_i(z)}{m}\sum_{j=1}^{m} D_\lambda^n F_j(z) + (\gamma+2)D_\lambda^n F_i(z) = (\gamma+1)D_\lambda^n f_i(z) \tag{2.5}$$

Differentiating the equality (2.5) with respect to z, we obtain

$$\frac{P_i'(z)}{m}\sum_{j=1}^{m} D_\lambda^n F_j(z) + \frac{P_i(z)}{m}\sum_{j=1}^{m}[D_\lambda^n F_j(z)]' + (\gamma+2)[D_\lambda^n F_i(z)]' = (\gamma+1)[D_\lambda^n f_i(z)]'.$$

From (2.4) we have

$$P_i'(z)\frac{\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}{m} + \frac{P_i(z)}{m}\frac{\sum\limits_{i=1}^{m} P_i(z)\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}{mz} + (\gamma+2)\frac{P_i(z)\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}{mz} = (\gamma+1)[D_\lambda^n f_i(z)]'.$$

Hence

$$P_i'(z) + \frac{P_i(z)}{mz}\sum_{i=1}^{m} P_i(z) + (\gamma+2)\frac{P_i(z)}{z} = \frac{(\gamma+1)[D_\lambda^n f_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)}.$$

Then

$$\frac{-zP_i'(z)}{\frac{1}{m}\sum\limits_{i=1}^{m} P_i(z)+\gamma+2} - P_i(z) = \frac{-(\gamma+1)z[D_\lambda^n f_i(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)} \cdot \frac{1}{\frac{1}{m}\sum\limits_{i=1}^{m} P_i(z)+\gamma+2}$$

$$= \frac{-(\gamma+1)z[D_\lambda^n f_i(z)]'}{\frac{1}{m}\left[\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)\cdot\sum\limits_{i=1}^{m} P_i(z) + (\gamma+2)\sum\limits_{j=1}^{m} D_\lambda^n F_j(z)\right]}.$$

From (2.5) we have

$$\frac{-zP_i'(z)}{\frac{1}{m}\sum_{i=1}^{m} P_i(z)+\gamma+2} - P_i(z) = \frac{-(\gamma+1)z[D_\lambda^n f_i(z)]'}{\frac{1}{m}(\gamma+1)\sum_{i=1}^{m} D_\lambda^n f_i(z)} \prec h(z), \tag{2.6}$$

since $f=\{f_1,f_2,...,f_m\}\in\sum(m;n;\lambda;h)$. Now we can write for any $z_0\in E$,

$$\frac{-\frac{1}{m}z_0P_i'(z_0)}{\frac{1}{m}\sum_{j=1}^{m} P_j(z_0)+\gamma+2} - \frac{1}{m}P_i(z_0) = \frac{1}{m}h(w_i),$$

for some $w_0\in E$. This is true for $i=\overline{1,m}$. Since h is convex, there exists a $w_i\in E$ such that

$$\frac{z_0Q'(z_0)}{-Q(z_0)+\gamma+2} + Q(z_0) = h(w_0),$$

where $Q(z) = -\frac{1}{m}\sum_{i=1}^{m} P_i(z)$.

Hence

$$\frac{zQ'(z)}{-Q(z)+\gamma+2} + Q(z) \prec h(z).$$

Since $\operatorname{Re} h$ is bounded and $\operatorname{Re}(\lambda+2) > \max \operatorname{Re} h(z)$, it follows by Lemma 1.1 that $Q(z)\prec h(z)$, $z\in E$.

From (2.6) we have

$$\frac{z[-P_i(z)]'}{-Q(z)+\gamma+2} + [-P_i(z)] \prec h(z),$$

where $Q(z)\prec h(z)$, $i=\overline{1,m}$. An application of Lemma 1.2 gives $-P_i(z)\prec h(z)$, $z\in E$, $i=\overline{1,m}$.

That is

$$\frac{-z[D_\lambda^n F_i(z)]'}{\frac{1}{m}\sum_{j=1}^{m} D_\lambda^n F_j(z)} \prec h(z),$$

Now

$$F_i(z) = \frac{\gamma+1}{z^{\gamma+2}}\int_0^z t^{\gamma+1} f_i(t)dt, \; \gamma\in C, \; \operatorname{Re}\gamma>0.$$

It can be proved, easily, that, for every i, $1\le i\le m$

$$D_\lambda^n F_i(z) = \frac{\gamma+1}{z^{\gamma+2}}\int_0^z t^{\gamma+1} D_\lambda^n f_i(t)dt,$$

and hence

$$\sum_{i=1}^{m} D_{\lambda}^{n} F_i(z) = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} \sum_{i=1}^{m} D_{\lambda}^{n} f_i(t) dt = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma} g(t) dt,$$

where $g(t) = t \sum_{i=1}^{m} D_{\lambda}^{n} f_i(t) \neq 0$,for $t \in E$ by assumption.
Now define

$$\Omega(z) = \sum_{n=1}^{\infty} \frac{\gamma+1}{\gamma+n} z^{n-1}, \ \mathrm{Re}\,\gamma > 0,$$

Then an easy calculations shows that

$$z \sum_{i=1}^{m} D_{\lambda}^{n} F_i(z) = (\Omega * g)(z) \neq 0.$$

Thus $F = \{F_1, F_2, ..., F_m\} \in \sum(m; n; \lambda; h)$. □

Theorem 2.3. *if* $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n+1; \lambda; h)$, $\lambda > 0$ *and* $\mathrm{Re}\, h$ *is bounded in* E*, then* $f = \{f_1, f_2, ..., f_m\} \in \sum(m; n; \lambda; h)$ *holds for* $(1 + \frac{1}{\lambda}) > d = \max_{z \in E} \mathrm{Re}\, h(z)$ *in* E.

Proof. Let

$$p_i(z) = \frac{mz[D_{\lambda}^{n} f_i(z)]'}{\sum_{j=1}^{m} D_{\lambda}^{n} f_j(z)}, \ z \in E. \tag{2.7}$$

From (1.1), we know that

$$z\,(D_{\lambda}^{n} f_i(z))' = \frac{1}{\lambda} D_{\lambda}^{n+1} f_i(z) - (1 + \frac{1}{\lambda})\, D_{\lambda}^{n} f_i(z). \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\frac{1}{m} p_i(z) \sum_{j=1}^{m} D_{\lambda}^{n} f_j(z) = \frac{1}{\lambda} D_{\lambda}^{n+1} f_i(z) - (1 + \frac{1}{\lambda})\, D_{\lambda}^{n} f_i(z).$$

Differentiating with respect to z, we get

$$\frac{z}{m}p_i'(z)\sum_{j=1}^{m}D_\lambda^n f_j(z)+\frac{z}{m}p_i(z)\sum_{j=1}^{m}(D_\lambda^n f_j(z))'$$
$$=\frac{z}{\lambda}\left(D_\lambda^{n+1}f_i(z)\right)'-z(1+\frac{1}{\lambda})\left(D_\lambda^n f_i(z)\right)'.$$

Using (2.7), we have

$$\frac{z}{m}p_i'(z)\sum_{j=1}^{m}D_\lambda^n f_j(z)+p_i(z)\left[\frac{z}{m}\sum_{j=1}^{m}(D_\lambda^n f_j(z))'+\frac{1}{m}(1+\frac{1}{\lambda})\sum_{j=1}^{m}D_\lambda^n f_j(z)\right]$$
$$=\frac{z}{\lambda}\left(D_\lambda^{n+1}f_i(z)\right)'.$$

Then

$$\frac{\frac{z}{m}p_i'(z)\sum\limits_{j=1}^{m}D_\lambda^n f_j(z)}{\frac{z}{m}\sum\limits_{j=1}^{m}\left(D_\lambda^n f_j(z)\right)'+\frac{1}{m}(1+\frac{1}{\lambda})\sum\limits_{j=1}^{m}D_\lambda^n f_j(z)}+p_i(z)$$
$$=\frac{\frac{z}{\lambda}\left(D_\lambda^{n+1}f_i(z)\right)'}{\frac{z}{m}\sum\limits_{j=1}^{m}\left(D_\lambda^n f_j(z)\right)'+\frac{1}{m}(1+\frac{1}{\lambda})\sum\limits_{j=1}^{m}D_\lambda^n f_j(z)}.$$

Using (2.8), we get

$$\frac{\frac{z}{m}p_i'(z)\sum\limits_{j=1}^{m}D_\lambda^n f_j(z)}{\frac{z}{m}\sum\limits_{j=1}^{m}\left(D_\lambda^n f_j(z)\right)'+\frac{1}{m}(1+\frac{1}{\lambda})\sum\limits_{j=1}^{m}D_\lambda^n f_j(z)}+p_i(z)=\frac{z\left(D_\lambda^{n+1}f_i(z)\right)'}{\frac{1}{m}\sum\limits_{j=1}^{m}D_\lambda^{n+1}f_j(z)}. \tag{2.9}$$

Using (2.7) in the left hand side of (2.9), we have

$$\frac{-zp_i'(z)}{\frac{1}{m}\sum\limits_{j=1}^{m}p_j(z)+1+\frac{1}{\lambda}}-p_i(z)=\frac{-z\left(D_\lambda^{n+1}f_i(z)\right)'}{\frac{1}{m}\sum\limits_{j=1}^{m}D_\lambda^{n+1}f_j(z)}.$$

Since $f=\{f_1,f_2,...,f_m\}\in\sum(m;n+1;\lambda;h)$, then

$$\frac{-zp_i'(z)}{\frac{1}{m}\sum\limits_{j=1}^{m}p_j(z)+1+\frac{1}{\lambda}}-p_i(z)=\frac{-z\left(D_\lambda^{n+1}f_i(z)\right)'}{\frac{1}{m}\sum\limits_{j=1}^{m}D_\lambda^{n+1}f_j(z)}\prec h(z), \tag{2.10}$$

for $i = \overline{1,m}$.
Therefore for any $z_0 \in E$, we have

$$\frac{-\frac{z_0}{m}p_i'(z_0)}{\frac{1}{m}\sum\limits_{j=1}^{m} p_j(z_0) + (1+\frac{1}{\lambda})} - \frac{1}{m}p_i(z_0) = \frac{1}{m}h(w_i),$$

for some $w_0 \in E$. Since h is convex, there exists a $w_i \in E$, such that

$$\frac{-\frac{z_0}{m}\sum\limits_{i=1}^{m} p_i'(z_0)}{\frac{1}{m}\sum\limits_{j=1}^{m} p_j(z_0) + (1+\frac{1}{\lambda})} - \frac{1}{m}\sum_{i=1}^{m} p_i(z_0) = \frac{1}{m}\sum_{i=1}^{m} h(w_i) = h(w_0).$$

Setting $Q(z) = -\frac{1}{m}\sum\limits_{i=1}^{m} p_i(z)$, we have

$$\frac{zQ'(z)}{-Q(z) + (1+\frac{1}{\lambda})} + Q(z) \prec h(z).$$

Which by Lemma 1.1, implies that $Q(z) \prec h(z)$.
From (2.10), we have

$$\frac{-zp_i'(z)}{-Q(z) + (1+\frac{1}{\lambda})} - p_i(z) \prec h(z),$$

where $Q(z) \prec h(z)$. An application of Lemma 1.2 gives $-p_i(z) \prec h(z)$, which implies that $f = \{f_1, f_2, ..., f_m\} \in \sum(m;n;\lambda;h)$ □

3. THE CLASS $Q\,(m;n;\lambda;h)$

Definition 3.1. Let $Q\,(m;n;\lambda;h)$ denote the class of all functions $f \in \Sigma$ such that

$$\frac{-mz[D_\lambda^n f(z)]'}{\sum\limits_{j=1}^{m} D_\lambda^n g_j(z)} \prec h(z),\ z \in E.$$

where $g = \{g_1, g_2, ..., g_m\} \in \sum(m;n;\lambda;h)$.

Theorem 3.1. *Let* $f \in Q(m;n;\lambda;h)$. *If* $\operatorname{Re} h$ *is bounded in* E *and* $\operatorname{Re}(c+2) > \max\limits_{z\in E} \operatorname{Re} h\,(z)$, *then*

$$F(z) = \frac{c+1}{z^{c+2}} \int_0^z t^{c+1} f(t)dt,\ z \in E,\ c \in C,\ \operatorname{Re} c > 0,$$

also belongs to $Q(m;n;\lambda;h)$.

Proof. Since $f \in Q(m;n;\lambda;h)$, there exists a $g = \{g_1, g_2, ..., g_m\} \in \sum(m;n;\lambda;h)$ such that

$$\frac{-z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n g_j(z)} \prec h(z), \ z \in E.$$

Let

$$G_i(z) = \frac{c+1}{z^{c+2}} \int\limits_0^z t^{c+1} g_i(t)dt, \ c > 0.$$

Then by Theorem 2.2 $G = \{G_1, G_2, ..., G_m\} \in \sum(m;n;\lambda;h)$.
Let

$$P(z) = \frac{z[D_\lambda^n F(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}. \tag{3.1}$$

Now, from the definition of G_i and F, we have

$$z[D_\lambda^n G_i(z)]' + (c+2)D_\lambda^n G_i(z) = (c+1)D_\lambda^n g_i(z), \tag{3.2}$$

and

$$z[D_\lambda^n F(z)]' + (c+2)D_\lambda^n F(z) = (c+1)D_\lambda^n f(z). \tag{3.3}$$

From (3.1) into (3.3), we obtain

$$\frac{1}{m}P(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + (c+2)D_\lambda^n F(z) = (c+1)D_\lambda^n f(z).$$

Differentiating with respect to z we obtain

$$\frac{1}{m}P'(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + \frac{1}{m}P(z)\sum_{j=1}^{m} [D_\lambda^n G_j(z)]' + (c+2)[D_\lambda^n F(z)]'$$
$$= (c+1)[D_\lambda^n f(z)]'.$$

Then

$$\frac{z}{m}P'(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + \frac{z}{m}P(z)\sum_{j=1}^{m} [D_\lambda^n G_j(z)]' + z(c+2)[D_\lambda^n F(z)]' \tag{3.4}$$
$$= z(c+1)[D_\lambda^n f(z)]'.$$

From (3.1) into (3.4) we have

$$\frac{z}{m}P'(z)\sum_{j=1}^{m} D_\lambda^n G_j(z) + \frac{z}{m}P(z)\sum_{j=1}^{m}[D_\lambda^n G_j(z)]' + (c+2)\frac{P(z)}{m}\sum_{j=1}^{m} D_\lambda^n G_j(z)$$
$$= z(c+1)[D_\lambda^n f(z)]'.$$

Hence

$$\frac{\frac{z}{m}P'(z)\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}{\frac{z}{m}\sum\limits_{j=1}^{m}[D_\lambda^n G_j(z)]' + \frac{c+2}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)} + P(z)$$
$$= \frac{(c+1)z[D_\lambda^n f(z)]'}{\frac{z}{m}\sum\limits_{j=1}^{m}[D_\lambda^n G_j(z)]' + \frac{(c+2)}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}.$$

From (3.2) we get

$$\frac{-zP'(z)}{\frac{1}{m}\sum\limits_{j=1}^{m} Q_j(z) + c + 2} - P(z) = \frac{-z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n g_j(z)} \prec h(z),$$

where $Q_j(z) = \frac{z[D_\lambda^n G_j(z)]'}{\frac{1}{m}\sum\limits_{j=1}^{m} D_\lambda^n G_j(z)}$.

Now $(-Q_j(z)) \prec h(z), j = \overline{1,m}$, since $G = \{G_1, G_2, ..., G_m\} \in \sum(m;n;\lambda;h)$ and h is a convex function. Since $\mathrm{Re}\,(c+2) > \mathrm{Re}\,h$, an application of Lemma 1.2 implies that $-P(z) \prec h(z)$, hence $F \in Q(m;n;\lambda;h)$. □

Theorem 3.2. *If $f \in Q(m;n+1;\lambda;h)$, $\lambda > 0$ and $\mathrm{Re}\,h$ is bounded in E, then $f \in Q(m;n;\lambda;h)$ holds for $(1+\frac{1}{\lambda}) > d = \max\limits_{z\in E}\mathrm{Re}\,h\,(z)$ in E.*

Proof. The proof of this theorem is similar to that of Theorem 2.3 and is therefore omitted. □

4. THE CLASS $T(m;n;\lambda;\alpha;h)$

Definition 4.1. Let $T(m;n;\lambda;\alpha;h)$, $\alpha \geq 0$ denote the class of functions $f \in \sum$ satisfying the condition

$$S(\alpha; f, g_1, g_2, ..., g_m) = \left\{ \frac{-\alpha z[D_\lambda^{n+1} f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^{n+1} g_i(z)} + \frac{-(1-\alpha) z[D_\lambda^{n} f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^{n} g_i(z)} \right\} \prec h(z),$$

where $g = \{g_1, g_2, ..., g_m\} \in \sum(m;n;\lambda;h)$ and $z \sum\limits_{i=1}^{m} D_\lambda^{n+1} g_i(z) \neq 0$ in E.

Remark 4.1. If we put $\alpha = 0$, we get $T(m;n;\lambda;0;h) = Q(m;n;\lambda;h)$.

Theorem 4.1. *If $f \in T(m;n;\lambda;\alpha;h)$, $\lambda > 0$ and $\operatorname{Re} h$ is bounded in E, then $f \in T(m;n;\lambda;0;h)$ hold for $(\frac{1}{\lambda}+1) \geq d = \max\limits_{z\in E} \operatorname{Re} h(z)$.*

Proof. For $\alpha = 0$, the theorem is trivial and hence we can assume that $\alpha \neq 0$. Let

$$P(z) = \frac{z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^n g_i(z)}.$$

Then an easy calculations shows that

$$\frac{zP'(z)}{\frac{1}{m}\sum\limits_{i=1}^{m} q_i(z) + \frac{1}{\lambda} + 1} + P(z) = \frac{z[D_\lambda^{n+1} f(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^{n+1} g_i(z)},$$

where $q_i(z) = \frac{z[D_\lambda^n g_i(z)]'}{\frac{1}{m}\sum\limits_{i=1}^{m} D_\lambda^n g_i(z)}$. Also $-\frac{1}{m}\sum\limits_{i=1}^{m} q_i(z) \prec h(z)$.

Hence

$$S(\alpha; f; g_1, g_2, ..., g_m) = \frac{-\alpha z P'(z)}{-\frac{1}{m}\sum\limits_{i=1}^{m} q_i(z) + \frac{1}{\lambda} + 1} - P(z) \prec h(z),$$

since $f \in T(m;n;\lambda;\alpha;h)$. Now an application of Lemma 1.2 gives $-P(z) \prec h(z)$ which complete the proof. □

Theorem 4.2. *For $\alpha > \beta \geq 0$, $\lambda > 0$, $\operatorname{Re} h$ is bounded in E and $\left(1+\frac{1}{\lambda}\right) \geq d = \max\limits_{z\in E} \operatorname{Re} h(z)$, $T(m;n;\lambda;\alpha;h) \subset T(m;n;\lambda;\beta;h)$.*

Proof. The case $\beta = 0$ was treated in the previous theorem. Hence we assume that $\beta \neq 0$, $f \in T(m; n; \lambda; \alpha; h)$ implies that

$$S(\alpha; f, g_1, g_2, ..., g_m) \prec h(z). \tag{4.1}$$

By Theorem 4.1, we have

$$\frac{-z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z)} \prec h(z). \tag{4.2}$$

Now

$$S(\beta; f, g_1, g_2, ..., g_m) = -(1 - \frac{\beta}{\alpha})\frac{z[D_\lambda^n f(z)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z)} + \frac{\beta}{\alpha} S(\alpha; f, g_1, g_2, ..., g_m).$$

From (4.1) and (4.2) it follows that

$$\frac{-z[D_\lambda^n f(z_1)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z_1)} \in h(E)$$

and

$$\frac{-\alpha z_1[D_\lambda^{n+1} f(z_1)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^{n+1} g_i(z_1)} - (1-\alpha)\frac{z_1[D_\lambda^n f(z_1)]'}{\frac{1}{m}\sum_{i=1}^{m} D_\lambda^n g_i(z_1)} \in h(E).$$

Now h is convex and $\frac{\beta}{\alpha} < 1$, hence we have $S(\beta; f, g_1, g_2, ..., g_m)(z_1) \in h(E)$, showing that $f \in T(m; n; \lambda; \beta; h)$. □

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BREAKDOWN OF SOLUTIONS OF A SYSTEM DESCRIBING HEAT PROPAGATION WITH SECOND SOUND

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ABSTRACT. In this work we consider a hyperbolic quasilinear system describing the propagation of heat waves for rigid solids at very low temperature. We establish a blow-up result for classical solutions.

1. INTRODUCTION

In the classical theory of thermodynamics, heat conduction is viewed as a purely diffusive process, typically described using Fourier's Law. As a result we get the usual heat equation. This equation provides a useful description of heat conduction under a large range of conditions and predicts an infinite speed of propagation, that is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox (infinite speed propagation) and disturbances which are almost entirely thermal, may propagate in a finite speed. This wave-like of heat propagation is known as second sound. It was first detected in the **He**, and then in high purity dielectric crystals of sodium fluoride, **NaF**, and bismuth **Bi**. The range of temperature, for which the second sound is detectable, is in fact quite small and normal diffusive propagation takes place above it.

In this theory it is assumed that the heat flux satisfies Cattaneo's law

$$\tau(\theta)q_t + q = -k(\theta)\,\theta_x, \tag{1.1}$$

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where θ is the absolute temperature, q is the heat flux, and τ and k are strictly positive functions depending on the absolute temperature. With this relation, the internal energy, given by

$$e = \hat{e}(\theta), \tag{1.2}$$

is no longer compatible with the second Law of thermodynamics. Coleman, Fabrizio and Owen [1] showed in 1982 that, if (1.1) is adopted then compatibility with thermodynamics requires that (1.2) should be replaced by

$$e = \tilde{e}(\theta, q) = a(\theta) + b(\theta)q^2,$$

where b is a function determined by τ and k. In particular $b(\theta) > 0$. As a consequence, we obtain the system governing the evolution of θ and q

$$\begin{aligned} q_x - (a'(\theta) + b'(\theta)q^2)\theta_t + 2b(\theta)q\ q_t = 0 \\ \tau(\theta)q_t + q + k(\theta)q_x = 0. \end{aligned} \tag{1.3}$$

Global existence and decay of classical solutions to the Cauchy problem, as well as to some initial boundary value problems have been established by Coleman, Hrusa, and Owen [2]. In their paper, the authors used a classical energy argument to prove their result. Concerning the formation of singularities, Messaoudi [4] studied the following system

$$\begin{aligned} \tau(\theta)q_t = q + k(\theta)\,\theta_x = 0 \\ c(\theta)\,\theta_t + q_x = 0 \end{aligned}$$

and showed, under the same restrictions on τ, c and k, that classical solutions to the Cauchy problem break down in finite time if the initial data are chosen small in L^∞ norm with large enough derivatives. This result has been generalized by Messaoudi [5], [6] to the system (1.3).

Another approach to second sound is the one presented in [3], [8] and [9], where the authors introduced an internal parameter which accounts for the history memory effects of the heat flux. This approach gave rise to a new theory of heat conduction which we discuss in the next section.

2. DERIVATION OF EQUATIONS

In this section, we investigate the breakdown of classical solutions for the following quasilinear system:

$$v_t - p_x = 0 \tag{2.1}$$

$$p_t - (\sigma(v))_x = f(v)p \tag{2.2}$$

This system describes the propagation of heat wave for rigid solids at very low temperature, below about 20^0K.

Equation (2.1) comes from the balance of energy which in the one-dimensional case takes the form

$$(\varepsilon(v))_t + q_x = 0, \tag{2.3}$$

where $v > 0$ is the absolute temperature, ε is the internal energy, and q is the one-dimensional heat flux. Equation (2.2) is the evolution equation for an internal parameter p, which is introduced to account for memory effects the heat flux. The effect of memory may be considered, for example, as a functional of a history of temperature gradient,

$$q = -\alpha(v) \int_{-\infty}^{t} e^{-\sigma(t-s)} v_x(x,s) ds, \quad \alpha(v) > 0, \quad \sigma > 0. \tag{2.4}$$

By setting

$$p = \int_{-\infty}^{t} e^{-\sigma(t-s)}\, v_x(x,s)\, ds \tag{2.5}$$

equation (2.4) can be equivalently replaced by

$$q = -\alpha(v)p,$$

and a simple derivation shows that

$$p_t = -\sigma p + v_x \tag{2.6}$$

Equation (2.6) is linear and does not fully describe the properties of heat propagation in solids. To improve the model one may generalize the history dependence of q by modifying equation (2.4) as in [7] or, by introducing a suitable nonlinear dependence in (2.6) as in [8]. Namely,

$$p_t = g_1(v)\, v_x + g_2(v)p, \tag{2.7}$$

The functions α, g_1, and g_2 are material dependent. The second law of thermodynamics imposes the restrictions that $\alpha(v) = \kappa v^2 g_1(v)$ and $g_2(v) < 0$, where the constant $\kappa > 0$ comes from the Helmoltz free energy ψ which has the form $\psi = \psi_1(v) + \frac{1}{2}\kappa v p^2$. We also make an assumption that $g_1(v) > 0$. Combining (2.3), (2.4), and (2.7) we get the following system.

$$\begin{aligned} (\varepsilon(v))_t - (\alpha(v)p)_x &= 0 \\ p_t + (G_1(v))_x = g_2(v)p, \quad G_1'(v) &= -g_1(v). \end{aligned}$$

If we set $\varepsilon(v) = v$, $\alpha(v) = 1$, $\sigma = -G_1$, and $f = g_2$, system (2.1), (2.2) follows.

3. FORMATION OF SINGULARITIES

This section is devoted to the statement and the proof of the blow-up of solutions for the problem

$$\text{(3.1)} \qquad \begin{cases} v_t - p_x = 0, & x \in \mathbb{R}, \quad t \geq 0 \\ p_t - (\sigma(v))_x = f(v)p, & x \in \mathbb{R}, \quad t \geq 0 \\ v(x,0) = v_0(x), \quad p(x,0) = p_0(x), & x \in \mathbb{R}. \end{cases}$$

We first begin with a theorem which gives a pointwise upper bound on the solution in terms of the initial data.

Theorem 2.1 Assume that σ' and f are C^1 functions, with

$$\sigma'(y) > 0, \qquad |f(y)| \leq c_0, \, \forall y \in \mathbb{R}.$$

Let $v_0, p_0 \in H^2(\mathbb{R})$ be given. Then any solution (v,p) of problem (3.1) satisfies

$$\text{(3.2)} \qquad \max_{x\in\mathbb{R},\ 0\leq t\leq T} \{|v(x,t)| + |p(x,t)|\} \leq \gamma \max_{x\in\mathbb{R}} \{|v_0(x)| + |p_0(x)|\},$$

where γ is a positive constant which depends only on c_0 and T.

Proof. We introduce the quantities

$$\text{(3.3)} \qquad r(x,t) = p(x,t) - A(v(x,t)), \qquad s(x,t) = p(x,t) + A(v(x,t)),$$

where

$$A(v) = \int_0^v \sqrt{\sigma'(\xi)}\ d\xi$$

and the differential operators

$$\partial_t^+ = \frac{\partial}{\partial t} + \sqrt{\sigma'(v)}\,\frac{\partial}{\partial x}, \qquad \partial_t^- = \frac{\partial}{\partial t} - \sqrt{\sigma'(v)}\,\frac{\partial}{\partial x}. \tag{3.4}$$

We then compute

$$\begin{aligned} \partial_t^+ r &= r_t + \sqrt{\sigma'(v)}\, r_x \\ &= \left[-\sqrt{\sigma'(v)}\, v_t + p_t\right] + \sqrt{\sigma'(v)}\left[-\sqrt{\sigma'(v)}\, v_x + p_x\right] \\ &= \left[-\sqrt{\sigma'(v)}\, v_t + \sqrt{\sigma'(v)}\, p_x\right] + \left[p_t - \sigma'(v)\, v_x\right] \\ &= \sqrt{\sigma'(v)}\left[-v_t + p_x\right] + \left[p_t - \sigma'(v)\, v_x\right]. \end{aligned} \tag{3.5}$$

By using (2.1), (2.2), estimate (3.5) becomes

$$\partial_t^+ r = f(v)p. \tag{3.6}$$

Similar computations also lead to

$$\partial_t^- s = f(v)p. \tag{3.7}$$

We then define

$$R(t) := \max_x |r(x,t)|, \qquad S(t) := \max_x |s(x,t)| \tag{3.8}$$

The maxima in (3.8) are attained because r and s go to 0 as $x \to \pm\infty$ since they are H^1-functions. In this case, for any $t \in (0,T)$, we can choose x_1 and x_2 so that

$$R(t) = |r(x_1,t)|, \qquad S(t) = |s(x_2,t)|.$$

Therefore, for any $h \in (0,t)$, we have

$$R(t-h) \geq |r(x_1 - h\sqrt{\sigma'(v)}, t-h)|$$

Consequently

$$\begin{aligned} R(t) - R(t-h) &\leq |r(x_1,t)| - |r(x_1 - h\sqrt{\sigma'(v)},\ t-h)| \\ &\leq |r(x_1,t) - r(x_1 - h\sqrt{\sigma'(v)},\ t-h)|. \end{aligned}$$

By dividing by h and letting h go to 0, we obtain, for almost each $t \in (0,T)$,

$$R'(t) \le \left|\partial_t^+ r\right| = |f(v)\, p|. \tag{3.9}$$

Similarly, we can show that

$$S'(t) \le \left|\partial_t^- s\right| = |f(v)\, p| \tag{3.10}$$

We then add (3.9) to (3.10) and use (3.3), (3.8), and the boundedness of f, to get

$$\begin{aligned} \frac{d}{dt}[R(t)+S(t)] &\le 2|f(v)p| = c_0[r(x,t)+s(x,t)] \\ &\le c_0\,[R(t)+S(t)], \end{aligned} \tag{3.11}$$

for almost every $t \in [0,T)$. Integration of both sides of (3.11) gives

$$R(t)+S(t) \le (R(0)+S(0)) + c\int_0^t (R(\eta)+s(\eta))\, d\eta$$

and Gronwall's inequality leads to

$$(R(t)+S(t)) \le (R(0)+S(0))\, e^{c_0 t}, \quad \forall\, t \in [0,T]. \tag{3.12}$$

Therefore, (3.2) follows by using (3.3) and (3.8).

Theorem 2.2 Let σ' and f be as in Theorem 2.1. Assume further that $\sigma'' > 0$. Then given any $L > 0$, there exist initial data $v_0, p_0 \in H^2(\mathbb{R})$, for which the solution (v,p) blows up in finite time $T^* < L$.

Proof. We take an $x-$partial derivative of (3.6) to get

$$\begin{aligned} \left(\partial_t^+ r\right)_x &= \left(r_t + \sqrt{\sigma'(v)}\, r_x\right)_x \\ &= r_{tx} + \frac{1}{2}\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\, v_x\, r_x + \sqrt{\sigma'(v)}\, r_{xx} \\ &= \left(r_{tx} + \sqrt{\sigma'(v)}\, r_{xx}\right) + \frac{1}{2}\sigma''(v)\,\sigma'^{-\frac{1}{2}}\, v_x\, r_x \end{aligned} \tag{3.13}$$

$$= \partial_t^+ r_x + \frac{1}{2}\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\,v_x\,r_x = (f(v)p)_x.$$

From (3.3), we have

$$v_x = \frac{s_x - r_x}{2\sqrt{\sigma'(v)}}, \qquad p_x = \frac{s_x + r_x}{2}. \tag{3.14}$$

We substitute (3.14) in (3.13) to obtain

$$\begin{aligned} \partial_t^+ r_x &= \frac{-1}{2}\sigma''(v)\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\,\sigma'^{-\frac{1}{2}}(v)\,r_x + f'(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right)\left(\frac{r+s}{2}\right) \\ &\quad + f(v)\left(\frac{r_x + s_x}{2}\right). \end{aligned} \tag{3.15}$$

Now, we introduce $w = \alpha(v)r_x$ and compute

$$\begin{aligned} \partial_t^+ w &= \partial_t^+(\alpha(v)r_x) = \left[\partial_t^+\alpha(v)\right] r_x + \alpha(v)\partial_t^+ r_x \\ &= \left[\alpha_t(v) + \sqrt{\sigma'(v)}\,\alpha_x(v)\right] r_x + \alpha(v)\left[\frac{-1}{2}\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right) r_x \right. \\ &\quad \left. + f'(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right)\left(\frac{r+s}{2}\right) + f(v)\left(\frac{r_x + s_x}{2}\right)\right] \qquad (3.16) \\ &= \alpha'(v)\left[v_t + \sqrt{\sigma'(v)}\,v_x\right] r_x + \alpha(v)\left(\left[\frac{-\sigma''(v)(s_x - r_x)}{4\sigma'(v)}\,r_x\right]\right. \\ &\quad \left. + \frac{f'(v)(s_r - r_x)}{4\sqrt{\sigma'(v)}}\,(r+s) + f(v)\left(\frac{r_x + s_x}{2}\right)\right). \end{aligned}$$

At this point we choose $\alpha(v)$ so that

$$\alpha'(v)\left[v_t + \sqrt{\sigma'(v)}\,v_x\right] r_x - \frac{\sigma''(v)\,s_x r_x\,\alpha(v)}{4\sigma'(v)} = 0$$

By using (2.1), we arrive at

$$\alpha'(v)\left[p_x + \sqrt{\sigma'(v)}\,v_x\right]\,r_x - \frac{\sigma''(v)s_x\,r_x\,\alpha(v)}{4\sigma'(v)} = 0$$

and exploiting (3.14) we get

$$\alpha'(v)\left[p_x+\sqrt{\sigma'(v)}\,v_x\right]r_x=\frac{\sigma''(v)\alpha(v)[p_x+\sqrt{\sigma'(v)}\,v_x]}{4\sigma'(v)}r_x.$$

Therefore α satisfies

$$\alpha'(v)-\frac{\sigma''(v)\alpha(v)}{4\sigma'(v)}=0;$$

hence

$$\alpha(v)=[\sigma'(v)]^{\frac{1}{4}}.$$

By substituting in (3.16) we obtain

$$(3.17)\quad \partial_t^+ w=\sigma'(v)^{\frac{1}{4}}\left[\frac{\sigma''(v)}{4\sigma'(v)}r_x^2+\frac{f'(v)(s_x-r_x)}{4\sqrt{\sigma'(v)}}(r+s)+f(v)\left(\frac{r_x+s_x}{2}\right)\right]$$

$$=\frac{[\sigma'(v)]^{\frac{-3}{4}}\sigma''(v)}{4}r_x^2-\sigma'(v)^{\frac{1}{4}}\left(\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)-\frac{f(v)}{2}\right)r_x+\sigma'(v)^{\frac{1}{4}}\left(\frac{f'(v)(r+s)}{4\sqrt{\sigma'[(v)}}+\frac{f}{2}\right)s_x$$

$$=\frac{\sigma''(v)}{4\sigma'(v)^{\frac{5}{4}}}w^2-\left(\frac{f(v)}{4\sqrt{\sigma'(v)}}(r+s)-\frac{f(v)}{2}\right)w+\sigma'(v)^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)+\frac{f}{2}\right]s_x$$

The last terms in (3.17) can be handled as follows:

$$(3.18)\quad [\sigma'(v)]^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)\right]s_x=\sigma'(v)^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(2r+2A(v))\right]s_x$$

By using

$$s_x = \frac{-1}{2\sqrt{\sigma'(\imath)}}\,\partial_t^+\,(r-s)$$

we obtain from (3.18),

$$\begin{aligned}\sigma'(v)^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)\right]s_x &= \sigma'^{\frac{1}{4}}(v)\frac{f'(v)}{4\sqrt{\sigma'(v)}}(2r+2A(v))\left(\frac{-\partial_t^+(r-s)}{2\sqrt{\sigma'(v)}}\right)\\ &= \frac{-1}{4}(\sigma'(v))^{\frac{-3}{4}}f'(v)(r+A(v)\partial_t^+(r-s).\end{aligned} \tag{3.19}$$

By recalling (3.3), direct calculations yield

$$\partial_t^+(r-s) = \partial_t^+(-2A(v)) = \partial_t^+\left(-2\int_0^v \sqrt{\sigma'(\xi)}\,d\xi\right) = -2\sqrt{\sigma'(v)}\;\partial_t^+\,v$$

Thus (3.19) becomes

$$\begin{aligned}&\frac{-1}{4}(\sigma'(v))^{-\frac{3}{4}}[f'(v)(r+A(v)]\,\partial_t^+(r-s)\\ =\;& \frac{1}{2}(\sigma'(v))^{\frac{-1}{4}}f'(v)r\,\partial_t^+v + \frac{\sigma'(v)^{-\frac{1}{4}}}{2}f'(v)A(v)\partial_t^+v\\ =\;& \frac{1}{2}\left[\partial_t^+\left(r\int_0^v \sigma'(\xi)^{\frac{-1}{4}}f'(\xi)d\xi\right) - (\partial_t^+r)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi)\,d\xi\right]\\ &+\partial_t^+\left[\frac{1}{2}\int_0^v \sigma'(\xi)^{\frac{-1}{4}}f'(\xi)A(\xi)\,d\xi\right]\end{aligned} \tag{3.20}$$

By substituting (3.20) in (3.17) we deduce

$$\begin{aligned}\partial_t w &= \frac{\sigma'' w^2}{4\sigma^{\frac{5}{4}}} - \left(\frac{f'}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right)w\\ &\quad+\frac{1}{2}\partial_t^+\left(r\int_0^v \sigma'(\xi)^{\frac{-1}{4}}f'(\xi)d\xi\right) - \frac{1}{2}\left(\partial_t^+r\right)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi)\,d\xi\\ &\quad+\partial_t^+\left[\frac{1}{2}\int_0^v \sigma'(\xi)^{\frac{-1}{4}}f'(\xi)A(\xi)\,d\xi\right]\\ &= \frac{\sigma'' w^2}{4\sigma^{\frac{5}{4}}} - \left(\frac{f'}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right)w\\ &\quad-f(v)\left(\frac{r+s}{2}\right)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi)\,d\xi + \partial_t^+g,\end{aligned} \tag{3.21}$$

where
$$g=\frac{1}{2}\left[r\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)d\xi+\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)A(\xi)\,d\xi\right]$$
By setting $k=w-g$, (3.21) takes on the form

$$(3.22)\qquad \begin{aligned}\partial_t^+ k \;=\;& \frac{\sigma''(v)(k+g)^2}{4\sigma^{\frac{5}{4}}(v)}-\left(\frac{f'(v)}{4\sqrt{\sigma}}(r+s)-\frac{f(v)}{2}\right)(k+g)\\
&-\frac{f(v)(r+s)}{2}\int_0^v\left(\sigma'^{\frac{-1}{4}}f'\right)(\xi)d\xi\\
=\;&\frac{\sigma''(v)}{4\sigma'(v)^{\frac{5}{4}}}k^2+\left(\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g-\frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}}+\frac{f(v)}{2}\right)k\\
&+\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g^2-\left(\frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}}-\frac{f(v)}{2}\right)g\\
&-\frac{f(v)(r+s)}{2}\int_0^v(\sigma'f')(\xi)\,d\xi.\end{aligned}$$
By choosing initial data small enough (in L^∞ norm), we are guaranteed to have
$$a:=inf\left(\frac{\sigma''(v)\,\sigma'^{\frac{-5}{4}}(v)}{4}\right)>0.$$
and by setting
$$m:=\max\left|\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g^2-\left(\frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}}-\frac{f(v)}{2}\right)g-\frac{f(v)(r+s)}{2}\int_0^v(\sigma'f')(\xi)\,d\xi\right|$$
$$M=\max\left|\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g-\frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}}+\frac{f}{2}\right|$$
we get, from (3.22),
$$\partial_t^+ k\geq ak^2-Mk-m.$$
We then exploit Young's inequality

$$Mk\leq\frac{a}{2}k^2+\frac{1}{2a}M^2$$

to arrive at

$$\begin{aligned}\partial_t^+ k(t) &\geq ak^2 - \frac{a}{2}k^2 - \frac{1}{2a}M^2 - m \\ &\geq \frac{a}{2}k^2(t) - (\frac{1}{2a}M^2 + m).\end{aligned} \tag{3.23}$$

It suffices to choose v_0 and p_0 small enough in L^∞ norm with derivatives large enough so that

$$\frac{a}{4}k^2(0) > \frac{1}{2a}M^2 + m, \qquad \frac{4}{ak(0)} < L$$

consequently, (3.23) reduces to:

$$\partial_t^+ k(t) \geq \frac{a}{4}k^2(t). \tag{3.24}$$

Integration along the forward characteristics then yields

$$k(t) \geq \frac{1}{\frac{1}{k(0)} - \frac{a}{4}t}$$

Therefore, $k(t) \longrightarrow \infty$, as $t \to T^* \leq 4/ak(0) < L$. Hence, r_x blows up in finite time.

Remark 2.1. The blow up of r_x implies that either u_x or v_x (hence v_t or u_t) blows up in finite time. However, the solution (u, v) remains bounded in the L^∞ norm.

Remark 2.2. A similar result can also be obtained for certain initial boundary value problem.

Remark 2.3. We obtain the same blow-up result, if $\sigma < 0$. In this case we study the evolution of s_x over the backward characteristic.

Remark 2.4. The blow-up result also holds for $\sigma(0) \neq 0$. In this case a slight modification in the proof is needed.

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مجلة علمية محكمة تصدر نصف سنوياً عن الجمعية السعودية للعلوم الرياضية

بالتعاون مع قسم الرياضيات بجامعة الملك سعود

المجلد الثاني عشر **جمادى الأولى ١٤٢٧هـ**

العدد الأول **يونيو/حزيران ٢٠٠٦م**

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ARAB JOURNAL OF MATHEMATICAL SCIENCES

TABLE OF CONTENTS

Volume 12 Number 1 June 2006

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