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The publication of this issue of the Arab Journal of Mathematical Sciences was unexpectedly delayed by administrative and financial constraints beyond the control of the editorial board. We apologize to our readers, subscribers and authors for any inconvenience caused.

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IMPROVED CONVERGENCE ORDER FOR FINITE VOLUME SOLUTIONS. PART I: 1D PROBLEMS

BILAL ATFEH AND ABDALLAH BRADJI

ABSTRACT. We improve the convergence order of finite volume solutions approximating some second order elliptic problems in one dimension. The problem domain is discretized using an admissible mesh $\mathcal{T}$ with size h. We introduce the finite volume scheme of three points. The finite volume solution $u^{\mathcal{T}}$, of order h, can be computed using a tridiagonal matrix $\mathcal{A}^{\mathcal{T}}$. To obtain a new $O(h^2)$-order finite volume approximation $u_1^{\mathcal{T}}$ of u, we use the same matrix used for computing $u^{\mathcal{T}}$ and we combine Fox's idea of difference correction with three facts namely: the finite volume solution $u^{\mathcal{T}}$ is of order $O(h)$ in H_0^1-norm, the finite volume approximation depends on the second derivatives of the unknown solution u and the second derivatives can be expanded as a combination of u and its first derivative. In order to obtain finite volume approximations of order $O(h^{k+1})$, we repeat the same procedure, for k correction starting from $u^{\mathcal{T}}$, and we use the same matrix with slight changes in the second member of the original system.

1. INTRODUCTION

We consider finite volume discretization, over **arbitrary admissible meshes** $\mathcal{T}$, of linear second order elliptic problems in $\mathbb{R}$. We introduce the finite volume scheme of three points. The approximation leads to a system of algebraic equations denoted by $\mathcal{A}^{\mathcal{T}} u^{\mathcal{T}} = f^{\mathcal{T}}$, where $\mathcal{A}^{\mathcal{T}}$ is a tridiagonal matrix. The convergence order of the finite volume solution $u^{\mathcal{T}}$ is $O(h)$ in discrete H_0^1-norm.

AMS subject classification: 65L10, 65B05.

Key words: Second order elliptic boundary problems, Finite volume solution, Finite volume scheme of three point, Admissible meshes, Higher convergence orders.

Our aim in this article is to improve the convergence order of $u^{\mathcal{T}}$ by using the same matrix $\mathcal{A}^{\mathcal{T}}$ and by changing the second member $f^{\mathcal{T}}$ only. Improving the convergence order by using lower order schemes seems to be worthy. There is well developed literature concerning this subject in the framework of both finite difference and finite element methods. However, this topic has not attracted the attention it merits in the framework of finite volume methods.

Let us mention some works related to finite difference/element methods that we believe closest to ours. In order to use low order scheme to produce higher accurate approximations in finite difference methods, Fox [13] introduced a difference correction technique. In brief, he considered the discretization, (by central second-order differences on an equidistant grid), of the following second order boundary value problem on the interval (a, b)

$$\begin{cases} -z_{xx}(x) + p(x)z(x) = q(x), \ \ x \in (a, b), \\ \\ z(a) \text{ and } z(b) \text{ given.} \end{cases}$$

The truncation error can be expanded as a sum of higher order central differences. He replaced the values of the unknown solution of this sum by the values of the basic finite difference solution. The resulting truncation error was then used to produce an iterative process to correct the basic approximation in such a way that in each iteration one obtains a new approximation of higher order, (called correction), of the unknown solution by using the same matrix of the original system and by changing only the second member. Pereyra (1967-1973) was interested in the theoretical justification of Fox's idea in order to see what order of accuracy could be gained for each correction. In a series of papers [16, 17, 18, 19], he established a general principle for difference correction, from both theoretical and practical viewpoints. These ideas have been revived by many other techniques, such as Lindberg's deferred correction, Zadunaisky's global and Frank's local defect corrections (for more information see [6] and [20]). The theoretical justifications of these methods are mainly based on the existence of a **smooth asymptotic expansion of the global discretization error**. While the maximum principle has been the main tool in proving the existence of error expansions. In finite element methods, the defect correction technique was used to produce higher convergence orders. It was introduced by Barrett and Moore [5] and Moore [14], (in a one dimensional space), by using **uniform meshes**. They proved that, starting from the linear finite element solution of order $O(h^2)$ in higher divided difference norms and using the same tridiagonal matrix, one can obtain linear finite element approximations of order $O(h^{2k+2})$. Recently Butcher et al. [7] defined a defect correction technique for second-order two boundary value problems on nonequidistant meshes, i.e. those satisfying the so-called **supra-convergent mesh conditions** . Their idea is based on mono-implicit Runge-Kutta formulae [8]. Moore's approach [14] was reconsidered later by Chibi and Moore [10] in the two dimensional case. They proved, (always on **uniform meshes**), that one can do only one correction on the rectangle, (i.e. linear finite element approximations of order $O(h^4)$), and as many corrections as desired for semi-periodic problems, (i.e. linear finite element approximations of order $O(h^{2k+2})$). It is clear that the finite volume method is quite different from, (but sometimes related to), finite difference/element methods. The above results cannot be extended directly to the context of finite volume methods. For example, one reason is that the mesh considered in finite volume methods is arbitrary admissible, (in the sense of [11]), and the convergence order of $u^{\mathcal{T}}$ is just $O(h)$ in L^2 and H_0^1 norms. Nevertheless, we propose here the following approach to improve the convergence order of the finite volume solution $u^{\mathcal{T}}$. Namely, we prove that the convergence error of the finite volume solution depends on the second derivatives of the unknown solution u. The classical idea for approximating the second derivative in finite difference/element methods is to use the fact that the divided difference of order two of the numerical solution (finite difference or finite element solution) converges to the second derivative of the unknown solution (see [14]). While in finite volume methods the divided difference of order two of the finite volume solution does not converge to the second derivative of the unknown solution (cf. subection 5.2 of [11]). However, the second derivative u_{xx} can be expanded as a combination of the solution itself, its first derivative as well as the given data. Combining this with the fact that the convergence order of $u^{\mathcal{T}}$ is $O(h)$ in H_0^1-norm and using the values of the basic finite volume solution $u^{\mathcal{T}}$, we obtain an optimal approximation of the second derivative. This approximation allows us to correct $u^{\mathcal{T}}$ and to obtain a new approximation $u_1^{\mathcal{T}}$ which can be computed by using $\mathcal{A}^{\mathcal{T}}$. It is called the first correction and it is of order $O(h^2)$ in the discrete H_0^1-norm. Another way to compute an optimal approximation of u_{xx} is to use the fact that it satisfies the same equation as the solution itself but with different second member and boundary conditions. This allows us to obtain an optimal approximation of the second derivative of the unknown solution by using the matrix $\mathcal{A}^{\mathcal{T}}$. We repeat this process successively to obtain, (for each integer k), a finite volume approximation $u_k^{\mathcal{T}}$ of order $O(h^{k+1})$ by using only the matrix $\mathcal{A}^{\mathcal{T}}$ of the original system. The new approximation $u_k^{\mathcal{T}}$ is called the k^{th} correction.

Our paper is organized as follows: in Section 2, we introduce the finite volume scheme of three points and we give the convergence order of the finite volume solution. In Section 3, we define the first correction and we prove that its convergence order is $O(h^2)$. The convergence analysis is done in the framework of Sobolev spaces. To simplify the presentation, we begin with desribing the first correction and then we give the general formulation of higher order corrections. In Theorems 3.1 and 4.1, we prove that on arbitrary admissible meshes the convegence order of the basic solution can be improved by $O(h)$ per correction. At the end of the paper, we provide several numerical tests showing that the first correction defined by (3.7) improves the basic convergence considerably. Finally, we should mention that the approach used here can be used, (work in progress), to improve the convergence order of finite element solutions on **arbitrary non uniform meshes**.

2. BASIC RESULTS AND PRELIMINARIES

The problem domain is the interval $I = (0, 1)$. The results of this article are presented in the spaces of continuous functions and in Sobolev spaces. For integers m, denote by $H^m(I)$ (resp. $\mathcal{C}^m(\bar{I})$) the Sobolev space of functions which together with their generalized derivatives, up to order m inclusive, are in $L^2(I)$ (resp. the space of continuous functions which together with their derivatives up to order m inclusive are in $\mathcal{C}(\bar{I})$). The norms are respectively defined by

$$\|w\|_{m,I} = \left(\sum_{|k| \leq m} \int_I (w^{(k)})^2 dx \right)^{\frac{1}{2}},$$

$$\|w\|_{m,\infty,\bar{I}} = \max_{|k| \leq m} (\max_{\bar{I}} |w^{(k)}|),$$

where $w^{(k)}$ denotes the usual derivative of order k of w. The space $H^1(I) \cap \{v : v(0) = v(1) = 0\}$ is denoted by $H_0^1(I)$. Basic results given here can be found in Eymard et al. [11]. Let f be a given function defined on $(0,1)$ and consider the following problem

$$
\text{(2.1)} \qquad \begin{cases} -u_{xx}(x) + \alpha u_x(x) + \beta u(x) = f(x), \; x \in I = (0,1), \\ u(0) = u(1) = 0, \end{cases}
$$

where $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Remark 2.1. For $\beta \geq 0$, Problem (2.1) admits always a unique solution $u \in H_0^1(I)$. Indeed,

1. In case $\alpha \geq 0$, thanks to Lax-Milgram theorem, there exists a unique variational solution $u \in H_0^1(I)$ such that

$$
\text{(2.2)} \qquad \int_I u_x(x)v_x(x)dx + \alpha \int_I u_x(x)v(x)dx + \beta \int_I u(x)v(x)dx = \int_I f(x)v(x)dx, \; \forall v \in\in H_0^1(I).
$$

2. When $\alpha < 0$, it suffices to change the variable x in (2.1) by $1-x$. The resulting equation satisfies hypothesis of Lax-Milgram theorem.

In order to compute the finite volume approximation of the solution u of (2.1), we use the following definition of the admissible mesh (cf. [11]).

Definition 2.1. **(Admissible one-dimensional mesh)** An admissible mesh of $(0,1)$, denoted by $\mathcal{T}$, is given by a family $(K_i)_{i=1,\dots,N}$, $N \in \mathbb{N}^\star$ with $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and a family $(x_i)_{i=0,\dots,N+1}$ such that

$$
x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < \dots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \dots < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1,
$$

and

$$
h_i = \mathrm{m}\,(K_i) = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \text{ for } i \in \{1, \dots, N\},
$$

$$
h_i^- = x_i - x_{i-\frac{1}{2}}, h_i^+ = x_{i+\frac{1}{2}} - x_i, \text{ for } i \in \{1, \dots, N\}, \; h_0^+ = h_{N+1}^- = 0,
$$

$$
h_{i+\frac{1}{2}} = x_{i+1} - x_i, i = 0, \dots, N \text{ and size } (\mathcal{T}) = h = \max\{h_i, i = 1, \dots, N\}.
$$

Let us introduce the space of piecewise constant functions and the discrete H_0^1 norm, in which we shall analyse the convergence of finite volume approximations.

Definition 2.2. (Finite volume space and discrete H_0^1 norm) Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 and $\mathcal{X}(\mathcal{T})$ be the space of functions from I to $\mathbb{R}$ which are constant over each K_i. For $v_\mathcal{T} \in \mathcal{X}(\mathcal{T})$, we define the discrete H_0^1-norm by

$$\|v_\mathcal{T}\|_{1,\mathcal{T}} = \left(\sum_{0 \le i \le N} \frac{(v_{i+1} - v_i)^2}{h_{i+\frac{1}{2}}} \right)^{\frac{1}{2}}, \tag{2.3}$$

where v_i denotes the value taken by $v_\mathcal{T}$ on the control volume K_i with $v_0 = v_{N+1} = 0$.

Remark 2.2. We shall naturally identify the set $\mathbb{R}^N$ with the set $\mathcal{X}(\mathcal{T})$.

The discrete unknowns of the finite volume solution $u^\mathcal{T}$ are denoted by u_i, $i = 1, ..., N$, and they are expected to be some approximation of the solution u (of (2.1)) in the control volume K_i. To define the equation satisfied by u_i, $i = 1, ..., N$, we integrate Equation (2.1) over each control volume K_i. This yields

(2.4)

$$-u_x(x_{i+\frac{1}{2}}) + u_x(x_{i-\frac{1}{2}}) + \alpha u(x_{i+\frac{1}{2}}) - \alpha u(x_{i-\frac{1}{2}}) + \int_{K_i} \beta u(x)dx, \ i = 1, ..., N.$$

We remark that

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+\frac{1}{2}}) - u(x_{i-\frac{1}{2}})}{h_{i+\frac{1}{2}}} + R_{i+\frac{1}{2}}, \ i = 0, ..., N, \tag{2.5}$$

where, provided that $u \in H^2(I)$, $R_{i+\frac{1}{2}}$ satisfies

$$|R_{i+\frac{1}{2}}|, \ ch, \ i = 0, ..., N. \tag{2.6}$$

This allows us to approximate $u_x(x_{i+\frac{1}{2}})$ by using the discrete flux $\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}}$, for any $i \in \{0, ..., N\}$ where, (thanks to the boundary condition $u(0) = u(1) = 0$), we have

$$u_0 = u_{N+1} = 0. \tag{2.7}$$

The convective term $\alpha u(x_{i+\frac{1}{2}})$ is approximated by αu_i because of stability consideration. Indeed, this choice with the fact that $\alpha \ge 0$ yield a stability result. The term $\int_{K_i} \beta u(x)dx$ is approximated by $\beta h_i u_i$. Therefore, the three point numerical scheme related to the approximation of Problem (2.1) reads as

$$\left\{ \begin{array}{l} \mathcal{A}^{\mathcal{T}} u^{\mathcal{T}} = b, \\ u_0 = u_{N+1} = 0, \end{array} \right. \tag{2.8}$$

where $u^{\mathcal{T}} = (u_1, ..., u_N)^t, b = (b_1, ..., b_N)^t$ with $\mathcal{A}^{\mathcal{T}}$ and b are defined by

$$(\mathcal{A}^{\mathcal{T}} u^{\mathcal{T}})_i = -\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} + \alpha(u_i - u_{i-1}) + \beta h_i u_i, \;\; i = 1, ..., N, \tag{2.9}$$

$$b_i = \int_{K_i} f(x)\, dx, \; i = 1, ..., N. \tag{2.10}$$

Remark 2.3. In case $\alpha < 0$ and in order to get the stability result, the previous choice αu_i to approximate $\alpha u(x_{i+\frac{1}{2}})$ should be replaced by αu_{i+1} (cf. Section 7 of [11]).

To make our exposition clearer, we introduce the following:

Definition 2.3. Define the linear mapping $\mathcal{A}^{\mathcal{T}}$ from $\mathcal{C}(\bar{I})$ into $\mathbb{R}^N$ by

$$\begin{aligned} (\mathcal{A}^{\mathcal{T}} g)_i = &-\frac{g(x_{i+1}) - g(x_i)}{h_{i+\frac{1}{2}}} + \frac{g(x_i) - g(x_{i-1})}{h_{i-\frac{1}{2}}} \\ &+ \alpha(g(x_i) - g(x_{i-1})) + \beta h_i g(x_i), \; i = 1, ..., N, \end{aligned} \tag{2.11}$$

where $(\mathcal{A}^{\mathcal{T}} g)_i, \;\; i = 1, ..., N$ denote the components of $\mathcal{A}^{\mathcal{T}} g$.

Remark 2.4. Throughout this paper, we use the following notations:

1. $\sum_i^1$ (resp. $\sum_i$) denotes $\sum_{i=1}^{i=N}$ (resp. $\sum_{i=0}^{i=N}$).
2. The letter c stands for a generic positive number possibly different at each appearance but 'constant' in that it is independent of discretization parameters $\{h, h_i, h_{i+\frac{1}{2}}, h_{i-\frac{1}{2}}, i, ...\}$.

In the sequal, we shall use the following properties of the discrete H_0^1-norm:

Lemma 2.1. (Discrete Sobolev's inequality) Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 and $v_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$, then

$$\|v_{\mathcal{T}}\|_{L^\infty(I)} \le \|v_{\mathcal{T}}\|_{1,\mathcal{T}}. \tag{2.12}$$

Proof. Let $v_{\mathcal{T}} = (v_i)_{i=1}^N$ and $v_0 = v_{N+1} = 0$, then

$$v_i = \sum_{1 \le j \le i} (v_j - v_{j-1}), \ \forall i \in \{1, ..., N\}. \tag{2.13}$$

The latter together with Cauchy-Schwarz inequality yield

$$|v_i| \le \left(\sum_i h_{i+\frac{1}{2}} \right) \|v_{\mathcal{T}}\|_{1,\mathcal{T}}. \tag{2.14}$$

Since $\sum_i h_{i+\frac{1}{2}} = 1$ in (2.14), one deduces (2.12). □

Lemma 2.1 with the fact that $\sum_i h_{i+\frac{1}{2}} = \sum_i h_i = 1$ imply

Lemma 2.2. (Discrete Poincaré's inequality) Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1 and $v_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$, then

$$\|v_{\mathcal{T}}\|_{0,I} \le \|v_{\mathcal{T}}\|_{1,\mathcal{T}}, \tag{2.15}$$

and

$$\left(\sum_{1 \le i \le N} h_{i+\frac{1}{2}} v_i^2 \right)^{1/2} \le \|v_{\mathcal{T}}\|_{1,\mathcal{T}}. \tag{2.16}$$

Remark 2.5. Inequality (2.15) means that

$$\left(\sum_i^1 h_i \, v_i^2 \right)^{1/2} \le \|v_{\mathcal{T}}\|_{1,\mathcal{T}}, \tag{2.17}$$

which is different from Inequality (2.16).

Combining Lemmata 2.1, 2.2 and Theorem 7.1 of [11], we get

Theorem 2.1. (Convergence order of the basic finite volume solution) Let $\alpha, \beta \ge 0$ and $f \in L^2(I)$. Let u be the (unique) solution $u \in H_0^1(I)$ of (2.1). Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1. Then there exists a unique solution $u^{\mathcal{T}}$ of (2.8). Assume that the solution u satisfies $u \in H^2(I)$. For each $K_i \in \mathcal{T}$, let $e_i = u(x_i) - u_i$, and define $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ by $e_{\mathcal{T}}(x) = e_i$, *a.e.* $x \in K_i$. Then the following error estimates hold:

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \le ch\|u\|_{2,I}, \tag{2.18}$$

$$\|e_{\mathcal{T}}\|_{0,I} \le ch\|u\|_{2,I}, \tag{2.19}$$

$$\left(\sum_i h_{i+\frac{1}{2}} e_i^2\right)^{\frac{1}{2}} \le ch\|u\|_{2,I}, \tag{2.20}$$

$$\|e_{\mathcal{T}}\|_{L^\infty(I)} \le ch\|u\|_{2,I}. \tag{2.21}$$

Remark 2.6.

1. Estimates (2.18), (2.19) with (2.20), and (2.21) can be respectively regarded as discrete H_0^1, L^2 and L^∞-estimates on the error.
2. Estimates (2.19) and (2.20) can be deduced from Estimate (2.18) using discrete Poincaré inequalities (2.15) and (2.16), whereas Estimate (2.21) can be deduced from Estimate (2.18) using the discrete Sobolev inequality (2.12).

Remark 2.7. The above Theorem gives an error estimate of order $O(h)$ in H_0^1 and L^2-norms. Indeed, in general, the convergence order is $O(h)$ in both norms. To justify this, consider $\alpha = \beta = 0$ and $u = \sin(\pi x)$ with an admissible mesh defined by $x_{i+\frac{1}{2}} = \frac{2x_i + x_{i+1}}{3}$ for $i = 1, ..., N-1$ and $h_{i+\frac{1}{2}} = h$. The numerical tests, (see tables 6 and 7 of Section 5), show that the order is $O(h)$ in both H_0^1 and L^2-norms. But, in case $\alpha = \beta = 0$, sometimes the error can be $O(h^2)$.

1. In case $u \in \mathcal{C}^4(\bar{I})$ and x_i is the center of K_i for all $i = 1, ...N$, Forsyth et al. [12] proved that the order is $O(h^2)$ in L^∞ and hence in L^2-norm too, while it is only $O(h)$ in H_0^1-norm (see tables 3 and 4 of Section 5).
2. If $x_{i+\frac{1}{2}}$ is the center of $[x_i, x_{i+1}]$, for all $i \in \{1, ..., N-1\}$, and $x_1 = 0, x_N = 1$, then the convergence order is $O(h^2)$ in H_0^1-norm and consequently in L^2 and L^∞-norms (cf. [11]).

In order to give general formulation of an arbitrary **correction**, i.e. finite volume approximation of u of order $O(h^{k+1})$ which can be computed by using the same matrix $\mathcal{A}^{\mathcal{T}}$, we are required to investigate the first **correction** .

3. THE FIRST CORRECTION

By integrating both sides Equation (2.1) over each finite volume K_i, we get

$$-u_x(x_{i+\frac{1}{2}}) + u_x(x_{i-\frac{1}{2}}) + \alpha u(x_{i+\frac{1}{2}}) - \alpha u(x_{i-\frac{1}{2}}) + \beta \int_{K_i} u\,dx = \int_{K_i} f\,dx. \tag{3.1}$$

Then, the convergence order of the finite volume solution depends on the orders of the approximation of the flux, $u(x_{i+\frac{1}{2}})$, $u(x_{i-\frac{1}{2}})$ and $\int_{K_i} u\,dx$. We shall use this idea combined with Fox's one to improve the convergence order of the basic solution $u^{\mathcal{T}}$ on the same scheme, i.e. using the same matrix $\mathcal{A}^{\mathcal{T}}$ that is used to compute the basic solution $u^{\mathcal{T}}$ and changing only the r.h.s of (2.8). Now, we look for an expansion of the error. Assuming $u \in H^3(I)$, by Sobolev's imbedding this implies that $u \in \mathcal{C}^2(\bar{I})$. Using Taylor's formula and the fact that $\mathcal{D}(\bar{I})$ is dense in $H^3(I)$, we obtain

$$\begin{aligned}(\mathcal{A}^{\mathcal{T}} u)_i = &\int_{K_i} f dx - \frac{h_{i+1}^- - h_i^+}{2} u_{xx}(x_{i+\frac{1}{2}}) \\ &+ \frac{h_i^- - h_{i-1}^+}{2} u_{xx}(x_{i-\frac{1}{2}}) - \alpha(h_i^+ u_x(x_i) - h_{i-1}^+ u_x(x_{i-1})) \\ &- \frac{\beta}{2}\left(h_i^{+2} u_x(x_i) - h_i^{-2} u_x(x_{i-1})\right) \\ &- (R^1_{i+\frac{1}{2}} - R^1_{i-\frac{1}{2}}) - \alpha(S^1_{i+\frac{1}{2}} - S^1_{i-\frac{1}{2}}) - \beta T^1_i.\end{aligned} \tag{3.2}$$

where

$$\left(\sum_i h_{i+\frac{1}{2}} (R^1_{i+\frac{1}{2}})^2\right)^{\frac{1}{2}} \le ch^2 \|u\|_{3,I}, \tag{3.3}$$

$$|S^1_{i+\frac{1}{2}}| \le ch_i^{+2} |u|_{2,\infty,\bar{I}},\ \forall i \in \{0, ..., N\}. \tag{3.4}$$

$$|T^1_i| \le ch^2 h_i |u|_{2,\infty,\bar{I}},\ \forall i \in \{1, ..., N\}. \tag{3.5}$$

Remark 3.1. In order to prove that the forthcoming improvements are valid for a general admissible mesh without additional conditions, we write Inequalities (3.4) and (3.5) as well as the forthcoming ones in the above manner (namely, they are estimated not only with respect to h but with respect to $h_i, h_i^+, ...$ as well). For instance, see the estimate of the second term of (3.14).

Because $u \in \mathcal{C}^2(\bar{I})$, Equation (2.1) makes sense for all x belonging to $\bar{I}$, thus Equality (3.2) becomes

$$
\begin{aligned}
(3.6) \qquad (\mathcal{A}^{\mathcal{T}}\, u)_i = & \int_{K_i} f\,dx \\
& - \frac{h_{i+1}^- - h_i^+}{2}\left(\alpha u_x(x_{i+\frac{1}{2}}) + \beta u(x_{i+\frac{1}{2}}) - f(x_{i+\frac{1}{2}})\right) \\
& + \frac{h_i^- - h_{i-1}^+}{2}\left(\alpha u_x(x_{i-\frac{1}{2}}) + \beta u(x_{i-\frac{1}{2}}) - f(x_{i-\frac{1}{2}})\right) \\
& - \alpha(h_i^+ u_x(x_i) - h_{i-1}^+ u_x(x_{i-1})) \\
& - \frac{\beta}{2}\left((h_i^+)^2 u_x(x_i) - (h_i^-)^2 u_x(x_{i-1})\right) \\
& - (R^1_{i+\frac{1}{2}} - R^1_{i-\frac{1}{2}}) - \alpha(S^1_{i+\frac{1}{2}} - S^1_{i-\frac{1}{2}}) - \beta T^1_i. \\
= & \int_{K_i} f\,dx + \mathbf{d_i} - (R^1_{i+\frac{1}{2}} - R^1_{i-\frac{1}{2}}) \\
& \qquad - \alpha(S^1_{i+\frac{1}{2}} - S^1_{i-\frac{1}{2}}) - \beta T^1_i.
\end{aligned}
$$

After having found an appropriate expansion of the error, straightforward approximation ${\mathbf{d_i}}^{\mathcal{T}}$ of the defect $\mathbf{d_i}$ can be done by replacing u by $u^{\mathcal{T}}$ and u_x with $(\frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}})_i$, where u_i are the components of $u^{\mathcal{T}}$. The resulting defect ${\mathbf{d_i}}^{\mathcal{T}}$ can be used to correct $u^{\mathcal{T}}$ in order to produce **first correction** $u_1^{\mathcal{T}} = (u_i^1)_{i=1}^N$ (which will improve the basic finite volume solution). Thus, $u_1^{\mathcal{T}}$ can be defined by the boundary conditions $u_0^1 = u_{N+1}^1 = 0$ and the equation, $\forall i \in \{1, ..., N\}$

$$
\begin{aligned}
(3.7) \qquad (\mathcal{A}^{\mathcal{T}}\, u_1^{\mathcal{T}})_i = & \int_{K_i} f\,dx - \frac{h_{i+1}^- - h_i^+}{2}\left(\alpha\partial_1 u_i + \beta u_i - f(x_{i+\frac{1}{2}})\right) \\
& + \frac{h_i^- - h_{i-1}^+}{2}\left(\alpha\partial_1 u_{i-1} + \beta u_{i-1} - f(x_{i-\frac{1}{2}})\right) \\
& - \alpha(h_i^+\partial_1 u_i - h_{i-1}^+\partial_1 u_{i-1}) \\
& - \frac{\beta}{2}\left((h_i^+)^2\partial_1 u_i - (h_i^-)^2\partial_1 u_{i-1}\right),
\end{aligned}
$$

where $\partial_1 u_i = \frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}}$ and $\mathcal{A}^{\mathcal{T}}$ is the matrix defined by (2.9).

3.1. THE CONVERGENCE ORDER OF THE FIRST CORRECTION.

To analyse the convergence of the first correction, we follow the same proof used in proving the convergence order of the basic solution $u^{\mathcal{T}}$ (see [11]).

Consider now the Error $e^1_{\mathcal{T}} = (e^1_i)_i$ in first correction

$$e^1_i = u^1_i - u(x_i),\ i \in \{1, ..., N\}. \tag{3.8}$$

By subtracting (3.6) from (3.7), side by side, we see that the error $e^1_i = u^1_i - u(x_i)$ satisfies

$$-\frac{e^1_{i+1} - e^1_i}{h_{i+\frac{1}{2}}} + \frac{e^1_i - e^1_{i-1}}{h_{i-\frac{1}{2}}} + \alpha(e^1_i - e^1_{i-1}) + \beta h_i e^1_i = \gamma^1_{i+\frac{1}{2}} - \gamma^1_{i-\frac{1}{2}} + \delta^1_i, \tag{3.9}$$

where

$$\gamma^1_{i+\frac{1}{2}} = -\frac{h^-_{i+1} - h^+_i}{2}\left(\alpha(\partial_1 u_i - u_x(x_{i+\frac{1}{2}})) + \beta(u_i - u(x_{i+\frac{1}{2}}))\right) - \alpha h^+_i(\partial_1 u_i - u_x(x_i)) + R^1_{i+\frac{1}{2}} + \alpha S^1_{i+\frac{1}{2}}. \tag{3.10}$$

$$\delta^1_i = -\frac{\beta}{2}\left((h^+_i)^2(\partial_1 u_i - u_x(x_i)) - (h^-_i)^2(\partial_1 u_{i-1} - u_x(x_{i-1}))\right) + \beta T^1_i. \tag{3.11}$$

Multiplying both sides of (3.9) by e^1_i and summing from $i = 1$ to $i = N$, we get

$$-\sum_i{}^1 \frac{e^1_{i+1} - e^1_i}{h_{i+\frac{1}{2}}} e^1_i + \sum_i{}^1 \frac{e^1_i - e^1_{i-1}}{h_{i-\frac{1}{2}}} e^1_i + \alpha \sum_i{}^1 (e^1_i - e^1_{i-1}) e^1_i + \beta \sum_i{}^1 h_i {e^1_i}^2 = \sum_i{}^1 \gamma^1_{i+\frac{1}{2}} e^1_i - \sum_i{}^1 \gamma^1_{i-\frac{1}{2}} e^1_i + \sum_i{}^1 \delta^1_i e^1_i. \tag{3.12}$$

Noting that $e_0 = e_{N+1} = 0$ and reordering by parts, this yields

$$\sum_i \frac{(e^1_{i+1} - e^1_i)^2}{h_{i+\frac{1}{2}}} + \alpha \sum_i{}^1 (e^1_i - e^1_{i-1}) e^1_i + \beta \sum_i{}^1 h_i {e^1_i}^2 = \sum_i{}^1 \gamma^1_{i+\frac{1}{2}} e^1_i - \sum_i{}^1 \gamma^1_{i-\frac{1}{2}} e^1_i + \sum_i{}^1 \delta^1_i e^1_i. \tag{3.13}$$

Remark that

$$\sum_i{}^1 e^1_i (e^1_i - e^1_{i-1}) = \frac{1}{2} \sum_{i=1}^{N+1} (e^1_i - e^1_{i-1})^2.$$

This together with the fact that $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$ in (3.13) imply

$$\|e^1_{\mathcal{T}}\|^2_{1,\mathcal{T}} \le \left(\sum_i h_{i+\frac{1}{2}} {\gamma_{i+\frac{1}{2}}}^2\right)^{\frac{1}{2}} \|e_{\mathcal{T}}\|_{1,\tau} + |\sum_i{}^1 \delta^1_i e^1_i|. \tag{3.14}$$

Inequalities (3.3) and (3.4) combined with the triangular inequality give the following estimate

$$\left(\sum_i h_{i+\frac{1}{2}} {\gamma_{i+\frac{1}{2}}}^2\right)^{\frac{1}{2}} \le ch\left(\sum_{j=1}^{3} \mathbb{E}_j + h\|u\|_{3,I}\right), \tag{3.15}$$

where

$$\mathbb{E}_1 = \left(\sum_i h_{i+\frac{1}{2}} (u_i - u(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}}, \tag{3.16}$$

$$\mathbb{E}_2 = \left(\sum_i h_{i+\frac{1}{2}} (\partial_1 u_i - u_x(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}}, \tag{3.17}$$

$$\mathbb{E}_3 = \left(\sum_i h_{i+\frac{1}{2}} (\partial_1 u_i - u_x(x_i))^2\right)^{\frac{1}{2}}. \tag{3.18}$$

Combining the triangular and Cauchy-Schwarz inequalities with Inequality (3.5) and using Sobolev imbeddings, we get

$$\begin{aligned} |\sum_i{}^1 \delta_i^1 e_i^1| \; &\le \; ch\left(\sum_i{}^1 h_i^+ (\partial_1 u_i - u_x(x_i))^2\right)^{\frac{1}{2}} \\ &\qquad \times \left(\sum_i{}^1 h_i^+ (e_i^1)^2\right)^{\frac{1}{2}} \\ &\quad + ch\left(\sum_i{}^1 h_i^- (\partial_1 u_{i-1} - u_x(x_{i-1}))^2\right)^{\frac{1}{2}} \\ &\qquad \times \left(\sum_i{}^1 h_i^- (e_i^1)^2\right)^{\frac{1}{2}} \\ &\quad + ch^2 \|u\|_{3,I} \|e^1\|_{L^2}. \end{aligned} \tag{3.19}$$

But

$$\left(\sum_i{}^1 h_i^+ (\partial_1 u_i - u_x(x_i))^2\right)^{\frac{1}{2}} \le \mathbb{E}_3, \tag{3.20}$$

and

$$(3.21)\qquad \left(\sum_i^1 h_i^-(\partial_1 u_{i-1} - u_x(x_{i-1}))^2\right)^{\frac{1}{2}} = \left(\sum_{i=0}^{N-1} h_{i+1}^-(\partial_1 u_i - u_x(x_i))^2\right)^{\frac{1}{2}} \leq \mathbb{E}_3.$$

Combining the fact that $h_i^-, h_i^+ \leq h_i$ with (3.20), (3.21) as well as discrete Poincaré's inequality (2.15), we obtain

$$(3.22)\qquad |\sum_i^1 \delta_i^1 e_i^1| \leq ch\left(\mathbb{E}_3 + h\|u\|_{2,I}\right)\|e_{\mathcal{T}}^1\|_{1,\mathcal{T}}.$$

Combining Inequalities (3.14), (3.15) and (3.22) yields

$$(3.23)\qquad \|e_{\mathcal{T}}^1\|_{1,\tau} \leq ch\left(\sum_{j=1}^3 \mathbb{E}_j + h\|u\|_{3,I}\right).$$

To estimate the r.h.s of Inequality (3.23), we need the following estimates

Lemma 3.1. *Let $u^{\mathcal{T}} = (u_i)$ be the basic solution defined by (2.8) and u be the solution of (2.1). Let $\mathbb{E}_j$ be the expansions defined by (3.16), (3.17) and (3.18). Assume that u satisfies $u \in H^2(I)$. Then, the following estimates hold*

$$(3.24)\qquad \mathbb{E}_j \leq ch\|u\|_{2,I},\ \forall j \in \{1, 2, 3\}.$$

Proof.

1. By the triangular inequality, we have

$$(3.25)\qquad \mathbb{E}_1 \leq \left(\sum_i h_{i+\frac{1}{2}}(u_i - u(x_i))^2\right)^{\frac{1}{2}} + \left(\sum_i h_{i+\frac{1}{2}}(u(x_i) - u(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \leq \left(\sum_i h_{i+\frac{1}{2}} e_i^2\right)^{\frac{1}{2}} + ch\|u\|_{2,I}.$$

Using Estimate (2.20) to majorize the first term of the the r.h.s. of (3.25), we obtain the desired estimate of $\mathbb{E}_1$.

2. Using the same technique yields

$$\begin{aligned}\mathbb{E}_2 \quad &\leq \quad \left(\sum_i h_{i+\frac{1}{2}}(\partial_1 u_i - \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}})^2\right)^{\frac{1}{2}} \\ &+ \quad \left(\sum_i h_{i+\frac{1}{2}}(\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - u_x(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \leq ch\|u\|_{2,I}.\end{aligned}$$

3. $\mathbb{E}_3$ can be estimated similarly; which completes the proof.

□

Combining Lemma 3.1 with Inequality (3.23) yields the following $O(h)$-improvement:

Theorem 3.1. *Let $\alpha, \beta \geq 0$ and $f \in L^2(I)$. Let u be the (unique) solution $u \in H_0^1(I)$ of (2.1). Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1. Then there exists a unique solution $u^{\mathcal{T}}$ of (2.8) and a unique solution $u^{\mathcal{T}}$, (called first correction), of Equation (3.7). Assume that the solution u satisfies $u \in H^3(I)$. For each $K_i \in \mathcal{T}$, let $e_i^1 = u(x_i) - u_i$ and define $e_{\mathcal{T}}^1 \in \mathcal{X}(\mathcal{T})$ by $e_{\mathcal{T}}^1(x) = e_i^1$, a.e. $x \in K_i$. Then the error $e_{\mathcal{T}}^1$ is of order $O(h^2)$. In other words, the following error estimates hold:*

$$\|e_{\mathcal{T}}^1\|_{1,\mathcal{T}} \leq ch^2\|u\|_{3,I}, \tag{3.26}$$

$$\|e_{\mathcal{T}}^1\|_{L^2} \leq ch^2\|u\|_{3,I}. \tag{3.27}$$

Remark 3.2. **1.** According to Remark 2.7, (i.e. the convergence order is $O(h)$ in general), Theorem 3.1 means that the first correction (3.7) considerably improves the basic finite volume solution (2.8) on general admissible mesh.

2. In the particular case $\alpha = \beta = 0$ and if $x_1 = 0, x_N = 1$ and if $x_{i+\frac{1}{2}}$ stands for the center of $[x_i, x_{i+1}]$ for $i \in \{1, ..., N-1\}$, the second member of Equation (3.7) does not contain any "correction", i.e. (3.7) and (2.8) have the same second member. This means that the first correction $u_1^{\mathcal{T}}$ concides with the finite volume solution $u^{\mathcal{T}}$ and therefore, (thanks to Theorem 3.1), the convergence order of the basic finite volume solution $u^{\mathcal{T}}$ is $O(h^2)$. This confirms the second item of Remark 2.7.

3.2. **ANOTHER WAY TO ESTIMATE THE SECOND DERIVATIVE OF THE UNKNOWN SOLUTION.** In some cases, like the model $-u_{xx} + \beta u = f$, we have another possibility to approximate the second derivative u_{xx} of u. Indeed, u_{xx} satisfies the following problem

$$\begin{cases} -v_{xx}(x) + \beta v(x) = f_{xx}(x), \ x \in I = (0,1), \\ v(0) = -f(0), \\ v(1) = -f(1). \end{cases}$$

Then, u_{xx} satisfies the same equation satisfied by u. This allows us to get a finite volume approximation of u_{xx}, provided that $u \in H^4(I)$ (see Theorem 2.1), by using the same scheme used to compute the basic solution $u^{\mathcal{T}}$. More precisely in order to compute a finite volume approximation of u_{xx}, we use the same matrix used to compute $u^{\mathcal{T}}$. This idea can be also used to compute higher order corrections.

Remark 3.3. The numerical results, (see Section 5), show that the use of the previous variant to define the formulation of the first correction gives better accuracies than the use of that one given by the formulation (3.7) in L^2-norm. Whereas in H_0^1-norm, the formulation (3.7) is better than the new variant (see Table 5 of Section 5).

Remark 3.4. In the finite element method, the convergence order of the linear finite element solution is $O(h^2)$ in discrete H_0^1-norm even if the mesh is non-uniform, i.e. $\|u^h - \pi u\|_{1,I} \leq ch^2|u|_{2,I}$, where u^h is the linear finite element solution and πu is the linear the interpolant of u. This implies that (thinks to an inverse inequality) $\|u^h - \pi u\|_{D^2} \leq ch|u|_{2,I}$ (cf. [14]), where $\|.\|_{D^2}$ is the divided difference norm of order two. Therefore, we can perform the corrections with $O(h)$-improvement, provided that u and f are smooth enough (see section Generalization in Moore [14]). Whereas, in the finite volume method, **we do not have this smoothness of the error**; as **Theorem 2.1** asserts we just have $\|u^h - \pi u\|_{1,I} \leq ch\|u\|_{2,I}$.

4. HIGHER ORDER CORRECTIONS

In this section, we give the general formulation of an arbitrary correction. The proof of the convergence order is the same as of the first correction. To do so, assuming that $u \in H^{k+2}(I)$ implies by using Sobolev imbedding that $u \in \mathcal{C}^{k+1}(\bar{I})$. By using the fact that $\mathcal{D}(\bar{I})$ is dense in $H^{k+2}(I)$ and Taylor's formula, we obtain

$$
\begin{aligned}
(4.1)\qquad (\mathcal{A}^T u)_i = \int_{K_i} f dx &- \sum_{m=2}^{k+1} a_m^i u^{(m)}(x_{i+\frac{1}{2}}) + \sum_{m=2}^{k+1} a_m^{i-1} u^{(m)}(x_{i-\frac{1}{2}}) \\
&- \alpha\left(\sum_{m=1}^{k} \frac{(h_i^+)^m}{m!} u^{(m)}(x_i) - \sum_{m=1}^{k} \frac{(h_{i-1}^+)^m}{m!} u^{(m)}(x_{i-1})\right) \\
&- \beta\left(\sum_{m=1}^{k} \frac{(h_i^+)^{m+1}}{(m+1)!} u^{(m)}(x_i)\right. \\
&\qquad \left. - \sum_{m=1}^{k} \frac{(-h_i^-)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h_{i-\frac{1}{2}}^j}{j!} u^{(m+j)}(x_{i-1})\right) \\
&- (R_{i+\frac{1}{2}}^k - R_{i-\frac{1}{2}}^k) - \alpha(S_{i+\frac{1}{2}}^k - \alpha S_{i-\frac{1}{2}}^k) - \beta T_i^k,
\end{aligned}
$$

where the coefficients a_m^i are defined by

$$
(4.2)\qquad a_m^i = \frac{1}{m!}\left(\sum_{j=0}^{m-1} (h_{i+1}^-)^j (-h_i^+)^{m-1-j}\right),
$$

and

$$
(4.3)\qquad \left(\sum_i h_{i+\frac{1}{2}} (R_{i+\frac{1}{2}}^k)^2\right)^{\frac{1}{2}} \leq c h^{k+1} \|u\|_{k+2,I},
$$

$$
(4.4)\qquad |S_{i+\frac{1}{2}}^k| \leq c h_i^{k+1} \|u\|_{k+1,\infty,\bar{I}},
$$

$$
(4.5)\qquad |T_i^k| \leq c h_i h^{k+1} \|u\|_{k+1,\infty,\bar{I}},
$$

After having found an appropriate expansion approximating Equation (3.1), we now need the following useful Lemma

Lemma 4.1. *Let u be the solution of Equation (2.1). Assume that $u \in H^{k+2}(I)$, then each m^{th} derivative, with $2 \leq m \leq k+1$, of the solution u*

can be expanded as a linear combination of the solution itself, its derivative and the derivatives of the given function f up to and including $(m-2)^{nd}$ derivative, i.e. there exist reals $\{\alpha_j^m\}_{j=0}^{m-2} \cup \{\overline{\alpha}_1^m, \overline{\alpha}_2^m\}$ such that

$$u^{(m)}(x) = \sum_{j=0}^{m-2} \alpha_j^m f^{(j)}(x) + \overline{\alpha}_1^m u(x) + \overline{\alpha}_2^m u_x(x). \tag{4.6}$$

Proof. It can be proved by induction on m together with the fact that u satisfies Equation (2.1). □

Remark 4.1. The coefficients $\{\alpha_j^m\}_{j=0}^{m-2} \cup \{\overline{\alpha}_1^m, \overline{\alpha}_2^m\}$ can be defined by induction on m.

Assume now that we have obtained the $(k-1)^{\text{st}}$ correction $u_{k-1}^{\mathcal{T}} = (u_i^{k-1})_i$, where $u_0^{\mathcal{T}} = u^{\mathcal{T}}$ is the basic solution. According to Equality (4.1), to obtain a correction of order $O(h^{k+1})$, we have to find approximations of the pointwise derivatives of the solution u up to and including the $k+1$-th order. The idea we will present is similar to that used in computing the first correction. At first, we seek "optimal" approximations of $u_{xx}(x_i)$,..., $u^{(k+1)}(x_i)$, (optimal in the sense that they provide a correction of order $O(h^{k+1})$). To do so, we assume that $u_{k-1}^{\mathcal{T}}$ is obtained and that certain approximations of $u_{xx}(x_i)$,...,$u^{(k)}(x_i)$ are also available. Such approximations are denoted by $(u_2^{i,k-1})_i$,...,$(u_k^{i,k-1})_i$ and their orders in discrete L^2-norm are respectively $O(h^{k-1})$, ..., $O(h)$. Because the coefficient of $u^{(k+1)}(x_i)$ in (4.1) is of order $O(h^k)$, it suffices to approximate it by an order $O(h)$. This can be done easily through Lemma 4.1, i.e. an approximation defined by

$$u_{k+1}^{i,k} = \sum_{j=0}^{k-1} \alpha_j^{k+1} f^{(j)}(x_i) + \bar{\alpha}_1^{k+1} u_i^{k-1} + \bar{\alpha}_2^{k+1} \partial_1 u_i^{k-1}, \forall i \in \{0, ..., N\}, \tag{4.7}$$

where we recall that $\partial_1 u_i^{k-1} = \frac{u_{i+1}^{k-1} - u_i^{k-1}}{h_{i+\frac{1}{2}}}$.

Note that in (4.7) we can use the basic solution $u^{\mathcal{T}}$ instead of the $(k-1)^{\text{st}}$. For any integer β such that $2 \le \beta \le k$, we seek now an $O(h^{k+2-\beta})$-order approximation $u_\beta^{\mathcal{T},k}$ of the pointwise derivative $(u^\beta(x_i))_i$. This is because the coefficients of this derivative in (4.1) are of order $O(h^{\beta-1})$. By Lemma 4.1, for

$i \in \{0, ..., N\}$ we have

$$u^{(\beta)}(x_i) = \sum_{j=0}^{\beta-2} \alpha_j^\beta f^{(j)}(x_i) + \bar{\alpha}_1^\beta u(x_i) + \bar{\alpha}_2^\beta u_x(x_i)$$
$$= \sum_{j=0}^{\beta-2} \alpha_j^\beta f^{(j)}(x_i) + \bar{\alpha}_1^\beta u(x_i)$$
$$+ \bar{\alpha}_2^\beta \left(\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k-\beta+2} \frac{u^{(j)}(x_i)}{j!} h_{i+\frac{1}{2}}^{j-1} \right) + r_i^k,$$

where

$$|r_i^k| \leq c h_{i+\frac{1}{2}}^{k+2-\beta} \|u\|_{k+3-\beta,\infty,\bar{I}}. \tag{4.8}$$

An obvious approximation of the pointwise derivative $(u^\beta(x_i))_i$ can be given by, for $i \in \{0, ..., N\}$

$$u_\beta^{i,k} = \sum_{j=0}^{\beta-2} \alpha_j^\beta f^{(j)}(x_i) + \bar{\alpha}_1^\beta u_i^{k-1} + \bar{\alpha}_2^\beta \left(\frac{u_{i+1}^{k-1} - u_i^{k-1}}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k-\beta+2} \frac{u_j^{i,k-1}}{j!} h_{i+\frac{1}{2}}^{j-1} \right). \tag{4.9}$$

At this stage, we shall need the following

Lemma 4.2. *Let u be the solution of Equation (2.1). Assume that $u \in H^{k+2}(I)$. Then the approximations $u_\beta^{\mathcal{T},k} = (u_\beta^{i,k})_i$, where $2 \leq \beta \leq k+1$, defined by the expansions (4.7) and (4.9) satisfy the following estimate*

$$\left(\sum_{i=0}^{N} h_{i+\frac{1}{2}} (u_\beta^{i,k} - u^{(\beta)}(x_i))^2 \right)^{\frac{1}{2}} \leq c h^{k-\beta+2} \|u\|_{k+2,I}. \tag{4.10}$$

Proof. It can be proved by induction on k. □

After having found optimal approximations of the "fundamental" pointwise derivatives $(u^\beta(x_i))_i$, we now derive optimal approximations of $(u^{(\beta)}(x_{i+\frac{1}{2}}))_{i=0}^N$, $2 \leq \beta \leq k+1$ and $(u_x(x_i))_{i=0}^N$. We have

$$u^{(\beta)}(x_{i+\frac{1}{2}}) = \sum_{j=0}^{k-\beta+1} \frac{(h_i^+)^j}{j!} u^{(\beta+j)}(x_i) + s_i^k, \ \forall i \in \{0, ..., N\}, \tag{4.11}$$

where

$$\left(\sum_i h_{i+\frac{1}{2}}(s_i^k)^2\right)^{\frac{1}{2}} \le ch^{k-\beta+2}\|u\|_{k+2,I}. \tag{4.12}$$

We suggest the following approximation of $(u^{(\beta)}(x_{i+\frac{1}{2}}))_i$

$$u_\beta^{i+\frac{1}{2},k} = \sum_{j=0}^{k-\beta+1} \frac{(h_i^+)^j}{j!} u_{\beta+j}^{i,k}, \ \forall i \in \{0, ..., N\}. \tag{4.13}$$

Finally, from the approximations of the fundamental pointwise derivatives $(u^\beta(x_i))_i$, $2 \le \beta \le k+1$, we deduce an approximation, (expected to be of order $O(h^k)$), of the pointwise first derivative $(u(x_i))_i$, namely

$$u_x(x_i) = \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k} \frac{h_{i+\frac{1}{2}}^{j-1}}{j!} u^{(j)}(x_i) + t_i^k, \tag{4.14}$$

where

$$|t_i^k| \le ch_{i+\frac{1}{2}}^k \|u\|_{k+1,\infty,\bar{I}}, \ \forall i \in \{0, ..., N\}. \tag{4.15}$$

This enables us to consider the following approximation of $(u_x(x_i))_i$

$$u_1^{i,k} = \frac{u_{i+1}^{k-1} - u_i^{k-1}}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k} \frac{h_{i+\frac{1}{2}}^{j-1}}{j!} u_j^{i,k}, \ \forall i \in \{0, ..., N\}. \tag{4.16}$$

We make use of the following useful lemma

Lemma 4.3. *Let u be the solution of Equation (2.1). Assume that $u \in H^{k+2}(I)$. Then the approximations $(u_\beta^{i+\frac{1}{2},k})_i$, where $2 \le \beta \le k+1$, and $(u_1^{i,k})_i$ defined respectively by the expansions (4.13) and (4.16) satisfy the following estimates*

1. $\left(\sum_{i=0}^{N} h_{i+\frac{1}{2}}(u_\beta^{i+\frac{1}{2},k} - u^{(\beta)}(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \le ch^{k-\beta+2}\|u\|_{k+2,I}.$
2. $\left(\sum_{i=0}^{N} h_{i+\frac{1}{2}}(u_1^{i,k} - u_x(x_i))^2\right)^{\frac{1}{2}} \le ch^{k}\|u\|_{k+2,I}.$

Proof. Combine the triangular inequality with (4.10)-(4.16). □

Now we are able to define the k^{th} correction $u_k^{\mathcal{T}} = (u_i^k)_{i=0}^{N+1}$, where $u_0^k = u_{N+1}^k = 0$ and for $i \in \{1, ..., N\}$, we have

$$(\mathcal{A}^{\mathcal{T}} u_k^{\mathcal{T}})_i = \int_{K_i} f dx \tag{4.17}$$
$$- \sum_{m=2}^{k+1} a_m^i u_m^{i+\frac{1}{2},k} + \sum_{m=2}^{k+1} a_m^{i-1} u_m^{i-\frac{1}{2},k}$$
$$- \alpha \left(\sum_{m=1}^{k} \frac{(h_i^+)^m}{m!} u_m^{i,k} - \sum_{m=1}^{k} \frac{(h_{i-1}^+)^m}{m!} u_m^{i-1,k} \right)$$
$$- \beta \left(\sum_{m=1}^{k} \frac{(h_i^+)^{m+1}}{(m+1)!} u_m^{i,k} \right.$$
$$\left. - \sum_{m=1}^{k} \frac{(-h_i^-)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h_{i-\frac{1}{2}}^j}{j!} u_{m+j}^{i-1,k} \right).$$

Next, consider the following expansions

$$\gamma_i^k = - \sum_{m=2}^{k+1} a_m^i (u_m^{i+\frac{1}{2},k} - u^{(m)}(x_{i+\frac{1}{2}})) \tag{4.18}$$
$$- \alpha \sum_{m=1}^{k} \frac{(h_i^+)^m}{m!} (u_m^{i,k} - u^{(m)}(x_i)) + R_{i+\frac{1}{2}}^k + \alpha S_{i+\frac{1}{2}}^k$$

$$\delta_i^k = -\beta \left(\sum_{m=1}^{k} \frac{(h_i^+)^{m+1}}{(m+1)!} \left(u_m^{i,k} - u^{(m)}(x_i) \right) \right. \tag{4.19}$$
$$\left. - \sum_{m=1}^{k} \frac{(-h_i^-)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h_{i-\frac{1}{2}}^j}{j!} \left(u_{m+j}^{i,k} - u^{(m+j)}(x_{i-1}) \right) \right) + \beta T_i^k.$$

Let $e_i^k = u_i^k - u(x_i)$ be the error in the k^{th} correction, thus

$$- \frac{e_{i+1}^k - e_i^k}{h_{i+\frac{1}{2}}} + \frac{e_i^k - e_{i-1}^k}{h_{i-\frac{1}{2}}} + \alpha e_i^k - \alpha e_{i-1}^k + \beta h_i e_i^k = \gamma_i^k - \gamma_{i-1}^k + \delta_i^k. \tag{4.20}$$

Using the convergence proof of the first corrections and Lemmata 4.2 and 4.3 together with (4.2)-(4.5), we get the following

Theorem 4.1. *Let $\alpha, \beta \geq 0$ and $f \in L^2(I)$. Let u be the (unique) solution $u \in H_0^1(I)$ of (2.1). Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1. Then, for any $k \geq 2$, there exists a unique solution $u_k^{\mathcal{T}} = (u_i^k)_i$ (called*

k^{th} correction) to Equation (4.17). Assume that the solution u satisfies $u \in H^{k+2}(I)$. For each $K_i \in \mathcal{T}$, let $e_i^k = u(x_i) - u_i^k$ and define $e_{\mathcal{T}}^k \in \mathcal{X}(\mathcal{T})$ by $e_{\mathcal{T}}^k(x) = e_i^k$, a.e. $x \in K_i$. Then the error $e_{\mathcal{T}}^k$ is of order $O(h^{k+1})$, i.e. the following error estimates hold:

$$\|e^k\|_{1,\mathcal{T}} \le ch^{k+1}\|u\|_{k+2,I}, \tag{4.21}$$

$$\|e^k\|_{L^2} \le ch^{k+1}\|u\|_{k+2,I}. \tag{4.22}$$

Remark 4.2.

1. If the admissible mesh $\mathcal{T}$ satifies the condition $\varpi h_i \le h_{i+\frac{1}{2}} \le \kappa h_i$, where ϖ, κ are two positive real numbers independent on i and h, the k^{th} correction $u_k^{\mathcal{T}}$ can be reduced to

$$\begin{aligned}(\mathcal{A}^{\mathcal{T}} u_k^{\mathcal{T}})_i = \int_{K_i} f dx - \sum_{m=2}^{k+1} a_m^i u_m^{i+\frac{1}{2},k} + \sum_{m=2}^{k+1} a_m^{i-1} u_m^{i-\frac{1}{2},k} \\ - \alpha \sum_{m=1}^{k} \frac{(h_i^+)^m}{m!} u_m^{i,k} + \alpha \sum_{m=1}^{k} \frac{(h_{i-1}^+)^m}{m!} u_m^{i-1,k} \\ - \beta \left(\sum_{m=1}^{k} \frac{(h_i^+)^{m+1} - (-h_i^-)^{m+1}}{(m+1)!} u_m^{i,k} \right).\end{aligned}$$

2. The results obtained above can be extended to the linear convection-diffusion equation $-(\lambda u_x)_x(x) + au_x(x) + bu(x) = f(x)$, where λ, a, b and$f$ are supposed to be smooth enough (about the finite volume discretization of this equation, we refer to [11]).

5. NUMERICAL TESTS

To justify the efficiency of our technique, we perform a few simple numerical experiments using two types of meshes, namely:

1. A uniform mesh: it is an admissible mesh $\mathcal{T}$, (in the sense of Definition 2.1), such that the following conditions are fulfilled:
   1.1. $(x_0, x_N) = (0, 1)$, (recall that the domain $I = (0, 1)$).
   1.2. $h_{i+\frac{1}{2}} = h$ and $h_{i+1}^- = h_i^+$, (this last condition means that $x_{i+\frac{1}{2}}$ is the center of $[x_i, x_{i+1}]$).

2. A cell centered mesh: it is an admissible mesh $\mathcal{T}$, (in the sense of Definition 2.1), such that the following conditions are fulfilled:

2.1.

$$h_i = \begin{cases} h, i \text{ is even}, \\ \frac{h}{2}, i \text{ is odd}. \end{cases}$$

2.2. $h_i^- = h_i^+$.

To find the convergence orders of the first correction and the basic finite volume solution, we compute the ratio

$$ratio = \frac{\log(e(h)) - \log(e(h_0))}{\log(h) - \log(h_0)},$$

where h_0 is the initial value of h in each numerical test and $e(h)$ is the error corresponding to h. In the uniform mesh case, we use the rule

$$ratio = -\frac{\log(e(1/2^{k+1})) - \log(e(1/2^k))}{\log 2}.$$

Remark 5.1. The numerical tests are performed using Matlab and the integrals $\int_{K_i} f(x)dx$ are computed without any numerical quadrature of integration. Of course, when we use numerical integration, we should choose convenient quadratures, (for instance see [14]), which allow us to obtain convergence orders expected from the corrections .

5.1. **First Test.** We consider Equation **(I)** : $-u_{xx} = f$ with homogeneous boundary conditions, namely $u(x) = \sin(\pi x)$ and $f(x) = \pi^2 \sin(\pi x)$.

Remark 5.2. In case of the cell-centered finite volume mesh defined above, the divided difference of order two of finite volume solution does not converge to the second derivative of the solution u of **(I)** (see Subsection 5.2 of [11]). The improvements we obtain, (on this type of meshes), show the efficiency of our technique in comparision with the classical techniques, (this fact is already pointed out in the Introduction).

Remark 5.3. In case of $(x_0, x_N) = (0, 1)$ and $x_{i+\frac{1}{2}}$ is the center of $[x_i, x_{i+1}]$, (for $-u_{xx} = f$), and in particular for a uniform mesh, the first correction concides with the basic finite volume solution (see Remark 3.2 (2)). Thus we are interested in the following tests, (for the case of the uniform mesh), with the second correction instead of the first one.

TABLE 1. The convergence orders of the first correction and the basic solution in L^2-norm on the uniform mesh.

h	second correction		basic solution	
	order	error $/h^4$	order	error $/h^4$
1/32	-	0.0359	-	0.0298e+04
1/64	4.0004	0.0359	2.0003	0.1191e+04
1/128	4.0000	0.0359	2.0001	0.4764e+04
1/256	4.0032	0.0390	2.0000	1.9057e+04

TABLE 2. The convergence orders of the first correction and the basic solution in H_0^1-norm on the uniform mesh.

h	second correction		basic solution	
	order	error $/h^4$	order	error $/h^4$
1/32	-	0.1127	-	0.0935e+04
1/64	3.9999	0.1127	1.9999	0.3742e+04
1/128	3.9999	0.1129	2.0000	1.4967e+04
1/256	4.0032	0.1224	2.0000	5.9869e+04

TABLE 3. The convergence orders of the first correction and the basic solution in L^2-norm on the cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
4/149	-	0.2474	-	0.3971
4/599	1.9903	0.2507	1.9986	0.3978
4/2999	1.9943	0.2516	1.9991	0.3981
4/14999	1.9961	0.2473	1.9994	0.3968

TABLE 4. The convergence orders of the first correction and the basic solution in H_0^1-norm on the cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
4/149	-	0.5856	-	0.0281e+03
4/599	1.9929	0.5914	1.0000	0.1131e+03
4/2999	1.9958	0.5930	1.0000	0.5664e+03
4/14999	1.9971	0.5828	1.0000	2.8329e+03

TABLE 5. Comparaison between the accuracy of the first correction that uses the first variant and the one using the second variant in H_0^1 and L^2-norms on the cell-centered mesh.

h	first variant		second variant	
	L^2-norm	H_0^1-norm	L^2-norm	H_0^1-norm
4/149	1.7827e-04	4.2204e-04	7.9569e-05	6.6012e-04
4/599	1.1180e-05	2.6374e-05	5.4358e-06	4.1591e-05
4/2999	4.4758e-07	1.0549e-06	2.2254e-07	1.6670e-06
4/14999	1.7591e-08	4.1450e-08	8.6276e-09	6.6477e-08

5.2. **Second Test.** In case of the cell-centered test, (Table 3), we have seen that the convergence order of the first correction is the same one of the basic solution in L^2-norm for the model (**I**). We present here an example of the mesh where the convergence order of the first correction considerably improves the one of the basic solution in both H_0^1 and L^2-norms for the model (**I**). We consider $h_{i+\frac{1}{2}} = h$ and $x_{i+\frac{1}{2}} = \frac{2x_i+x_{i+1}}{3}$ for all $i = 1, ..., N-1$.

TABLE 6. The convergence orders of the first correction and the basic solution in L^2-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
$1/2^8$	-	0.3701	-	20.2232
$1/2^9$	1.9668	0.3788	1.0000	40.4472
$1/2^{10}$	1.9633	0.3832	1.0000	80.8948
$1/2^{11}$	1.9915	0.3855	1.0000	161.7899
$1/2^{12}$	2.0073	0.3835	1.0000	323.5799
$1/2^{13}$	1.9864	0.3871	1.0000	647.1598

TABLE 7. The convergence orders of the first correction and the basic solution in H_0^1-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
$1/2^8$	-	1.4588	-	129.5822
$1/2^9$	2.1180	1.3443	0.9999	259.1881
$1/2^{10}$	2.0675	1.2828	1.0000	518.3882
$1/2^{11}$	2.0363	1.2509	1.0000	1.0368e+03
$1/2^{12}$	2.0301	1.2251	1.0000	2.0736e+03
$1/2^{13}$	1.9985	1.2263	1.0000	4.1471e+03

5.2.1. ***Third Test.*** We consider the Equation (**II**) : $-u_{xx} + u_x + u = f$, with homogeneous boundary conditions, namely $u(x) = \sin(\pi x)$ and $f(x) = (\pi^2 + 1)\sin(\pi x) + \pi\cos(\pi x)$.

TABLE 8. The convergence orders of the first correction and the basic solution in L^2-norm on the cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
2/75	-	0.1415	-	0.0099e+03
1/150	2.0074	0.1401	0.9796	0.0407e+03
1/750	2.0043	0.1398	0.9880	0.2051e+03
1/3750	2.0020	0.1402	0.9918	1.0270e+03

TABLE 9. The convergence orders of the first correction and the basic solution in H_0^1-norm on the cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
2/75	-	0.4441	-	0.0377e+03
1/150	1.9936	0.4480	0.9887	0.1530e+03
1/750	1.9963	0.4491	0.9934	0.7681e+03
1/3750	1.9970	0.4502	0.9955	3.8438e+03

5.3. Some Comments about the Numerical Results.

1. In **Table 1** and **Table 2**, the numerical results show that on uniform meshes and for the model **(I)**, we can gain an $O(h^2)$- improvement in both H_0^1-norm and L^2-norm by the second correction. To justify that the second correction is of order $O(h^4)$ not only $O(h^3)$, (as in Theorem 4.1), it suffices to remark that in (4.17), (with $k = 2$, $\alpha = \beta = 0$), the second correction concides with the third one; because the coefficients a_4^i given by (4.2)) are all zero. In fact, in Formula (4.2) it is easily seen that $a_{2k+2}^i = 0$ for all $i \in \{1, ..., N\}$ and $k \in \mathbb{N}$. Then, for any $k \in \mathbb{N}^*$, the correction $u_{2k}^{\mathcal{T}}$ concides with $u_{2k+1}^{\mathcal{T}}$. This means that the correction $u_{2k}^{\mathcal{T}}$ is of order $O(h^{2k+2})$, (in case $\alpha = \beta = 0$ and the mesh $\mathcal{T}$ is uniform).
2. In **Table 3**, the numerical results show that for the model **(I)**, we do not have an improvement in L^2-norm by the first correction when the mesh is cell-centered. Nevertheless, the coefficents of the error in the first correction are better than those of the basic solution. Then to improve the order in the L^2-norm, we should compute the second correction.
3. In **Table 4**, the numerical results show that for the model **(I)**, we gain an $O(h)$-improvement by the first correction in H_0^1-norm.
4. In **Table 5**, the numerical results show that the accuracy of the error in the first correction defined by the second variant, (see Subsection 2.3.1), is better than that of (3.7) in L^2-norm in contrast to the H_0^1-norm.

5. In **Table 6** and **Table 7**, the numerical results show that the convergence of the first correction improves that of the basic solution in both H_0^1 and L^2 norms for **(I)**. This implies that, on an arbitrary admissible mesh, the first correction improves the basic solution.
6. In **Table 8** and **Table 9**, the numerical results show that for the model **(II)**, we gain an $O(h)$-improvement in both norms, L^2-norm and H_0^1-norm, by just the first correction of the cell-centered mesh.

Remark 5.4. The number of operations required for the first correction is comparable to that of the basic solution. Indeed, recall that we compute the basic finite volume solution $u^{\mathcal{T}}$ via an algebraic system defined by:

$$\mathcal{A}^{\mathcal{T}} u^{\mathcal{T}} = f^{\mathcal{T}}, \tag{5.1}$$

where $\mathcal{A}^{\mathcal{T}}$ is a tridigonal matrix of dimension $N \times N$ (recall that $N = O(\frac{1}{h})$). The first correction can also be obtained via an algebraic system defined by:

$$\mathcal{A}^{\mathcal{T}} u_1^{\mathcal{T}} = f^{\mathcal{T}} + d^{\mathcal{T}}, \tag{5.2}$$

where $d^{\mathcal{T}}$ is defined by the second member of Equation (3.7) without the source term $\left(\int_{K_i} f(x)dx\right)_i$. Thus to compute $d^{\mathcal{T}}$, (and therefore $u_1^{\mathcal{T}}$), we have to compute $u^{\mathcal{T}}$. One remarks that the matrix $\mathcal{A}^{\mathcal{T}}$ is symmetric, then it is useful to use Cholesky's method. The number of operations required for the factorization $\mathcal{A}^{\mathcal{T}} = L.L^t$ are of order $O(N)$. Then, the number of operations required to compute the first correction is of order $O(N)$. In general, the number of operations $\mathcal{N}_k$ required for the k^{th} correction is comparable to that of the basic solution $\mathcal{N}_0$, more precisely:

$$\mathcal{N}_0(\mathcal{T}) \le \mathcal{N}_k(\mathcal{T}) \le C(k)\mathcal{N}_0(\mathcal{T}), \tag{5.3}$$

where $C(k)$ is a positive increasing function (recall that $\mathcal{N}_0(\mathcal{T}) = O(N)$).

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ON UNIFORM ACCELERATED FLUID FLOW PAST A ROTATING CIRCULAR CYLINDER

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ABSTRACT. The non-steady flow, due to the uniformly accelerated and rotating circular cylinder from rest in a stationary viscous incompressible fluid is considered. This numerical experiment is made for various values of the Reynolds number in the range from 1 to 350 and several values of rotation parameter α in the range from 0 to 4.5. In this numerical attempt, we have adopted a scheme which consists of two steps. In the first step, the special finite-difference method Dennis [10] is used to approximate the constitutive equations. This method transforms the governing partial differential equations to a system of finite-difference equations which are then solved numerically by S.O.R. iterative method. In the second step, the results obtained are further refined and upgraded by Richardson Extrapolation method. For the purpose of verification, the obtained results are compared on five different grid sizes as well as with those of Collins and Dennis [9] for α=0. The comparison is very favourable.

1. INTRODUCTION

The problem of the fluid flow due to uniformly accelerated and rotating infinite circular cylinder in the stationary incompressible and viscous fluids is of fundamental interest owing to its valuable and large number of applications. For example, in lift enhancement Ackeret [1] and Sayers [24], in the Flettner rotor ship, where the rotating vertical cylinders were employed to develop a thrust normal to any wind blowing past the ship. Secondly, the rotation effects are dominant to control the boundary layer separation discovered by Prandtl [21] and Prandtl and Tietjens [22], Tennant et al [28], and Moore [20]. Thirdly, the concept of rotating cylinder is very famous and has practical importance in "breakaway" phenomenon Riley [23], Tennant [27], and Walker [29]. Now-a-days, this problem has great dominance in boundary layer control devices such as on the flaps of V/STOL aircraft Cichy [7] and upstream of ship rudders and to overcome separation effects in a subsonic diffuser. Moreover, it has also importance in geophysical motion which is of interest to meteorologists and oceanographers and others who are studying topographic effects in geophysical flow fields.

In present studies, we have discussed the flow behaviour by using two parameters; the rotation parameter α, and R, the Reynolds number. For α=0 (i.e., flow without rotation) the flow becomes accelerated flow past a circular cylinder. This problem has a long history. In 1908, Blasius [2] for the first time considered the general problem for two cases of motion from rest. In case 1, he studied the flow past the impulsively started circular cylinder with uniform velocity, while in case 2, he investigated the flow due to impulsively started circular cylinder with uniform acceleration. For the second case, Goldstein et al [14-15] showed also their efforts numerically. Görlter [16-17], and Watson [30] generalised this theory to other types of variation of the initial velocity of the cylinder. The numerical methods used to solve Navier-Stokes equations for these types of flows were very approximate and valid for all Reynolds number only for leading term in the series solution which is expanded in powers of time. But subsequent terms in the expansion are valid only for infinite Reynolds number.

In 1974, Collins and Dennis [9] studied the second case of Blasius [2] very well and improved and solved them by two techniques. In the first, by the spectral method while in the second, direct integration of Navier-Stokes equations was made which was the extension of the first method. Experimentally this problem was also examined by Tameda [26] in 1972. He measured the time of separation of flow and the growth of the separated wake for Reynolds numbers R^2=97.5, 5850 and 122x10^3.

However, the flow due to uniform translation and rotation with zero acceleration has been investigated numerically as well as experimentally by several researchers for example Badr and Dennis [3-4], Chang and Chern [6], Ece [13] etc. But unfortunately the flow with non-zero acceleration, that is, the flow due to uniformly accelerated and rotating circular cylinder problem did not receive due attention. It is perhaps due to its sensitivity and complications. It is the one of the reasons for numerical researchers in adopting low order numerical schemes which can provide results and does not converge for large value of t (see for example Collins and Dennis[9]). To overcome this deficiency, we are putting a minute contribution by adopting higher order numerical scheme. Therefore, our scheme is valid for all time, for all Reynolds numbers and there is no problem of divergence. Moreover, our results are in good agreement with those of Collins and Dennis[9] for low values of t when α=0. We shall also try to answer the questions such as: what will be the flow behaviour in the process of vortex shedding in the wake and on the surface of cylinder, the variation of surface forces and pressure development on the surface of cylinder as time passes if we vary rotation speed and its translational speed?

Although the calculations are made for various values of the Reynolds number R in the range 1 $\leq$R $\leq$350 and several values of α in the range $0 \leq \alpha \leq 4.5$, yet we focus on three values of R namely 1, 10, 349.285 and three values of α namely 2.5, 3, 4.5 and compare the results with Collins and Dennis[9] for small value of t when α=0. The results are presented in graphical form in section 5 in which we have examined the streamlines behaviour, vorticity contours, vorticity variation with time, the variation of surface forces with t for different values of R and α and especially the vortex shedding over the surface cylinder and pressure development in the wake of the cylinder. The formulation and basic analysis of the problem under consideration is given in section 2. The section 3 contains the analysis of transformation of governing equations into a system of finite-difference equations by using special finite difference method and the computational procedure is briefly described in section 4.

2. BASIC ANALYSIS

The continuity and the Navier-Stokes equations for incompressible fluid in dimensional form, in the absence of body force are given as follows:

$$\nabla.\mathbf{V} = 0 \tag{2.1}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\mathbf{V}\right] = -\nabla p + \mu\nabla^2\mathbf{V} \tag{2.2}$$

where **V** is the fluid velocity vector, ρ the density, p the pressure, and μ the viscosity coefficient.

The mechanics of the problem under consideration can briefly be stated as, the flow is normal to an infinite circular cylinder which is uniformly accelerated and rotating from rest in an infinite stationary viscous incompressible fluid such that the circular cylinder is rotating with angular speed Ω while advancing from right to left with uniform acceleration f. The rotation and translation are started at the same time. Obviously, cylindrical co-ordinate system will be used, where the reference frame is fixed on the cylinder such that origin coincides with the cylinder's centre. However, the coordinate system will be modified further by taking s=lnr. Moreover, the flow is being considered as non-steady, two-dimensional and laminar for all time.

On deforming equation (2.2) into vorticity transport equation, we get

$$-\mu\nabla \times \nabla \times \boldsymbol{\omega} = \rho\left[\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{V} \times \boldsymbol{\omega})\right] \tag{2.3}$$

where

$$\boldsymbol{\omega} = \nabla x \mathbf{V}. \tag{2.4}$$

In order to normalize the field variables, the following dimensionless parameters can be introduced,

$$x_i^* = \frac{x_i}{c}, \mathbf{V}^* = \frac{\mathbf{V}}{\sqrt{fc}}, \boldsymbol{\omega}^* = \sqrt{\frac{c}{f}}\boldsymbol{\omega}, \quad \text{and} \quad t^* = \sqrt{\frac{f}{c}}t. \tag{2.5}$$

where "*" stands for dimensionless parmeters while f and c signify for uniform acceleration and the radius of the cylinder respectively.

On the introduction of equation (2.5) into equations (2.1) and (2.3), we yield

$$\nabla^*.\mathbf{V}^* = 0 \tag{2.6}$$

$$-\nabla^* \times \nabla^* \times \boldsymbol{\omega}^* = \frac{\rho c\sqrt{fc}}{\mu}\left[\frac{\partial \boldsymbol{\omega}^*}{\partial \bar{t}^*} - \nabla^* \times (\mathbf{V}^* \times \boldsymbol{\omega}^*)\right] \tag{2.7}$$

On dropping "*", equations (2.6) and (2.7) can be expressed in components form in the cylindrical polar coordinates system if we use

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, u_\theta = -\frac{\partial \psi}{\partial r}, \tag{2.8}$$

and $\boldsymbol{\omega} = [0, 0, E]$ to obtain

$$E = -\nabla^2\psi \tag{2.9}$$

$$\nabla^2 E = \frac{R}{2}\left[\frac{\partial E}{\partial t} + \frac{1}{r}\left\{\frac{\partial\psi}{\partial\theta}\frac{\partial E}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial E}{\partial\theta}\right\}\right] \tag{2.10}$$

where R$=\frac{2c\sqrt{fc}}{v}$, beingReynolds number and $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$.

Our next target is to solve equations (2.9), and (2.10) w. r. t. the following boundary conditions,

$$\begin{cases} \psi = 0, \frac{\partial\psi}{\partial r} = -\alpha, E_w = -\frac{6\psi_1 + H^2E_1}{2(H^2+H^3)} + \frac{3\alpha H}{2H^2+H^3}, \text{ at } r = 1, \\ \frac{1}{r}\frac{\partial\psi}{\partial r} \to t\sin\theta, \frac{\partial\psi}{\partial\theta} \to t\cos\theta, E \to 0, \text{as } r \to \infty, \end{cases} \tag{2.11}$$

where $\alpha = \Omega\sqrt{\frac{c}{f}}$.

Earlier numerical attempts (e.g., Collins and Dennis[9]) are related to the problem under consideration without rotation of circular cylinder and are studied by using boundary layer techniques which has two major deficiencies: one is that it is valid for high Reynolds number flow and secondly solution no longer converges for large values of time t. The later type of difficulty has been encountered by various authors, for example Sears and Telionis[25], Collins and Dennis[8], Cebeci [5], and Dommelen and Shen[11] and Dommelen[12]. To overcome these difficulties, we adopt higher order numerical scheme which is valid for all Reynolds number, rotation parameter, and specially for all time. Detail of this scheme is given in the following section. This numerical scheme reduces the highly non-linear system of partial differential equations to a system of difference equations, which then can be solved by direct and indirect methods. But we have chosen the SOR-iterative method which accelerates the convergence of the iterative scheme. Henceforth, this numerical procedure is efficient for studies of such type of sensitive flow problems and is straightforward, economical in core storage requirements of a computer, and easy to programme.

3. FINITE-DIFFERENCE METHOD

For convenience, let,

$$s = \ln r. \tag{3.1}$$

Then equations (2.9) and (2.10) would deform as

$$E = -e^{-2s}\Theta^2\psi, \tag{3.2}$$

$$\Theta^2 E = \frac{R}{2}\left[e^{2s}\frac{\partial E}{\partial t} + \frac{\partial \psi}{\partial \theta}\frac{\partial E}{\partial s} - \frac{\partial \psi}{\partial s}\frac{\partial E}{\partial \theta}\right], \tag{3.3}$$

where

$$\Theta^2 \equiv \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \theta^2}. \tag{3.4}$$

Obviously the boundary conditions given in Eq. (2.11) would deform as,

$$\begin{cases} \psi = 0, \frac{\partial \psi}{\partial s} = -\alpha, E_w = -\frac{6\psi_1 + H^2 E_1}{2(H^2+H^3)} + \frac{3\alpha H}{2H^2+H^3}, & \text{at } s = 0, \\ e^{-s}\frac{\partial \psi}{\partial r} \to t\sin\theta, e^{-s}\frac{\partial \psi}{\partial \theta} \to t\cos\theta, E \to 0, & \text{as } s \to \infty, \end{cases} \tag{3.5}$$

where $s = 0$ represents the surface of the cylinder. E_w is for vorticity on the surface of the cylinder while the subscript 1 denotes the point one cell away to the cylinder's surface.

We shall use the following notation. The grid size along s-, θ-, and t-directions, are taken by H, K_1, K_2 respectively, and the points (s_0, θ_0, t_0), (s_0+H, θ_0, t_0), (s_0, θ_0+K_1, t_0), (s_0-H, θ_0, t_0), (s_0, θ_0-K_1, t_0), (s_0, θ_0, t_0+K_2), (s_0, θ_0, t_0-K_2) are represented by the subscripts 0, 1, 2, 3, 4, 5, 6 respectively. Moreover, the grid which we adopted to discreatize the whole domain of computation is given in figure 1.

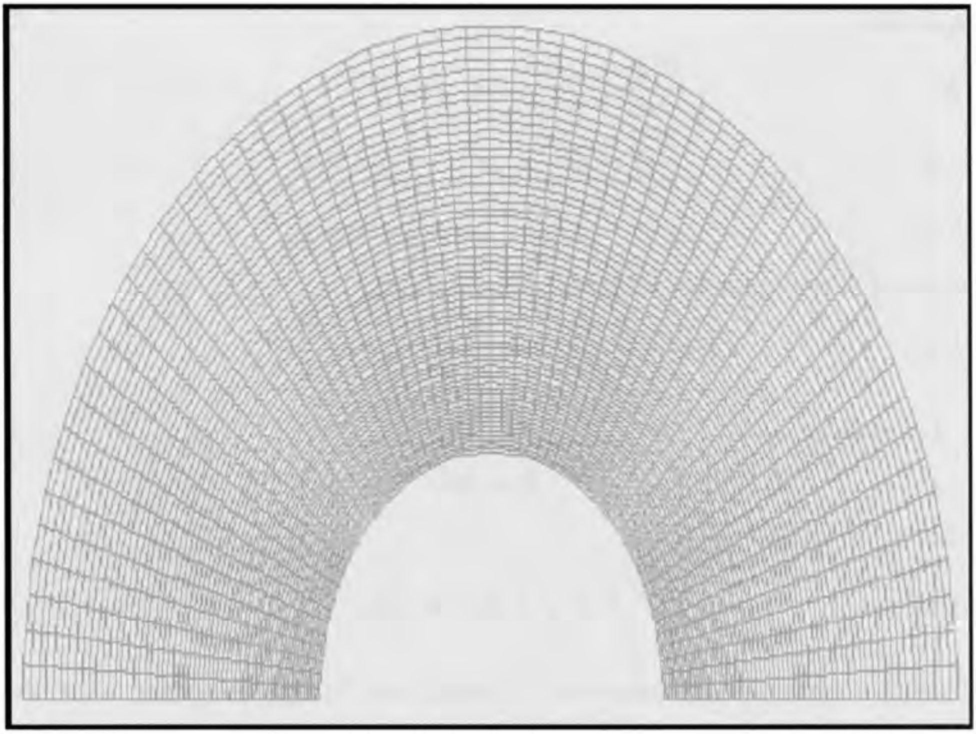

Figure 1. The meshing of computational domain.

In order to approximate equation (3.2), we employ standard central difference formulation at the point "0", that is,

$$\frac{1}{H^2}\psi_1 + \frac{1}{K_1^2}\psi_2 + \frac{1}{H^2}\psi_3 + \frac{1}{K_1^2}\psi_4 - \left(\frac{2}{H^2} + \frac{2}{K_1^2}\right)\psi_0 = -e^{2s_0}E_0 \tag{3.6}$$

The variation in finite-difference formulation, at the point "0", appears in approximating equation (3.3), which can be splitted into the following three equations,

$$\frac{\partial^2 E}{\partial s^2} + B\frac{\partial E}{\partial s} = A(s,\theta,t) \tag{3.7}$$

$$\frac{\partial^2 E}{\partial \theta^2} + C\frac{\partial E}{\partial \theta} = -\frac{1}{2}A(s,\theta,t) \tag{3.8}$$

$$L\frac{\partial E}{\partial t} = -\frac{1}{2}A(s,\theta,t) \tag{3.9}$$

equations,

$$\frac{\partial^2 E}{\partial s^2} + B\frac{\partial E}{\partial s} = A(s,\theta,t) \tag{3.7}$$

$$\frac{\partial^2 E}{\partial \theta^2} + C\frac{\partial E}{\partial \theta} = -\frac{1}{2}A(s,\theta,t) \tag{3.8}$$

$$L\frac{\partial E}{\partial t} = -\frac{1}{2}A(s,\theta,t) \tag{3.9}$$

where A is the unknown arbitrary function and

$$B = -\frac{R}{2}\frac{\partial\psi}{\partial\theta}, C = \frac{R}{2}\frac{\partial\psi}{\partial s}, L = -\frac{R}{2}e^{2s}. \tag{3.10}$$

Let us introdue

$$E = \lambda e^F \tag{3.11}$$

where

$$F = \frac{1}{2}\int_{s_0}^{s} B(x,\theta,t)dx. \tag{3.12}$$

Then the equation (3.7) takes the following form by approximating the derivatives involved by central-differences:

$$(\lambda_1 + \lambda_3 - 2\lambda_0) - \left[\frac{1}{2}\left(\frac{\partial B}{\partial s}\right)_0 + \frac{1}{4}B_0^2\right]\lambda_0 H^2 = A_0 H^2. \tag{3.13}$$

To approximat equation (3.8) along $\theta-$direction, let

$$E = \mu e^G \tag{3.14}$$

where

$$G = \frac{1}{2}\int_{\theta_0}^{\theta} C(s,x,t)dx \tag{3.15}$$

and the equation (3.8) will take the form

$$\frac{H^2}{K_1^2}(\mu_2 + \mu_4 - 2\mu_0) - \left[\frac{1}{2}\left(\frac{\partial C}{\partial \theta}\right)_0 + \frac{1}{4}C_0^2\right]\mu_0 H^2 = \frac{A_0}{2}H^2 \tag{3.16}$$

Next if we approximate equation.(3.9) simply by standard central difference approximation, we get,

$$\frac{L_0 H^2}{2K_2}[E_5 - E_6] = -\frac{A_0}{2}H^2 \tag{3.17}$$

By adding Equations (3.10), (3.11), and (3.12) and using argument that $\lambda_0 = \mu_0 = E_0$, we get

$$\left[\lambda_1 + \lambda_3 - 2E_0 - \frac{1}{4}E_0 B_0^2 H^2\right] + \left[\frac{H^2}{K_1^2}(\mu_2 + \mu_4 - 2E_0) - \frac{1}{4}E_0 C_0^2 H^2\right] + \frac{L_0 H^2}{2K_2}[E_5 - E_6] = 0, \tag{3.18}$$

In order to express λ and μ back in terms of E, it can be done from the definitions. It is found that

$$\lambda_i = E_i e^{F_i}, \mu_j = E_j e^{G_j} \tag{3.19}$$

where

$$\begin{cases} F_i = \frac{1}{2}\int\limits_{s_0}^{s_i} B(x, \theta_0, t_0)dx, \\ \\ G_j = \frac{1}{2}\int\limits_{\theta_0}^{\theta_j} C(s_0, x, t_0)dx \end{cases} \tag{3.20}$$

such that i=1, 3, while j=2, 4

Now we can replace λ_i, and μ_j by expressions involving E_i, and E_j respectively, but they will involve exponential coefficients.

We, next, expand the above exponentials in powers of their arguments keeping the truncation error of order H^4, and $H^2K_1^2$. In expanding the exponents we neglect the terms of order H^4, and $H^2K_1^2$and higher order. After some simplifications under above arguments and using the Taylor theorem, we obtain:

$$\lambda_1 + \lambda_3 = \left[1 + \frac{H^2}{4}\left(\frac{\partial B}{\partial s}\right)_0 + \frac{B_0^2 H^2}{8}\right][E_1 + E_3] + \frac{B_0 H}{2}[E_1 - E_3] \tag{3.21}$$

and

$$\mu_2 + \mu_4 = \left[1 + \frac{K_1^2}{4}\left(\frac{\partial C}{\partial \theta}\right)_0 + \frac{C_0^2 K_1^2}{8}\right][E_2 + E_4] + \frac{C_0 K_1}{2}[E_2 - E_4] \tag{3.22}$$

Introducing equations (3.21) and (3.22) into equation (3.18), we get,

$$\left[1+\frac{B_0H}{2}+\frac{B_0^2H^2}{8}\right]E_1+\left[\frac{H^2}{K_1^2}+\frac{C_0H^2}{2K_1}+\frac{C_0^2H^2}{8}\right]E_2 + \left[1-\frac{B_0H}{2}+\frac{B_0^2H^2}{8}\right]E_3+\left[\frac{H^2}{K_1^2}-\frac{C_0H^2}{2K_1}+\frac{C_0^2H^2}{8}\right]E_4 + \frac{L_0H^2}{2K_2}\left[E_5-E_6\right]-\left[2+\frac{2H^2}{K_1^2}+\frac{B_0^2H^2}{4}+\frac{C_0^2H^2}{4}\right]E_0=0 \tag{3.23}$$

4. COMPUTATIONAL PROCEDURE

Equations (3.6) and (3.23) can be written in the following form,

$$R_1\psi_1+R_2\psi_2+R_3\psi_3+R_4\psi_4+R_5E_0-\psi_0 = 0 \tag{4.1}$$

$$T_1E_1+T_2E_2+T_3E_3+T_4E_4+T_5E_5+T_6E_6-E_0 = 0 \tag{4.2}$$

where

$$R_1=R_3=\frac{1}{H^2Z_1},\qquad R_2=R_4=\frac{1}{K_1^2Z_1},$$

$$R_5=\frac{e^{2s_0}}{Z_1},\qquad Z_1=\left[\frac{2}{H^2}+\frac{2}{K_1^2}\right],$$

$$T_1=\frac{1+\frac{B_0H}{2}+\frac{B_0^2H^2}{8}}{Z_2},\qquad T_2=\frac{\frac{H^2}{K_1^2}+\frac{C_0H^2}{2K_1}+\frac{C_0^2H^2}{8}}{Z_2},$$

$$T_3=\frac{1-\frac{B_0H}{2}+\frac{B_0^2H^2}{8}}{Z_2},\qquad T_4=\frac{\frac{H^2}{K_1^2}-\frac{C_0H^2}{2K_1}+\frac{C_0^2H^2}{8}}{Z_2},$$

$$T_5=\frac{L_0H^2}{2K_2}=-T_6,\qquad Z_2=2+\frac{2H^2}{K_1^2}+\frac{B_0^2H^2}{4}+\frac{C_0^2H^2}{4}.$$

The set of equations (4.1) and (4.2) is solved then iteratively by the point S. O. R iterative procedure Hildebrand [18], subject to the appropriate boundary conditions given in (3.5).

The above procedure is repeated until convergence is obtained according to the criterion,

$$\max\left|E_0^{(m+1)}-E_0^{(m)}\right|<10^{-5},\max\left|\psi_0^{(m+1)}-\psi_0^{(m)}\right|<10^{-5},$$

where superscript 'm' represents the number of iteration.The results obtained are further refined and enhanced up to the order four by Richardson's extrapolation method, see Jain [19].

5. CALCULATED RESULTS AND DISCUSSION

The calculations were made for various values of Reynolds number R in the range $1 \leq R \leq 350$ and the rotation parameter in the range $0 \leq \alpha \leq 4.5$. The results have been found for the following five grid sizes

(a) $H=1/10$, $K_1=\pi/10$, $K_2=1/130$

(b) $H=1/20$, $K_1=\pi/20$, $K_2=1/130$

(c) $H=1/30$, $K_1=\pi/30$, $K_2=1/130$

(d) $H=1/40$, $K_1=\pi/40$, $K_2=1/130$

(e) $H=1/60$, $K_1=\pi/60$, $K_2=1/130$

But for the purpose of discussion, we focus on four values of R namely 1, 10, 100, and 349.285. The accuracy of the numerical results is checked by comparing the results on different grid sizes for vorticity E and stream function ψ, for various values of R and as mentioned above. In the early stage of flow development, the highly viscous effects and generation of the secondary flows have been observed. That is why, we focus ourselves not only to study in detail the streamlines and eddies behaviour in the time $t<1$ but also we examine their behaviour for time $t>1$. Streamlines are presented graphically in figures 2-10, which show how secondary flow is produced and how it takes the shape with the variation of rotation parameter and time t, specially in the early stages of time. From the studies of these graphs, we observe that secondary flows are produced in the early stages and they vary with R. If we fix and vary R we observe that closed streamlines will remain dominant for longer time. In the early stages of the flow it seems to be very complex and formation of secondary flows is very progressive, for example see figures 5(a), 6(a), 7(a), 8(a), and 9(a). So one can conclude that the secondary flows, which occur in form of closed streamlines and which produce in early stage of fluid motion, remain dominant for longer time as R increases, for fix value of α.

If we fix R and vary α and examine the flow behaviour then illustrations show that the radius of closed streamlines increases for earlier stage while it will continue to decrease as time passes.

The variation of maximum value of streamfunction, ψ_{max} will fluctuate for fixed value of R and t at different values of α. But ψ_{max} also increases at fix value of t and on increasing R as shown in figures 2 to 10. Moreover, it is observed that ψ_{max} varies linearly with t for all values of α.

The behaviour of curves of constant vorticity with the change of R, α, and t is also examined. For the purpose of discussion, we have selected four time levels t=0.046, 0.1, 0.154, and 0.993 for the values of R and α mentioned above. Graphically their behaviour is given in figures 11 to 13. On analysis of the graphs, one can observe that there is no secondary vortex for low Reynolds number for all time before and after the rotation of the circular cylinder as shown in figure 6. When α=0 and R is moderate the situation is different, a pair of secondary vortices produces and these vortices remain isolate before t=0.069 as shown in figure 11(a). But at and after this time this pair by passing through the variety of shapes becomes single vortex at t=0.993 as shown in figure 11(b-d). Moreover, when R increases further the secondary vortices remain isolate till t=0.154. If more increase in R is made i.e., R^2=1.22x10^5 then the time of dissolving of vortices also increases viz., t=1.438. Thus it is found that the secondary vortices in pair are produced in early stage of flows and are dissolved and merged into single big diameter vortex depend upon R and t. When R increases the merge time increases and diameter vortex increases as time passes sometime and then again they occur after certain time. This process varies with the variation with the variation of R. When α increases then one can observe that the vortices are damped and compressed and their radius decreases with the increase of time as shown in figures 12 and 13. The maximum values of vorticity E_M shifts to higher value of time t, that is, E_M occurs at larger time as rotation parameter is increased as shown in figure 13 for α=4.5 respectively.

Figure 14 indicates the variation of minimum vorticity for θ=15° for R=1 and 349.285 and α=2.5 and 4.5. It is observed that the minimum value of vorticity E_{min} fluctuates as time passes. It decreases as either rotation parameter or Reynolds number R increases.

The variation of drag and lift on the surface of the cylinder with respect to time t is also examined for different values of R and α as shown in figures 15 and 16. It can be observed that drag increases as R increases for fix value of α as shown in figure 15, while situation is vice versa for lift. Here to overcome lift decline, the rotation acts as a vital role. If the rotation parameter increases keeping R as constant we observe that the drag decreases while the lift increases as shown in figures 15 and 16.

The loading effects on the surface of the cylinder is one of our main points of studies, so we have also examined the variation of pressure on the cylinder's surface for various values of the flow parameters and the results are displayed in figures 17. The figure 17(a) illustrates the variation of the pressure on the surface of the cylinder when the cylinder is not rotating while figures17(b-d) represent pressure's variation when rotation is imposed and varies. We also examine that the pressure in the wake and it is observed that it decreases as time passes. This situation does not depend upon whether cylinder is rotating or not. Moreover, the pressure in the wake also decreases strictly and non-linearly as the rotation increases. However, the pressure at front stagnation point remains nearly unaltered as the flow parameters R and α vary and as well as time passes. Figures 17(b-d) show that if the rotation of cylinder begins to fluctuate while it does not occur when rotation tends to stop. Moreover, when α=0 the variation of the pressure on the surface of a cylinder agrees well with that of Collins and Dennis [8].

Finally, the variation of vorticity over the surface of the cylinder with respect to θ is investigated for various values of time t and different values of Reynolds number R and rotation parameter α as shown in figures 18 and 19. In Collins and Dennis[9] at page 479 , it is stated, "The integration terminated soon after (at t=3.2) because of its failure of converge." But our numerical scheme converges for all values of Reynolds number R as well as for $t \geq 0$ including t=3.2, when α =0. Figures 18 and 19 show that our results are in good agreement with those of Collins and Dennis[9]. For large values of t, the results are also displayed graphically for α=0 as well as for α=4.5 in figures 19 and 20(b).

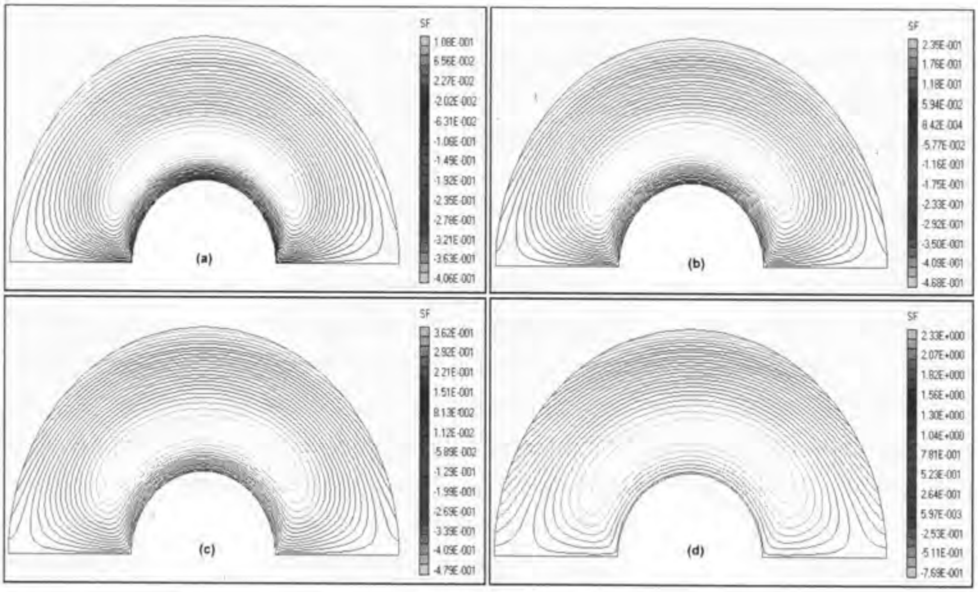

Figure 2. Streamlines for R=1, α=2.5 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

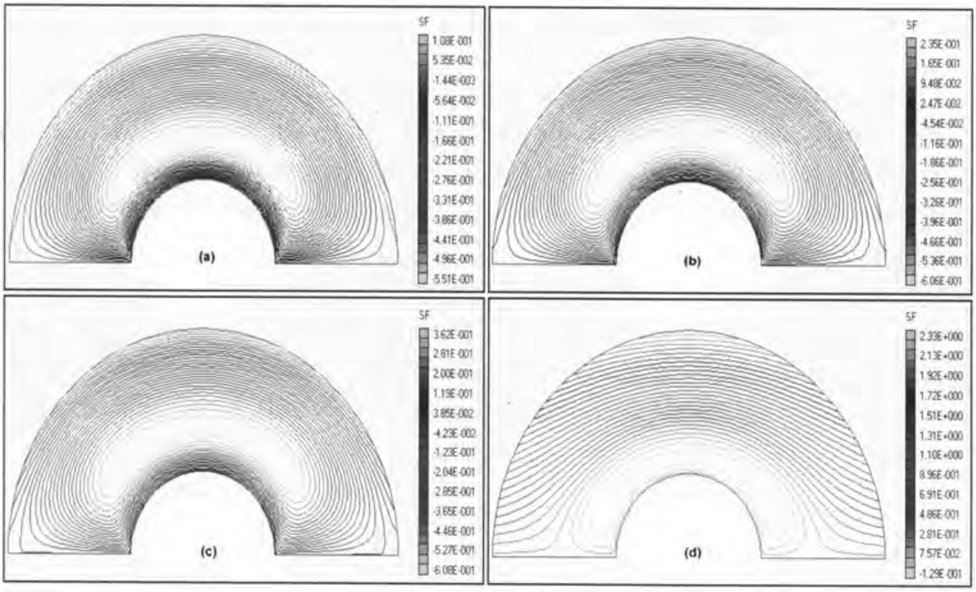

Figure 3. Streamlines for R=1, α=3.0 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

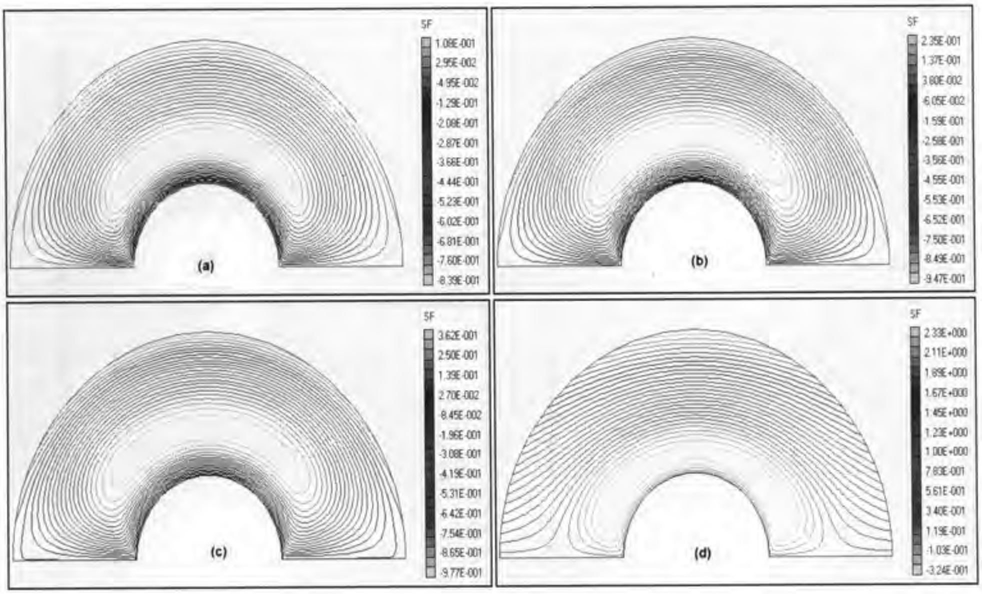

Figure 4. Streamlines for R=1, α=4.5 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

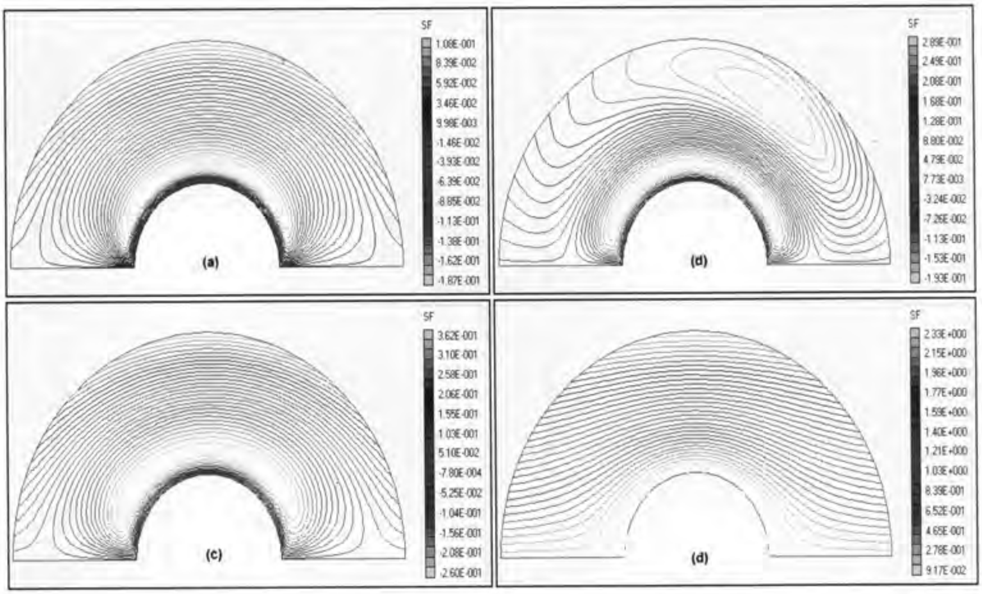

Figure 5. Streamlines for R=10, α=2.5 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

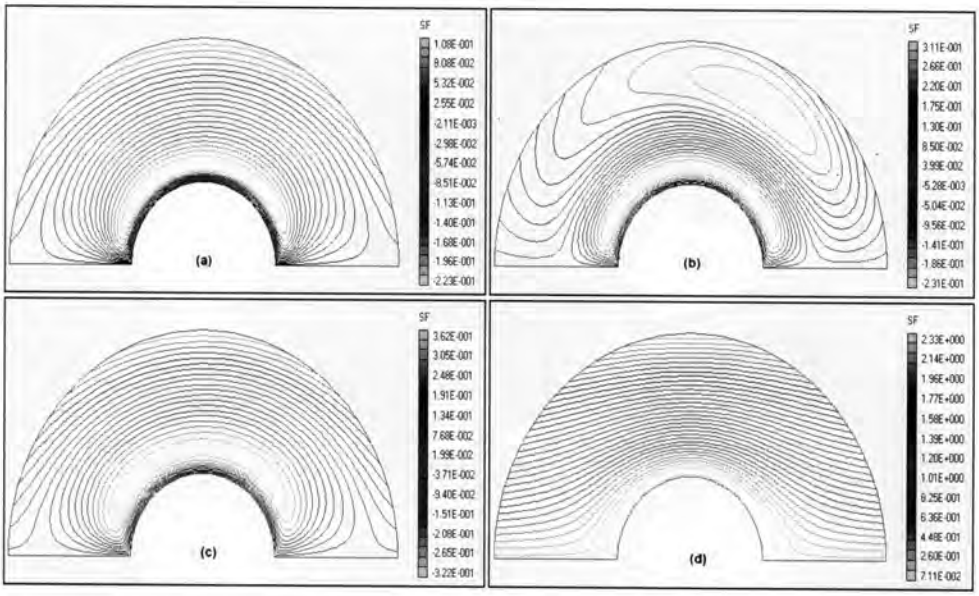

Figure 6. Streamlines for R=10, α=3.0 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

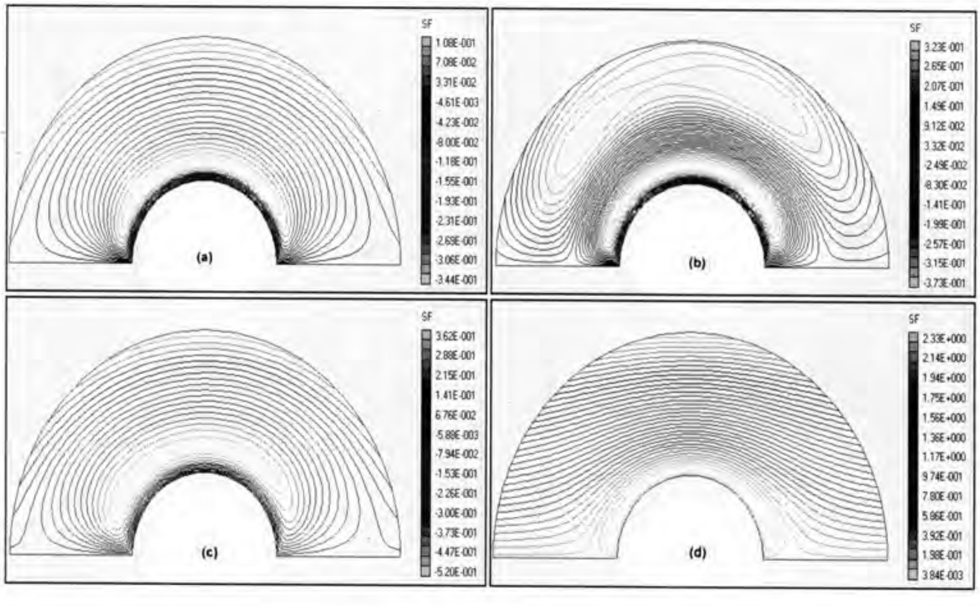

Figure 7. Streamlines for R=10, α=4.5 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

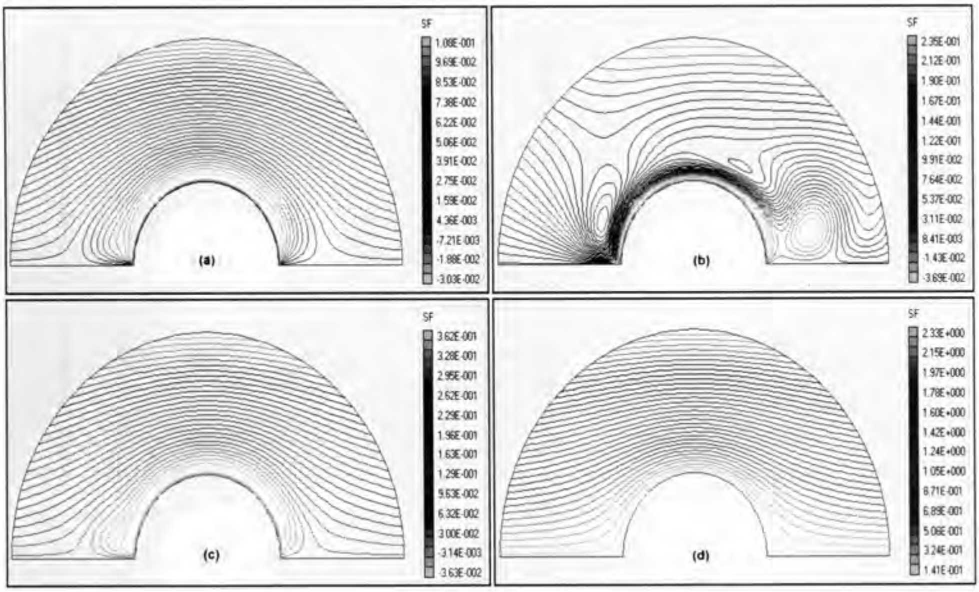

Figure 8. Streamlines for R=349.285, α=2.5 at time levels (a)t=0.046 (b)t=0.1 (c)t=0.154 (d)t=0.993.

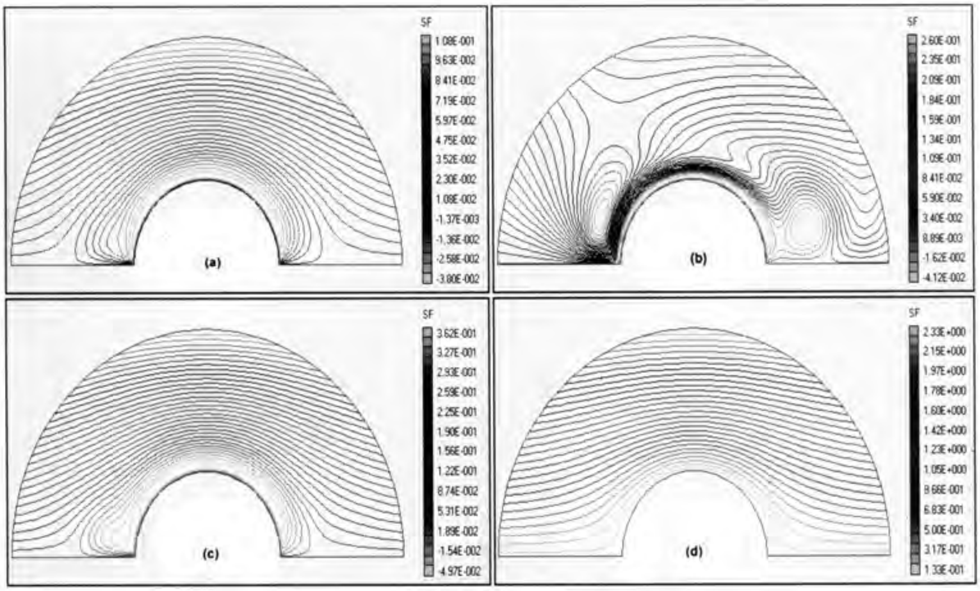

Figure 9. Streamlines for R=349.285, α=3 at time levels (a)t=0.046 (b)t=0.1 (c)t=0.154 (d)t=0.993.

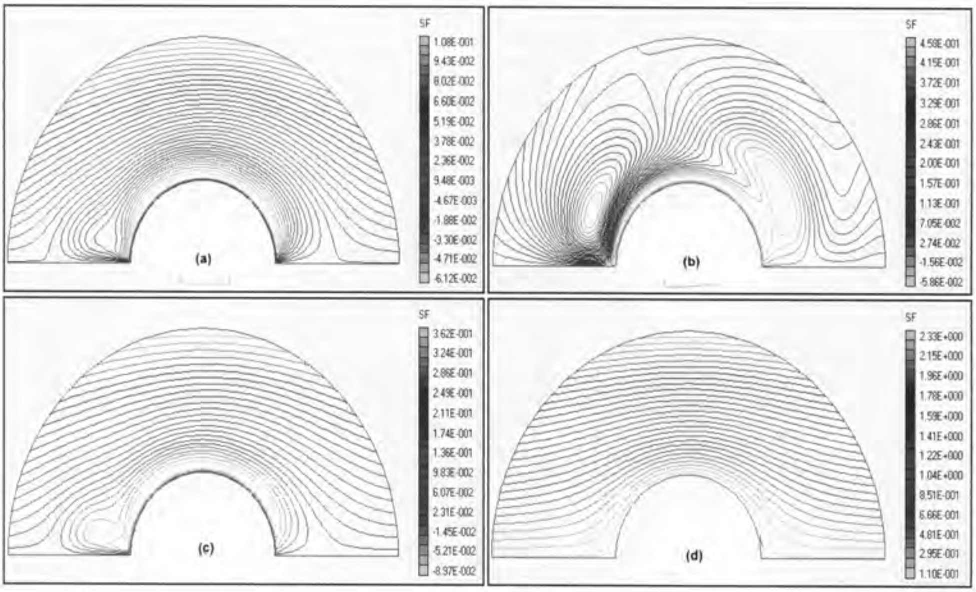

Figure 10. Streamlines for R=349.285, =4.5 at time levels (a)t=0.046 (b)t=0.1 (c)t=0.154 (d)t=0.993.

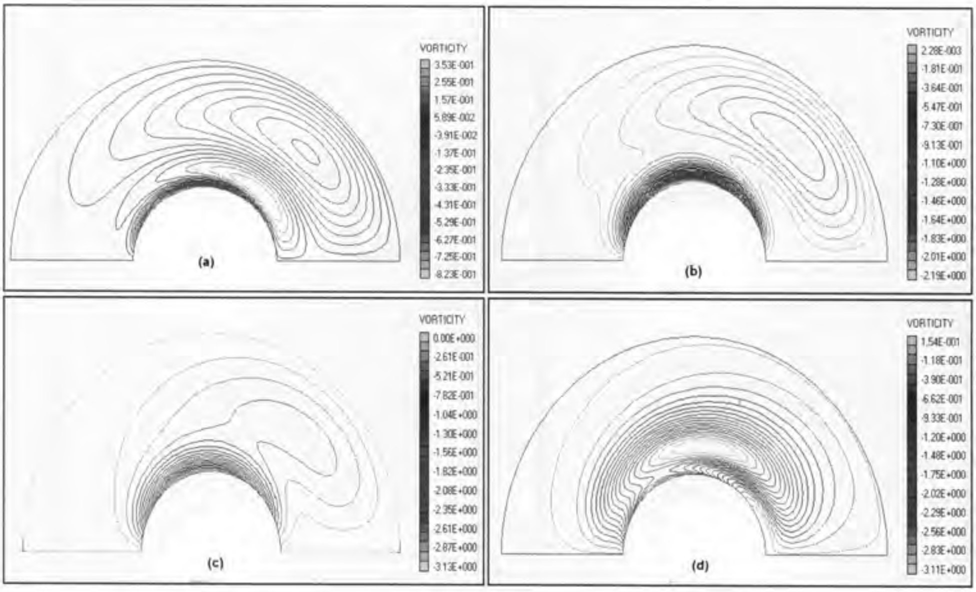

Figure 11. The lines of constant vorticity for R=10, α=0 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

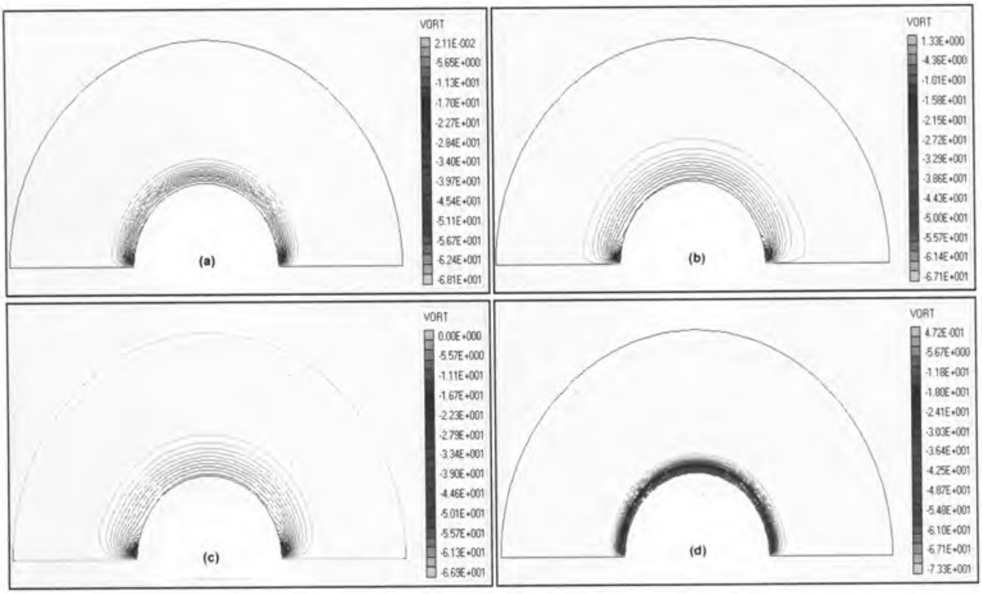

Figure 12. The lines of constant vorticity for R=10, α=2.5 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

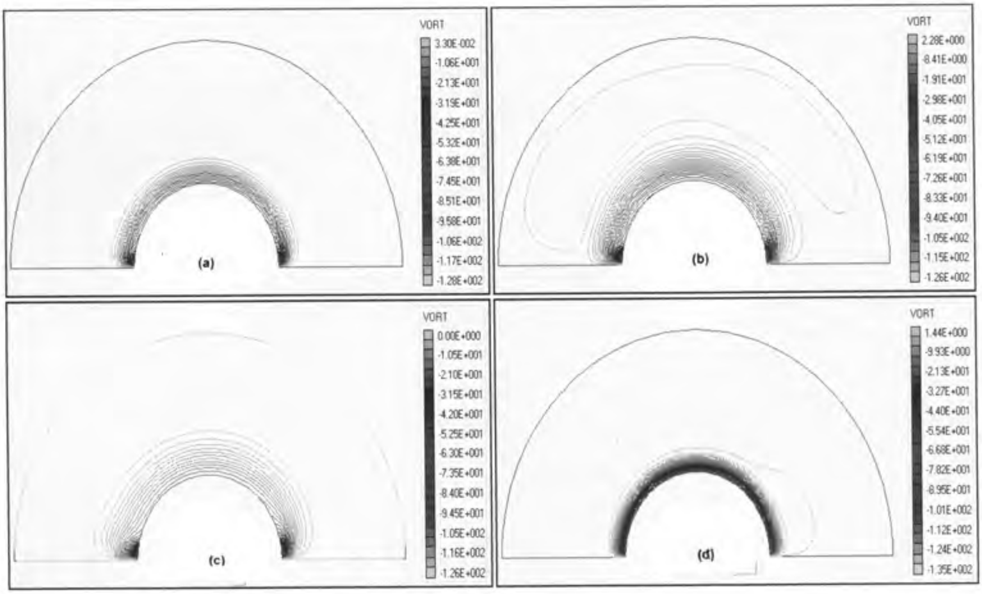

Figure 13. The lines of constant vorticity for R=10, α=4.5 at time levels (a) t=0.046 (b) t=0.1 (c) t=0.154 (d) t=0.993.

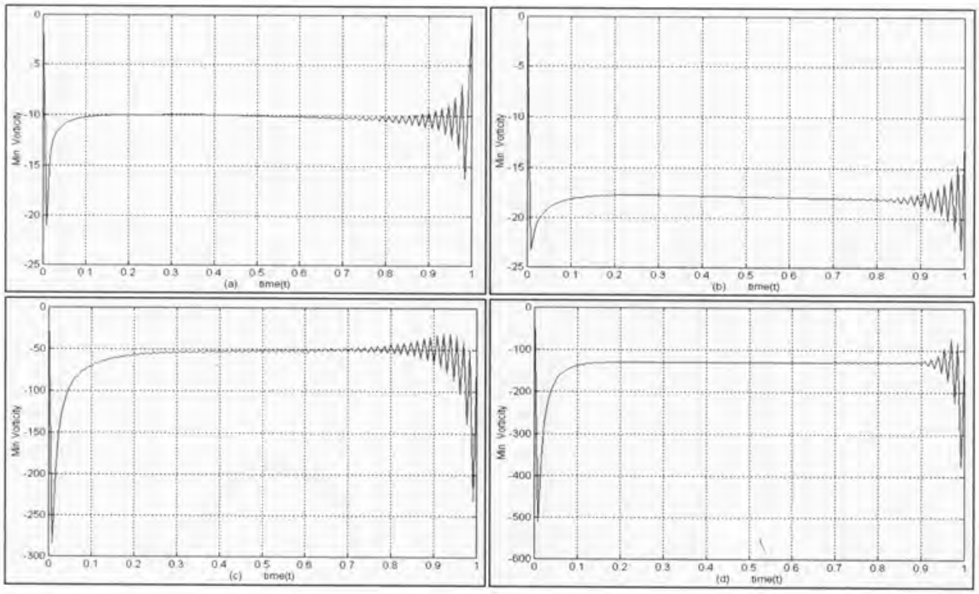

Figure 14. The variation of minimum value of vorticity E_{min} for (a)R=1, α=0 (b) R=1, α=4. 5 (c)R=349.285, α=2.5 (d) R=349.285, α=4. 5.

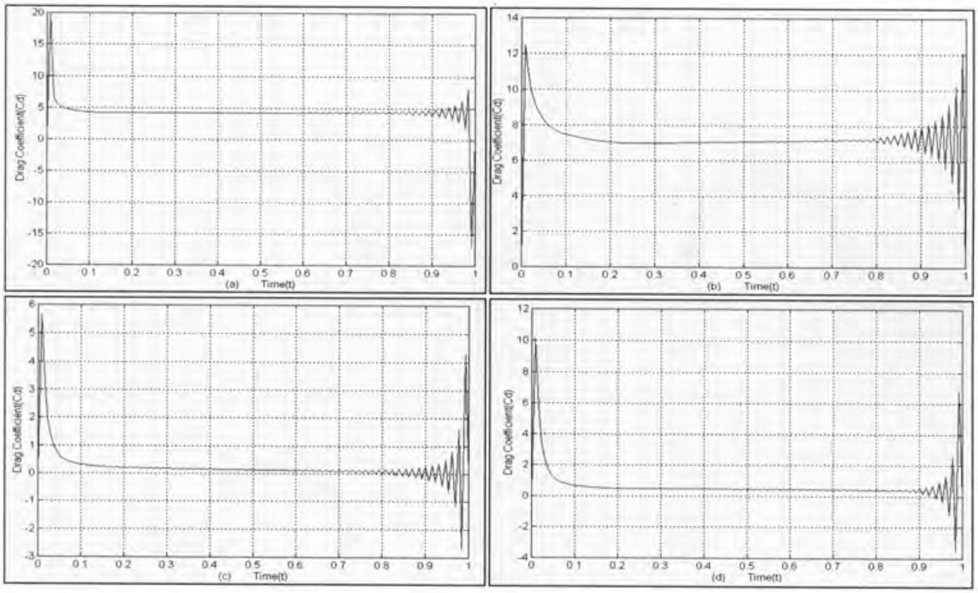

Figure 15. The variation of the drag over the surface of the cylinder for (a) α=0 (b) α=4. 5(c)R=349.285, α=2.5 (d) R=349.285, α=4. 5.

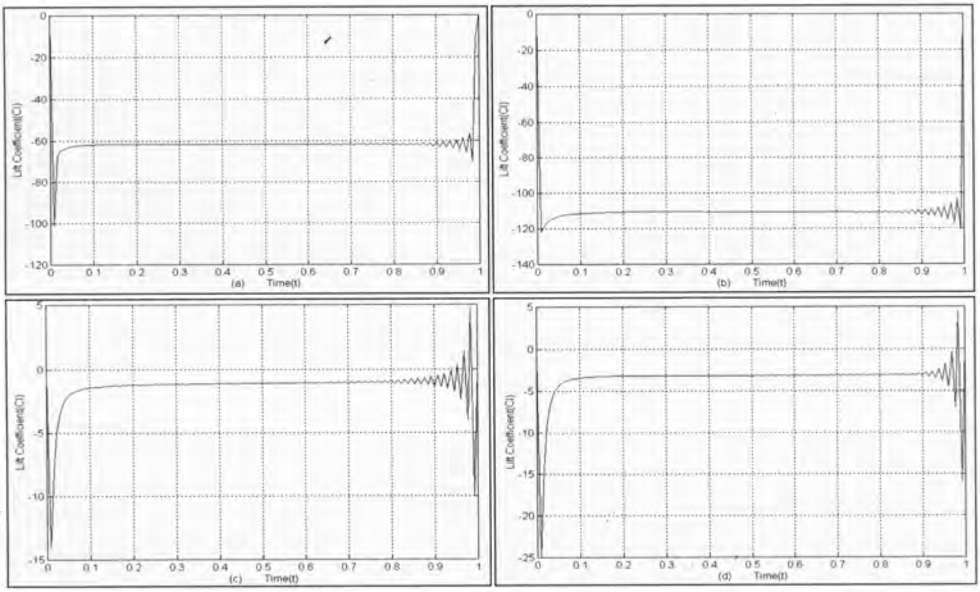

Figure 16. The variation of the lift over the surface of the cylinder for (a) α=0 (b) α=4. 5(c)R=349.285, α=2.5 (d) R=349.285, α=4.5.

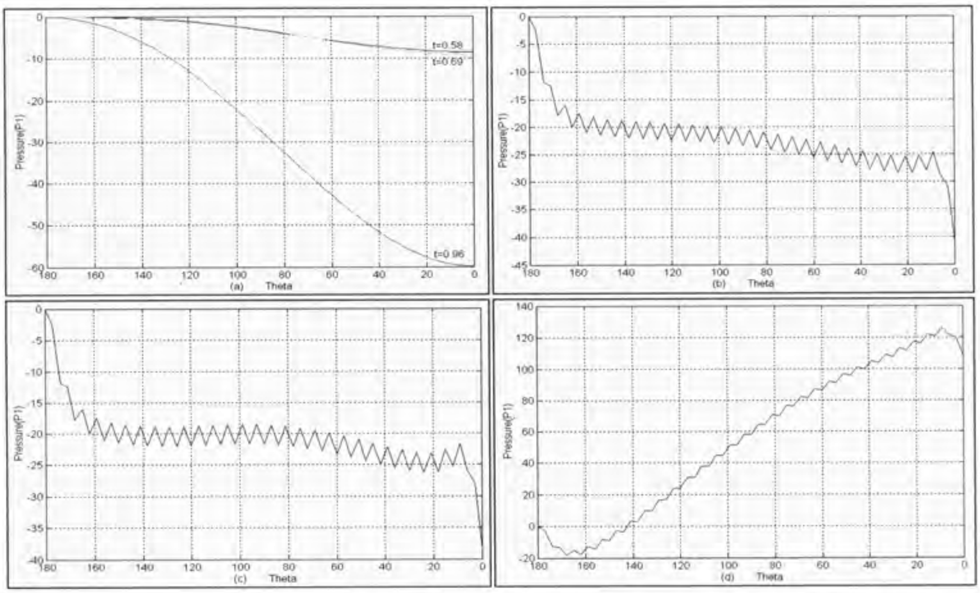

Figure 17. The variation of the pressure over the surface of the cylinder for R=349.285 (a) α=0 (b-d) α=4.5 at time levels t=0.58, 0.69, 0.96 respectively.

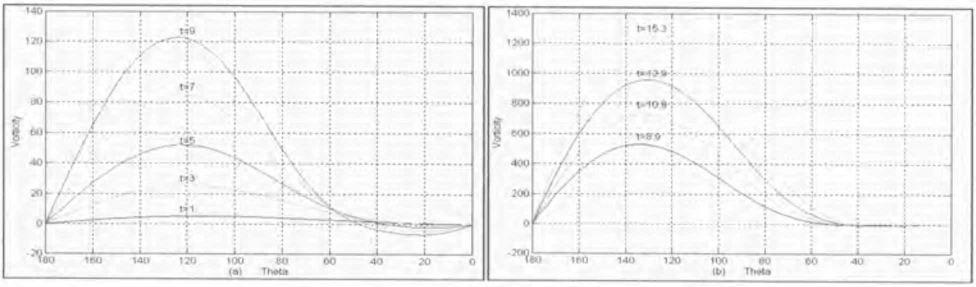

Figure 18. The variation of vorticity over the surface of a cylinder for smaller time for (a) R=10 (b) R=349.285 when α=0.

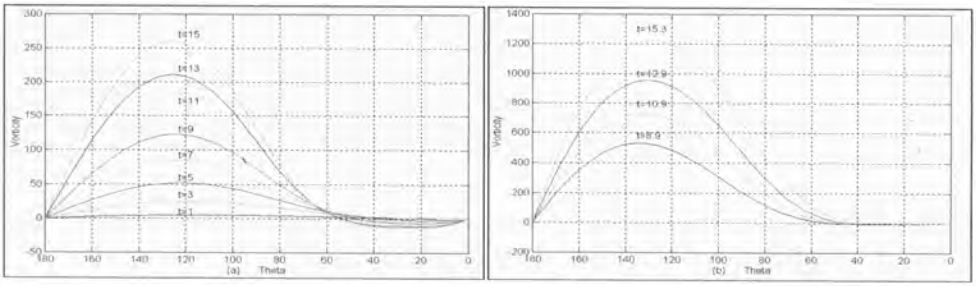

Figure 19. The variation of vorticity over the surface of a cylinder for larger time for (a) R=10 (b) R=349.285 when α=0.

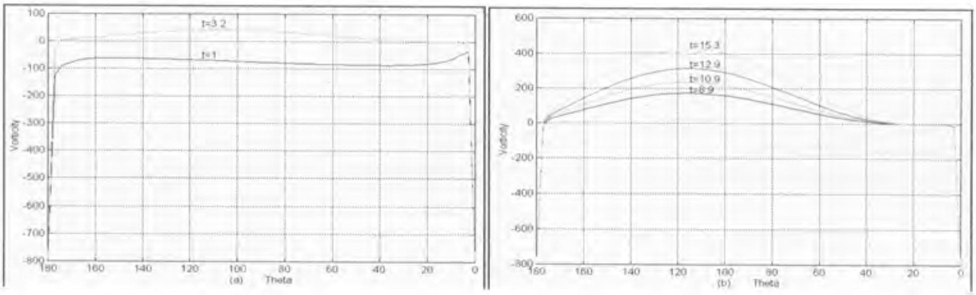

Figure 20. The variation of vorticity over the surface of a cylinder for R=349.285, α=4.5 (a) for smaller time (b) for larger time.

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المجلة العربية للعلوم الرياضية

مجلة علمية محكمة تصدر نصف سنوياً عن الجمعية السعودية للعلوم الرياضية

بالتعاون مع قسم الرياضيات بجامعة الملك سعود

المجلد الحادي عشر **جمادى الأولى ١٤٢٦هـ**

العدد الأول **يونيو/حزيران ٢٠٠٥م**

قواعد النشر

١) تعني المجلة العربية للعلوم الرياضية بنشر البحوث الأصيلة في جميع فروع العلوم الرياضية ، بما في ذلك الرياضيات البحته والتطبيقية ، والإحصاء وبحوث العمليات ، والرياضيات الحاسبة. يتكون كل مجلد من عددين تصدر نصف سنوياً.

٢) تخضع جميع الأبحاث التي تقدم إلى المجلة لعملية تحكيم ، ويشترط فيها أنه لم يسبق نشرها وليست في طريقها إلى النشر لدى أية جهة أخرى. يقدم المؤلف (المؤلفون) البحث مطبوعاً على هيئة LaTeX format بالشكل المتبع في أي عدد من أعداد المجلة ، ومذيلاً برقم التصنيف في أسفل الصفحة الأولى، ويرسل للمجلة من ثلاث نسخ على العنوان التالي:

رئيس التحرير ــ المجلة العربية للعلوم الرياضية

قسم الرياضيات بجامعة الملك سعود

ص ٢٤٥٥ الرياض ١١٤٥١

المملكة العربية السعودية

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٣) يعاد البحث بعد قبوله للنشر إلى المؤلف لإجراء التعديلات والتصويبات المطلوبة ، ويفضل أن يصل البحث إلى المجلة في صيغته الأخيرة بواسطة البريد الالكتروني على هيئة LaTeX format.

٤) لا تتقاضى المجلة أي رسوم على نشر البحوث ، وتزود الباحث (الباحثين) بعشرين نسخة من كل بحث منشور.

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