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Two computational algorithms for the numerical solution for system of fractional differential equations

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Abstract. In this paper, two efficient numerical methods for solving system of fractional differential equations (SFDEs) are considered. The fractional derivative is described in the Caputo sense. The first method is based upon Chebyshev approximations, where the properties of Chebyshev polynomials are utilized to reduce SFDEs to system of algebraic equations. Special attention is given to study the convergence and estimate the error of the presented method. The second method is the fractional finite difference method (FDM), where we implement the Grünwald–Letnikov's approach. We study the stability of the obtained numerical scheme. The numerical results show that the approaches are easy to implement implement for solving SFDEs. The methods introduce a promising tool for solving many systems of linear and non-linear fractional differential equations. Numerical examples are presented to illustrate the validity and the great potential of both proposed techniques.

Keywords: System of fractional differential equations; Caputo derivative; Chebyshev approximation; Convergence analysis; Grünwald–Letnikov's approach; Fractional FDM

1. Introduction

Fractional differential equations have recently been applied in various applications of engineering, science, finance, applied mathematics, bio-engineering and others. However, many researchers remain unaware of this field. Ordinary and partial fractional

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differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [23]. Consequently, considerable attention has been given to the solutions of FDEs of physical interest. Most FDEs do not have exact solutions, so approximate and numerical techniques [6,8,31], must be used. Recently, several numerical methods to solve the FDEs have been given such as, variational iteration method [10,30], homotopy perturbation method [29], homotopy analysis method [9], collocation method [7,14,16,17,19,24] and finite difference method [1,2,13,20,21,27,32,33].

We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1. The Riemann–Liouville fractional derivative operator D_R^α of order α is defined by [23]

$$D_R^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x \frac{f(t)}{(x-t)^{\alpha-m+1}} dt, & m-1 < \alpha < m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \end{cases}$$

where m is a positive integer and Γ is the Gamma function.

Definition 2. The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \qquad \alpha > 0,$$

where $m-1 < \alpha \leqslant m, m \in \mathbb{N}, x > 0$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x),$$

where λ and μ are constants. For Caputo's derivative we have [23]

$$D^\alpha C = 0, \qquad C \text{ is a constant}, \tag{1}$$

$$D^\alpha x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geqslant \lceil \alpha \rceil. \end{cases} \tag{2}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivative definitions and their properties see [22,23,25].

The main aim of the present work is to apply the Chebyshev collocation method and the fractional finite difference method to solve numerically the system of fractional differential equations. The two proposed methods discretize the introduced problem to system of algebraic equations thus greatly simplifying the problem. Chebyshev polynomials are well known family of orthogonal polynomials on the interval $[-1,1]$ that have many applications [18,28]. They are widely used because of their good properties in the approximation of functions. Khader [15] introduced a new approximate formula of the fractional derivative and used it to solve numerically the fractional diffusion equation. Ashyralyev and Cakir [4] presented stable difference schemes for the fractional parabolic equation with Dirichlet and Neumann boundary conditions. Also, stability estimates and almost coercive stability estimates for the solution of these difference schemes are obtained. A procedure of the modified Gauss elimination method is used for solving these difference schemes of one dimensional fractional parabolic PDEs. In this work, we will extend this formula to solve SFDEs and prove the error estimate of the introduced formula.

In this article, we consider the following general form of the non-linear system of differential equations

$$D^{v_i}u_i(x) = f_i(x, u_1, u_2, \dots, u_n), \quad u_i^{(r)}(0) = c_i^r, \quad 0 \leqslant r \leqslant \lceil v \rceil. \tag{3}$$

The existence and the uniqueness of this initial value problem for the system of FDEs (3) have been proved in [5]. Many authors considered this system to solve it using different numerical methods, for example, differential transform method [8] and Adomian decomposition method [11].

The organization of this paper is as follows. In the next section, the approximation of fractional derivative $D^{\alpha}x(t)$ is derived, study the convergence analysis and estimate the error of the derived formula. Section 3, summarizes the definitions of Grünwald–Letnikov's approaches to Caputo's derivative. Section 4, is assigned to implement the two proposed methods to solve numerically two systems of FDEs. Also, a conclusion is given in Section 5.

2. Derivation of an approximate formula for fractional derivatives using Chebyshev series expansion

The well-known Chebyshev polynomials [28] are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad T_0(z) = 1, \; T_1(z) = z, \quad n = 1, 2, \dots .$$

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by

$$T_n(z) = n\sum_{i=0}^{[\frac{n}{2}]}(-1)^i \, 2^{n-2i-1} \frac{(n-i-1)!}{(i)!(n-2i)!} z^{n-2i}, \qquad n = 3, 4, \dots . \tag{4}$$

where $[n/2]$ denotes the integer part of $n/2$. The orthogonality condition is

$$\int_{-1}^{1} \frac{T_i(z)T_j(z)}{\sqrt{1-z^2}} dz = \begin{cases} \pi, & \text{for } i = j = 0; \\ \frac{\pi}{2}, & \text{for } i = j \neq 0; \\ 0, & \text{for } i \neq j. \end{cases}$$

In order to use these polynomials on the interval $[0,1]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2t - 1$.

The shifted Chebyshev polynomials are defined as $T_n^*(t) = T_n(2t-1) = T_{2n}(\sqrt{t})$.

The analytic form of the shifted Chebyshev polynomial $T_n^*(t)$ of degree n is given by

$$T_n^*(t) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!}t^k, \quad n = 1, 2, \ldots. \tag{5}$$

The function $x(t)$, which belongs to the space of square integrable functions in $[0, 1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$x(t) = \sum_{i=0}^{\infty}c_i\ T_i^*(t), \tag{6}$$

where the coefficients c_i are given by

$$c_0 = \frac{1}{\pi}\int_0^1 \frac{x(t)T_0^*(t)}{\sqrt{t-t^2}}\ dt, \qquad c_i = \frac{2}{\pi}\int_0^1 \frac{x(t)\ T_i^*(t)}{\sqrt{t-t^2}}\ dt, \qquad i = 1, 2, \ldots. \tag{7}$$

In practice, only the first $(m+1)$-terms of shifted Chebyshev polynomials are considered. Then we have

$$x_m(t) = \sum_{i=0}^{m}\ c_i\ T_i^*(t). \tag{8}$$

Theorem 1. *(Chebyshev truncation theorem)* [28]

The error in approximating $x(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$x_m(t) = \sum_{k=0}^{m}c_k\ T_k(t), \tag{9}$$

then

$$E_T(m) \equiv |x(t) - x_m(t)| \leqslant \sum_{k=m+1}^{\infty}|c_k|, \tag{10}$$

for all $x(t)$, all m, and all $t \in [-1, 1]$.

The main approximate formula of the fractional derivative of $x(t)$ is given in the following theorem.

Theorem 2. *Let $x(t)$ be approximated by Chebyshev polynomials as (8) and also suppose $\alpha > 0$, then*

$$D^{\alpha}(x_m(t)) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i}c_i\ w_{i,\ k}^{(\alpha)}\ t^{k-\alpha}, \tag{11}$$

where $w_{i,\ k}^{(\alpha)}$ is given by

$$w_{i,\ k}^{(\alpha)} = (-1)^{i-k}\frac{2^{2k}\ i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}. \tag{12}$$

Proof. Since Caputo's fractional differentiation is a linear operation we have

$$D^{\alpha}(x_m(t)) = \sum_{i=0}^{m} c_i D^{\alpha}\left(T_i^*(t)\right). \tag{13}$$

Employing Eqs. (1) and (2) we have

$$D^{\alpha}T_i^*(t) = 0, \qquad i = 0, 1, \ldots, \lceil\alpha\rceil - 1, \quad \alpha > 0. \tag{14}$$

Also, for $i = \lceil\alpha\rceil, \lceil\alpha\rceil + 1, \ldots, m$, and by using Eqs. (1) and (2), we get

$$\begin{aligned} D^{\alpha}T_i^*(t) &= i\sum_{k=\lceil\alpha\rceil}^{i} (-1)^{i-k} \frac{2^{2k}(i+k-1)!}{(i-k)!(2k)!} D^{\alpha}t^k \\ &= i\sum_{k=\lceil\alpha\rceil}^{i} (-1)^{i-k} \frac{2^{2k}(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)} t^{k-\alpha}. \end{aligned} \tag{15}$$

A combination of Eqs. (14) and (15) and (13) leads to the desired result (11). □

Theorem 3. *The Caputo fractional derivative of order* α *for the shifted Chebyshev polynomials can be expressed in terms of the shifted Chebyshev polynomials themselves in the following form*

$$D^{\alpha}\left(T_i^*(t)\right) = \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} T_j^*(t), \tag{16}$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i-k} 2i(i+k-1)!\Gamma(k-\alpha+\frac{1}{2})}{h_j \Gamma\left(k+\frac{1}{2}\right)(i-k)!\Gamma(k-\alpha-j+1)\Gamma(k+j-\alpha+1)}, \quad h_0 = 2,\ h_j = 1,\ j = 1, 2, \ldots.$$

Proof. See [7,18]. □

Theorem 4. *The error* $|E_T(m)| = |D^{\alpha}x(t) - D^{\alpha}x_m(t)|$ *in approximating* $D^{\alpha}x(t)$ *by* $D^{\alpha}x_m(t)$ *is bounded by*

$$|E_T(m)| \leqslant \left| \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \right) \right|. \tag{17}$$

Proof. A combination of Eqs. (6), (8) and (16) leads to

$$|E_T(m)| = |D^{\alpha}x(t) - D^{\alpha}x_m(t)| = \left| \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} T_j^*(t) \right) \right|,$$

but $|T_j^*(t)| \leqslant 1$, so, we can obtain

$$|E_T(m)| \leqslant \left| \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \right) \right|,$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem. □

Remark 1. The difference between the Caputo derivative and Riemann-Liouville derivative is given by the equation [23]

$$D^{\alpha}y(t) - D^{\alpha}_{R}y(t) = \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} y^{(k)}(0), \; m-1 \leqslant \alpha \leqslant m. \tag{18}$$

Under natural conditions on the function $y(t)$, for $\alpha \longrightarrow n$, Caputo's derivative becomes a conventional n-th derivative of the function $y(t)$.

3. Grünwald–Letnikov's approach to Caputo's derivative

In this section, we introduce the definition of Grünwald-Letnikov fractional derivative [23].

Definition 3. The Grünwald–Letnikov's approach is defined as

$$\frac{d^{\alpha}y(t)}{dt^{\alpha}} = \lim_{h \longrightarrow 0} \frac{1}{h^{\alpha}} \sum_{i=0}^{\left[\frac{t}{h}\right]} (-1)^{i} \binom{\alpha}{i} y(t-ih), \tag{19}$$

and the shifted Grünwald–Letnikov fractional derivative is defined as

$$\frac{d^{\alpha}y(t)}{dt^{\alpha}} = \lim_{h \longrightarrow 0} \frac{1}{h^{\alpha}} \sum_{i=0}^{\left[\frac{t}{h}\right]+1} (-1)^{i} \binom{\alpha}{i} y(t-(i-1)h), \tag{20}$$

where $[t]$ means the integer part of t. Here, we state a lemma given in [12].

Lemma 1. *Assume that y satisfies some smoothness conditions e.g., $y(t)$ can be written in the form of a power series for $|t| < \rho$. The Grünwald–Letnikov formula holds for each $0 < r < \rho$ and a series of step size h with $\frac{\tau}{h} \in \mathbb{N}$,*

$$D^{\alpha}_{R}y(\tau) = \frac{1}{h^{\alpha}} \Delta^{\alpha}_{h} y(nh) + O(h), \quad (h \longrightarrow 0),$$

where

$$\Delta^{\alpha}_{h} y(nh) = \sum_{i=0}^{n} (-1)^{i} \binom{\alpha}{i} y(t_{n-i}). \tag{21}$$

In the case of Caputo's operator, we have according to Eq. (18), for $0 < \alpha < 1$,

$$D^{\alpha}y(\tau) = \frac{1}{h^{\alpha}} \Delta^{\alpha}_{h} y(nh) - \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} y(0) + O(h), \quad (h \longrightarrow 0). \tag{22}$$

The most favorable case is when the initial values for Caputo's differential equation are given to be zero.

The numerical formula (22) is used to solve numerically more problems, for example this formula is implemented to solve the Bagley–Torvik equation [23]. Also, it has been applied to solve the fractional-order heat equation [26]. In this paper, we extend this implementation of this formula to solve the system of fractional differential equations.

4. Numerical implementation

In this section, we solve numerically system of FDEs using the two approches, the Chebyshev spectral method and the Grünwald-Letnikov finite difference approach. To achieve this purpose we will consider the following two examples.

Example 1. We consider the following system of linear fractional differential equations

$$\begin{aligned} D^{\alpha}\, x(t) &= x(t) + y(t), \\ D^{\beta}\, y(t) &= -x(t) + y(t), \end{aligned} \tag{23}$$

the parameters α and β refer to the fractional order of time derivative with $0 < \alpha, \beta \leqslant 1$.

We also assume the following initial conditions

$$x(0) = 0, \qquad y(0) = 1. \tag{24}$$

1.I: Implementation of Chebyshev approximation

Consider the systems of fractional differential Eq. (23). In order to use the Chebyshev collocation method, we first approximate $x(t)$ and $y(t)$ as

$$x_m(t) = \sum_{i=0}^{m} a_i\; T_i^*(t), \quad y_m(t) = \sum_{i=0}^{m} b_i\; T_i^*(t). \tag{25}$$

From Eq. (23) and Theorem 2 we have

$$\begin{aligned} \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} t^{k-\alpha} &= \sum_{i=0}^{m} a_i T_i^*(t) + \sum_{i=0}^{m} b_i T_i^*(t), \\ \sum_{i=\lceil\beta\rceil}^{m}\sum_{k=\lceil\beta\rceil}^{i} b_i w_{i,k}^{(\beta)} t^{k-\beta} &= -\sum_{i=0}^{m} a_i T_i^*(t) + \sum_{i=0}^{m} b_i T_i^*(t). \end{aligned} \tag{26}$$

We now collocate Eq. (26) at $(m+1-\lceil v\rceil)$ points $t_p (p = 0, 1, \ldots, m-\lceil v\rceil)$ as

$$\begin{aligned} \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} t_p^{k-\alpha} &= \sum_{i=0}^{m} a_i T_i^*(t_p) + \sum_{i=0}^{m} b_i T_i^*(t_p), \quad p = 0, 1, \ldots, m-\lceil\alpha\rceil, \\ \sum_{i=\lceil\beta\rceil}^{m}\sum_{k=\lceil\beta\rceil}^{i} b_i w_{i,k}^{(\beta)} t_p^{k-\beta} &= -\sum_{i=0}^{m} a_i T_i^*(t_p) + \sum_{i=0}^{m} b_i T_i^*(t_p), \quad p = 0, 1, \ldots, m-\lceil\beta\rceil. \end{aligned} \tag{27}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $T^*_{m+1-\lceil v\rceil}(t)$.

Also, by substituting Eq. (25) in the initial conditions (24) we can find

$$\sum_{i=0}^{m}(-1)^i a_i = 0, \quad \sum_{i=0}^{m}(-1)^i b_i = 1. \tag{28}$$

Eq. (27), together with the equations of the initial conditions (28), give $(2m+2)$ of linear algebraic equations which can be solved using the conjugate gradient method, for the unknowns a_i and $b_i, i = 0, 1, \ldots, m$.

1.II: Implementation of the fractional finite difference method

Here we will discretize the considered system (23) using the approximate formula (22) as follows. First, for system, we consider

$$\begin{aligned} D^{\alpha}x(t_n) &= x(t_n) + y(t_n), \\ D^{\beta}y(t_n) &= -x(t_n) + y(t_n). \end{aligned} \tag{29}$$

Second, we use the uniform grid $t_n = nh$, where $n = 0, 1, \ldots, M, Mh = T$ and use the abbreviations x_n and y_n for approximation of the true solutions $x(t_n)$ and $y(t_n)$ in the grid point t_n. Applying the shifted definition of the Grünwald–Letnikov fractional derivative and (22) to our system (29), we obtain

$$\begin{aligned} &\frac{1}{h^{\alpha}}\sum_{i=0}^{n+1}(-1)^i\theta_i^{\alpha}x_{n+1-i} - \frac{(nh)^{-\alpha}}{\Gamma(1-\alpha)}x_0 = x_n + y_n, \\ &\frac{1}{h^{\beta}}\sum_{i=0}^{n+1}(-1)^i\theta_i^{\beta}y_{n+1-i} - \frac{(nh)^{-\beta}}{\Gamma(1-\beta)}y_0 = -x_n + y_n, \end{aligned} \tag{30}$$

where $\theta_i^p = \binom{p}{i}, p = \alpha, \beta$.

To study the stability of the numerical scheme in (30), we state and prove the following two theorems.

Theorem 5. [3] *Numerical approximation (30) is consistent with fractional-order differential Eqs. (29), $x_m - x(t_m) = O(h^{1+\alpha})$ and $y_m - y(t_m) = O(h^{1+\beta})$.*

Theorem 6. *The numerical scheme (30) of the system of fractional differential Eqs. (29) is stable.*

Proof. Let (x_n, y_n) and $(\bar{x}_n, \bar{y}_n)$ be two solutions of the numerical scheme (30). Let $e_n = x_n - \bar{x}_n$ and $E_n = y_n - \bar{y}_n$, we have

$$\frac{1}{h^{\alpha}}\sum_{i=0}^{n+1}(-1)^i\theta_i^{\alpha}x_{n+1-i} - \frac{(nh)^{-\alpha}}{\Gamma(1-\alpha)}x_0 = x_n + y_n, \tag{31}$$

$$\frac{1}{h^{\beta}}\sum_{i=0}^{n+1}(-1)^i\theta_i^{\beta}x_{n+1-i} - \frac{(nh)^{-\beta}}{\Gamma(1-\beta)}y_0 = -x_n + y_n, \tag{32}$$

$$(1-h^{\alpha})e_n = -\sum_{i=1}^{n+1}(-1)^i\theta_i^{\alpha}e_{n-i} + h^{\alpha}E_n \leqslant -\sum_{i=0}^{\infty}(-1)^i\theta_i^{\alpha}e_{n-i}, \tag{33}$$

$$(1-h^{\beta})E_n = -\sum_{i=1}^{n+1}(-1)^i\theta_i^{\beta}E_{n-i} - h^{\beta}e_n \leqslant -\sum_{i=0}^{\infty}(-1)^i\theta_i^{\beta}E_{n-i}. \tag{34}$$

We have $(-1)^0\theta_0^p = 1, (-1)^i\theta_i^p < 0, i = 1, 2, , \ldots$ and $\sum_{i=0}^{\infty}(-1)^i\theta_i^p = 1$, $p = \alpha, \beta$.

Thus from (33) and (34), we have

$$\|e_n\| \leqslant \max(\|e_0\|, \|e_1\|, \dots, \|e_{n-1}\|) \leqslant \dots \leqslant \|e_0\|,$$
$$\|E_n\| \leqslant \max(\|E_0\|, \|E_1\|, \dots, \|E_{n-1}\|) \leqslant \dots \leqslant \|E_0\|.$$

Therefore, the numerical approximation (30) for solving fractional differential Eq. (29) is stable. □

The obtained numerical results of this example using the two proposed methods are presented in Figs. 1 and 2. Where in Fig. 1, we presented the behavior of the exact solution ($\alpha = \beta = 1$) with the numerical solutions using the Chebsyhev collocation method (at $m = 6$) and the fractional finite difference method with $h = 0.1$. But, in Fig. 2, we presented the behavior of numerical solutions using the two proposed methods at ($\alpha = 0.7, \beta = 0.9$). From these figures, we can see that our numerical results are in excellent agreement with the exact solution, this gives us induction that these two methods are well to implement for solving such a system of fractional differential equations.

Example 2. We consider the following system of non-linear fractional differential equations

$$\begin{aligned} &D^{\alpha}x(t) = 2y^2, \qquad 0 < \alpha \leqslant 1,\\ &D^{\beta}y(t) = tx, \qquad 0 < \beta \leqslant 1,\\ &D^{\gamma}z(t) = yz, \qquad 0 < \gamma \leqslant 1, \end{aligned} \tag{35}$$

with the initial conditions

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = 1. \tag{36}$$

2.I: Implementation of Chebyshev approximation

In order to use the Chebyshev collocation method, we first approximate $x(t), y(t)$ and $z(t)$ as

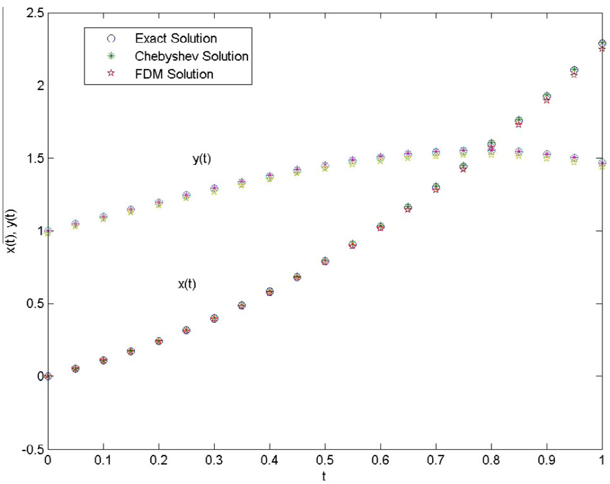

Fig. 1 The behavior of the exact solution (at $\alpha = 1, \beta = 1$), the numerical solution using the Chebyshev collocation method with $m = 6$ and the fractional FDM with $h = 0.1$.

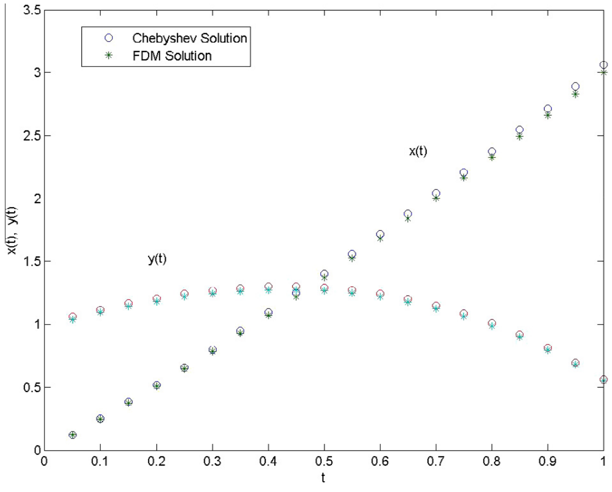

Fig. 2 The behavior of the numerical solution (at $\alpha = 0.7, \beta = 0.9$) using the Chebyshev collocation method with $m = 6$ and the fractional FDM with $h = 0.1$.

$$x_m(t) = \sum_{i=0}^{m} a_i T_i^*(t), \quad y_m(t) = \sum_{i=0}^{m} b_i T_i^*(t), \quad z_m(t) = \sum_{i=0}^{m} c_i T_i^*(t). \tag{37}$$

From Eqs. (35)–(37) and Theorem 2 we have

$$\begin{aligned} &\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} t^{k-\alpha} = 2\left(\sum_{i=0}^{m} b_i T_i^*(t)\right)^2, \\ &\sum_{i=\lceil\beta\rceil}^{m} \sum_{k=\lceil\beta\rceil}^{i} b_i w_{i,k}^{(\beta)} t^{k-\beta} = t\sum_{i=0}^{m} a_i T_i^*(t), \\ &\sum_{i=\lceil\gamma\rceil}^{m} \sum_{k=\lceil\gamma\rceil}^{i} c_i w_{i,k}^{(\gamma)} t^{k-\gamma} = \left(\sum_{i=0}^{m} b_i T_i^*(t)\right)\left(\sum_{i=0}^{m} c_i T_i^*(t)\right). \end{aligned} \tag{38}$$

We now collocate Eq. (38) at $(m + 1 - \lceil v\rceil)$ points $t_p (p = 0, 1, \ldots, m - \lceil v\rceil)$ as

$$\begin{aligned} &\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} a_i w_{i,k}^{(\alpha)} t_p^{k-\alpha} = 2\left(\sum_{i=0}^{m} b_i T_i^*(t_p)\right)^2, \qquad p = 0, 1, \ldots, m - \lceil\alpha\rceil, \\ &\sum_{i=\lceil\beta\rceil}^{m} \sum_{k=\lceil\beta\rceil}^{i} b_i w_{i,k}^{(\beta)} t_p^{k-\beta} = t_p\sum_{i=0}^{m} a_i T_i^*(t_p), \qquad p = 0, 1, \ldots, m - \lceil\beta\rceil, \\ &\sum_{i=\lceil\gamma\rceil}^{m} \sum_{k=\lceil\gamma\rceil}^{i} c_i w_{i,k}^{(\gamma)} t_p^{k-\gamma} = \left(\sum_{i=0}^{m} b_i T_i^*(t_p)\right)\left(\sum_{i=0}^{m} c_i T_i^*(t_p)\right), \qquad p = 0, 1, \ldots, m - \lceil\gamma\rceil. \end{aligned} \tag{39}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $T^*_{m+1-\lceil v\rceil}(t)$.

Also, by substituting Eq. (37) in the initial conditions (36) we can find

$$\sum_{i=0}^{m}(-1)^i a_i = 0, \qquad \sum_{i=0}^{m}(-1)^i b_i = 1, \qquad \sum_{i=0}^{m}(-1)^i c_i = 1. \tag{40}$$

Eq. (39), together with the equations of the initial conditions (40), give $(3m+3)$ of non-linear algebraic equations which can be solved using the Newton iteration method for the unknowns a_i, b_i and $c_i, i = 0, 1, \ldots, m$.

2.II: Implementation of the fractional finite difference method

Here we will discretize the considered system (35) using (22) as follows. First, for system, we consider

$$\begin{aligned} &D^\alpha x(t_n) = 2y^2(t_n), \\ &D^\beta y(t_n) = t_n x(t_n), \\ &D^\gamma z(t_n) = y(t_n) z(t_n). \end{aligned} \tag{41}$$

Second, we use the uniform grid $t_n = nh$, where $n = 0, 1, \ldots, M, Mh = T$ and use the abbreviations x_n, y_n and z_n for approximation of the true solutions $x(t_n)$, $y(t_n)$ and $z(t_n)$ in the grid point t_n.

Applying the shifted definition of the Grünwald-Letnikov fractional derivative and (22) to system (41), we obtain

$$\begin{aligned} &\frac{1}{h^\alpha}\sum_{i=0}^{n+1}(-1)^i\theta_i^\alpha x_{n+1-i} - \frac{(nh)^{-\alpha}}{\Gamma(1-\alpha)}x_0 = 2y_n^2, \\ &\frac{1}{h^\beta}\sum_{i=0}^{n+1}(-1)^i\theta_i^\beta y_{n+1-i} - \frac{(nh)^{-\beta}}{\Gamma(1-\beta)}y_0 = t_n x_n, \\ &\frac{1}{h^\gamma}\sum_{i=0}^{n+1}(-1)^i\theta_i^\gamma z_{n+1-i} - \frac{(nh)^{-\gamma}}{\Gamma(1-\gamma)}z_0 = y_n z_n, \end{aligned} \tag{42}$$

where $\theta_i^p = \binom{p}{i}, p = \alpha, \beta, \gamma$.

The obtained numerical results of this example using the two proposed methods are presented in Figs. 3 and 4. Where in Fig. 3, we presented the behavior of the exact solution $(\alpha = \beta = \gamma = 1)$ with the numerical solutions using the Chebsyhev collocation method (at $m = 6$) and the fractional FDM with $h = 0.1$. But, in Fig. 4, we presented the behavior of numerical solutions using the two proposed methods at $(\alpha = 0.8, \beta = 0.7, \gamma = 0.6)$. From these figures, we can see that our numerical results are in excellent agreement with the exact solution, this gives us induction that these two methods are well to implement for solving such a system of FDEs.

5. Conclusion

In this article, we implemented two computational methods, the Chebyshev spectral method and the fractional FDM for solving system of FDEs. The work emphasized our belief that the methods are reliable techniques to handle linear and non-linear system of FDEs. We derived an approximate formula of the fractional derivative. The properties of the Chebyshev polynomials are used to reduce FDEs to the solution of

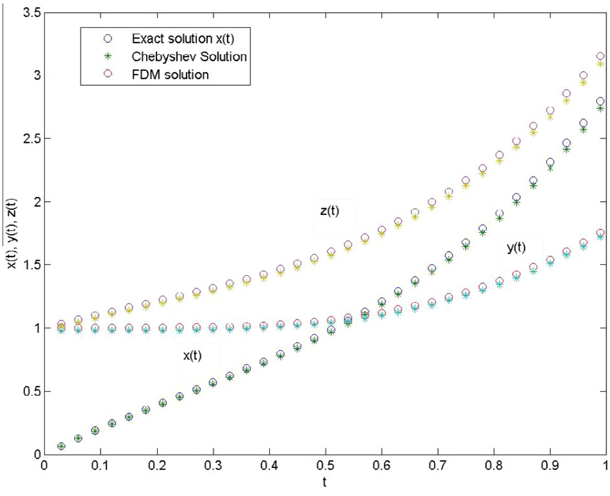

Fig. 3 The behavior of the exact solution (at $\alpha = \beta = \gamma = 1$), the numerical solution using the Chebyshev collocation method with $m = 6$ and the fractional FDM with $h = 0.1$.

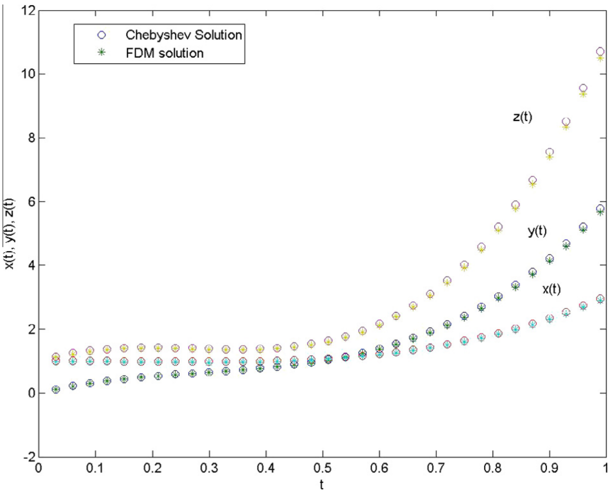

Fig. 4 The behavior of the numerical solution (at $\alpha = 0.8, \beta = 0.7, \gamma = 0.6$) using the Chebyshev collocation method with $m = 6$ and the fractional FDM with $h = 0.1$.

system of algebraic equations. Special attention is given to study the convergence analysis and estimate the upper bound of the error of the derived formula. Also, we studied the stability of the numerical scheme which was obtained from the fractional FDM using the Grünwald–Letnikov's approach. From the solutions obtained using the suggested methods, we can conclude that these solutions are in excellent agreement with the exact solution and show that these approaches can solve the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the Chebyshev series (25). All numerical results are obtained using Matlab 7.5.

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