Arab Journal of Mathematical Sciences

Arab J Math Sci 21 (2015) 159–169

Original article

Triangle-free graphs which are minimal for some nonstable 4-vertex subset

Mohammad Alzohairi

Department of Mathematics, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

Received 14 September 2014; received in revised form 11 February 2015; accepted 16 February 2015
Available online 25 February 2015

Abstract. In a graph G, a *module* is a vertex subset M such that every vertex outside M is adjacent to all or none of M. A graph G is *prime* if ϕ, the single-vertex sets, and $V(G)$ are the only modules in G. A prime graph G is *k-minimal* if there is some k-set U of vertices such that no proper induced subgraph of G containing U is prime.

Cournier and Ille in 1998 characterized the 1-minimal and 2-minimal graphs. Recently, Alzohairi and Boudabbous characterized 3-minimal triangle-free graphs. We characterize the triangle-free graphs which are minimal for some nonstable 4-vertex subset.

Keywords: Module; Prime; Minimal; Isomorphism

1. Introduction and basic notations

A *graph* G is a pair consisting of a finite *vertex set* $V(G)$ and an *edge set* $E(G)$ such that $E(G)$ is a subset of the set of the 2-element subsets of $V(G)$. We denote the edge $\{u, v\}$ by uv. Two distinct vertices u and v are *adjacent* if $uv \in E(G)$; otherwise u and v are *nonadjacent*. The set of *neighbors* of a vertex u, denoted by $N_G(u)$, is the set of vertices which are adjacent to u, and the *degree* of u, denoted by $d_G(u)$, equals $|N_G(u)|$. A vertex subset of a graph is *stable* if its elements pairwise are nonadjacent; otherwise it is *nonstable*. For a subset A of $V(G)$ and a vertex u outside A, we write $u \sim A$ if u is adjacent to all or none of A; otherwise we write $u \not\sim A$. Two distinct edges of a graph are *adjacent* if they have a common vertex.

For a positive integer k, the path P_k is the graph whose vertex set is $\{1, \ldots, k\}$ such that two distinct vertices are adjacent if and only if they are consecutive. For an integer k,

E-mail address: zohairi@ksu.edu.sa.

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.02.003

with $k \geq 3$, the cycle C_k is the graph whose vertex set is $\{1, \ldots, k\}$ and the edge set is $\{i(i+1) : 1 \leq i \leq k-1\} \cup \{1k\}$.

For distinct vertices u and v of a graph G, a uv-path is a sequence of distinct vertices $v_1, \ldots, v_k$ such that $v_1 = u$, $v_k = v$, and $v_i v_{i+1} \in E(G)$ for each i with $i \leq k-1$. A cycle in a graph G is a sequence of distinct vertices $v_1, \ldots, v_k$ such that $k \geq 3$ and $\{v_i v_{i+1}, : 1 \leq i \leq k-1\} \cup \{v_1 v_k\}$ is a subset of $E(G)$; such a cycle is denoted by $(v_1, \ldots, v_k)$. The *length* of a path, respectively cycle, is the number of its edges. A graph is *triangle-free* if it has no cycles of length three.

A graph G is *connected* if there is a uv-path for any distinct vertices u and v of G; otherwise G is *disconnected*. A graph with no cycles is *acyclic*. A *tree* is a connected acyclic graph. A *leaf* of a tree is a vertex of degree 1. The distance between two vertices u and v, $d(u, v)$, in a connected graph G is the length of a shortest uv-path.

For a vertex subset X of a graph G, the *subgraph of G induced by X*, $G[X]$, is the graph whose vertex set is X such that two vertices are adjacent in $G[X]$ if they are adjacent in G. For a vertex subset X of G, the subgraph of G induced by $V(G) - X$ is denoted by $G - X$. For a vertex v of G, the subgraph $G - \{v\}$ is denoted by $G - v$.

An *isomorphism* from a graph G onto a graph H is a bijection f from $V(G)$ onto $V(H)$ such that for any two vertices u and v of G, u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. Two graphs G and H are *isomorphic* if there is an isomorphism from G onto H, in which case we write $G \cong H$.

In a graph G, a subset M of the vertex set $V(G)$ is a *module* in G if every vertex outside M is adjacent to all or none of M. This concept was introduced in [6] and independently under the name *interval* in [4] and *autonomous set* in [5]. The empty set, the singleton sets, and the full set of vertices are *trivial modules*. A graph is *indecomposable* if all its modules are trivial; indecomposable graphs with at least three vertices are *prime graphs*. All graphs with at most two vertices are indecomposable, while all with three vertices are decomposable.

A prime graph G is *minimal* for a vertex subset U if no proper induced subgraph of G containing U is prime. A graph G is *k-minimal* if it is minimal for some set of k vertices. Analogous concepts were introduced by Cournier and Ille [2] for digraphs. They characterized the 1-minimal and 2-minimal graphs. Recently, Alzohairi and Boudabbous [1] characterized 3-minimal triangle-free graphs.

Our main goal in this paper is to characterize the triangle-free graphs which are minimal for some nonstable 4-vertex subset. To do so, we distinguish a particular 4-vertex nonstable subset $\{x, y, z, w\}$ of a graph G, and determine what structure G must have to be minimal for $\{x, y, z, w\}$. As a corollary, we show that there are exactly $[\frac{(n-1)^2}{12}] - \lfloor \frac{n-4}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n-4}{2} \rfloor - 1$ nonisomorphic n-vertex triangle-free graphs which are minimal for some nonstable 4-vertex subset when $n \geq 7$, where $[x]$ denotes the nearest integer to x, and $\lfloor x \rfloor$ denotes the floor of x.

In order to state our result, we introduce notation of special graphs. For $i, j \in \{1, \ldots, k\}$ with $k - 1 > |j - i| > 1$, obtain the graph $P_k^{i,j}$ from P_k, respectively $C_k^{i,j}$ from C_k, by adding the edge ij.

For positive integers k, m, n with $k \leq m \leq n$, let $S_{k,m,n}$ be the tree with $k + m + n + 1$ vertices that is the union of the paths of lengths k, m, and n having common endpoint r. Let $a_1, \ldots, a_k$, $b_1, \ldots, b_m$, and $c_1, \ldots, c_n$ denote the other vertices on these paths, indexed by their distance from r. (See Fig. 1).

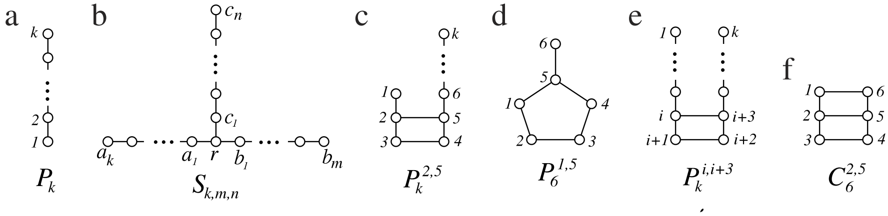

Fig. 1. Illustrates the graphs P_k, $S_{k,m,n}$, $P_k^{2,5}$, $P_6^{1,5}$, $P_k^{i,i+3}$ and $C_6^{2,5}$.

Our main result is:

Theorem 1.1. *Let x, y, z and w be distinct vertices in a triangle-free graph G such that $\{x,y,z,w\}$ is a nonstable set of G. The graph G is minimal for $\{x,y,z,w\}$ if and only if G and the nonstable set $\{x,y,z,w\}$ of G have one of the following forms:*

(i) $G \cong P_4$.
(ii) $G \cong P_k$ *with* $k \geq 5$ *such that* $\{x,y,z,w\}$ *contains the leaves.*
(iii) $G \cong P_6^{1,5}$ *such that* $\{x,y,z,w\} \in \{\{1,2,4,6\},\{1,4,5,6\}\}$.
(iv) $G \cong P_6^{2,5}$ *such that* $\{x,y,z,w\} \in \{\{1,2,3,5\},\{2,3,4,5\}\}$.
(v) $G \cong C_6^{2,5}$ *such that* $\{x,y,z,w\} \in \{\{1,2,3,5\},\{2,3,4,5\}\}$.
(vi) $G \cong P_k^{2,5}$ *with* $k \geq 6$ *such that* $\{x,y,z,w\} \in \{\{1,2,3,k\},\{2,3,4,k\},\{1,3,k-1,k\}\}$.
(vii) $G \cong P_k^{i,i+3}$ *with* $k \geq 6$ *and* $2 \leq i \leq \lfloor \frac{k-4}{2} \rfloor + 1$ *such that* $\{x,y,z,w\} = \{1,i+1,i+2,k\}$.
(viii) $G \cong S_{1,2,2}$ *such that* $\{x,y,z,w\} = \{r,a_1,b_1,c_1\}$.
(ix) $G \cong S_{1,2,n}$ *such that* $\{a_1,b_1,c_n\} \subset \{x,y,z,w\}$.
(x) $G \cong S_{k,m,n}$ *with* $m \geq 2$ *such that* $\{x,y,z,w\}$ *contains the leaves.*

Corollary 1.2. *The number of nonisomorphic triangle-free graphs which are minimal for some nonstable* 4*-vertex subset with n vertices equals:*

- 1 *if* $n \in \{4,5\}$.
- 5 *if* $n = 6$.
- $[\frac{(n-1)^2}{12}] - \lfloor \frac{n-4}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n-4}{2} \rfloor - 1$ *if* $n \geq 7$.

The following problem remains open:

Open problem. Characterize the triangle-free graphs which are minimal for some stable 4-vertex subset.

2. Proof of Theorem 1.1

First, we recall two essential results, due to Ehrenfeucht and Rozenberg, which are tools in the main studies of prime graphs.

Theorem 2.1 ([3]). *Let X be a vertex subset of a prime graph G such that $G[X]$ is prime. If G has at least two vertices outside X, then it has two distinct vertices x and y outside X such that $G[X \cup \{x,y\}]$ is prime.*

Theorem 2.1 follows from Proposition 2.2, which uses the following notations.

For a graph G and a vertex subset X such that $G[X]$ is prime, define the following subsets of $V(G) - X$.

1. $Ext(X)$ is the set of y outside X such that $G[X \cup \{y\}]$ is prime.
2. $\langle X \rangle$ is the set of y outside X such that X is a module of $G[X \cup \{y\}]$.
3. For each u in X, $X(u)$ is the set of y outside X such that $\{u, y\}$ is a module of $G[X \cup \{y\}]$.

Proposition 2.2 ([3]). *Let X be a proper vertex subset of a prime graph G such that $G[X]$ is prime.*

(i) *The family of nonempty sets among Ext(X), $\langle X \rangle$ and $\{X(u) \colon u \in X\}$ forms a partition of $V(G) - X$.*
(ii) *For distinct elements y and z of $Ext(X)$, the subgraph $G[X \cup \{y, z\}]$ is decomposable if and only if $\{y, z\}$ is a module of $G[X \cup \{y, z\}]$.*
(iii) *Given u in X, for y in $X(u)$ and for z outside $X \cup X(u)$, the subgraph $G[X \cup \{y, z\}]$ is decomposable if and only if $\{y, u\}$ is a module of $G[X \cup \{y, z\}]$.*
(iv) *For y in $\langle X \rangle$ and for z outside $X \cup \langle X \rangle$, the subgraph $G[X \cup \{y, z\}]$ is decomposable if and only if $X \cup \{z\}$ is a module of $G[X \cup \{y, z\}]$.*

The following lemma, which will be used in the proof of Lemma 2.4, is contained in Lemmas 3.1 and 3.3 in [1].

Lemma 2.3 ([1]).

(i) *If $k \geq 4$, then P_k is prime.*
(ii) *The graph $P_6^{1,5}$ is prime.*
(iii) *If $k \geq 6$, then $P_k^{2,5}$ is prime.*
(iv) *$S_{k,m,n}$ is prime if and only if $m \geq 2$.*
(v) *$S_{1,2,2}$ is minimal for $\{a_1, b_1, c_1\}$.*
(vi) *$S_{1,2,n}$ is minimal for $\{a_1, b_1, c_n\}$.*
(vii) *If $m \geq 2$, then $S_{k,m,n}$ is minimal for the set of its leaves.*

Lemma 2.4. (i) *P_4 is minimal for its vertex set.*
(ii) *If $k \geq 5$, then P_k is minimal for any 4-element vertex subset containing its leaves.*
(iii) *$P_6^{1,5}$ is minimal for each element of $\{\{1, 2, 4, 6\}, \{1, 4, 5, 6\}\}$.*
(iv) *$P_6^{2,5}$ is minimal for each element of $\{\{1, 2, 3, 5\}, \{2, 3, 4, 5\}\}$.*
(v) *$C_6^{2,5}$ is minimal for each element of $\{\{1, 2, 3, 5\}, \{2, 3, 4, 5\}\}$.*
(vi) *If $k \geq 6$, then $P_k^{2,5}$ is minimal for each element of $\{\{1, 2, 3, k\}, \{2, 3, 4, k\}, \{1, 3, k - 1, k\}\}$.*
(vii) *If $k \geq 6$ and $2 \leq i \leq \lfloor \frac{n-4}{2} \rfloor + 1$, then $P_k^{i,i+3}$ is minimal for $\{1, i + 1, i + 2, k\}$.*
(viii) *$S_{1,2,2}$ is minimal for each vertex subset A containing $\{a_1, b_1, c_1\}$.*
(ix) *$S_{1,2,n}$ is minimal for each vertex subset A containing $\{a_1, b_1, c_n\}$.*
(x) *If $m \geq 2$, then $S_{k,m,n}$ is minimal for each vertex subset A containing its leaves.*

Proof. The proofs of parts (i) and (ii) are easy. Moreover, it is clear that if a prime graph G is minimal for a vertex subset A, then G is minimal for any vertex subset containing A. Thus, the assertions (viii), (ix) and (x) are immediate consequences of part (vi), (vii) and (viii) of Lemma 2.3.

(iii) By part (ii) of Lemma 2.3, $P_6^{1,5}$ is prime. First, we will prove that $P_6^{1,5}$ is minimal for $\{1,2,4,6\}$. Each of the subgraphs $P_6^{1,5}-5$ and $P_6^{1,5}-\{3,5\}$ is decomposable because it is disconnected with at least three vertices. Furthermore, $P_6^{1,5}-5$ is decomposable because it is $S_{1,1,2}$. Therefore, $P_6^{1,5}$ is minimal for $\{1,2,4,6\}$.

Second, we will prove that $P_6^{1,5}$ is minimal for $\{1,4,5,6\}$. The subgraphs $P_6^{1,5}-\{2,3\}$ is $S_{1,1,1}$. By part (iv) of Lemma 2.3, $P_6^{1,5}-\{2,3\}$ is decomposable. Furthermore, each of the subgraphs $P_6^{1,5}-2$ and $P_6^{1,5}-3$ is $S_{1,1,2}$. By part (iv) of Lemma 2.3, each of the subgraphs $P_6^{1,5}-2$ and $P_6^{1,5}-3$ is decomposable. Therefore, $P_6^{1,5}$ is minimal for $\{1,4,5,6\}$.

(iv) By part (iii) of Lemma 2.3, $P_6^{2,5}$ is prime.

First, we will prove that $P_6^{2,5}$ is minimal for $\{1,2,3,5\}$. The subgraph $P_6^{2,5}-\{4,6\}$ is decomposable because it is $S_{1,1,1}$. Moreover, the subgraphs $P_6^{2,5}-4$ and $P_6^{2,5}-6$ are decomposable because $P_6^{2,5}-4$ is $S_{1,1,2}$, and $\{3,5\}$ is a module $P_6^{2,5}-6$. Therefore, $P_6^{2,5}$ is minimal for $\{1,2,3,5\}$.

Second, we will prove that $P_6^{2,5}$ is minimal for $\{2,3,4,5\}$. The subgraphs $P_6^{2,5}-\{1,6\}$ and $P_6^{2,5}-1$ are decomposable because each has the module $\{2,4\}$. Moreover, the subgraph $P_6^{2,5}-6$ is decomposable because it has the module $\{3,5\}$. Therefore, $P_6^{2,5}$ is minimal for $\{2,3,4,5\}$.

(v) First, we will prove that $C_6^{2,5}$ is prime. Denote $C_6^{2,5}$ by H. Let $X=\{1,2,3,4,6\}$. Notice that $H[X]$ is prime because $H[X]$ is P_5. Clearly, $5\notin\langle X\rangle$. It is not difficult to verify that $5\notin X(u)$ for any $u\in X$. Thus, by part (i) of Proposition 2.2, $5\in Ext(X)$. Therefore, H is prime.

Second, we will prove that H is minimal for $\{1,2,3,5\}$. The subgraph $H-\{4,6\}$ is decomposable because it is $S_{1,1,1}$. Moreover, the subgraphs $H-4$ and $H-6$ are decomposable because $\{1,5\}$ is a module of $H-4$, and $\{3,5\}$ is a module $H-6$. Therefore, H is minimal for $\{1,2,3,5\}$.

Third, we will prove that H is minimal for $\{2,3,4,5\}$. The subgraphs $H-\{1,6\}$ and $H-1$ are decomposable because each of them has the module $\{2,4\}$. Moreover, the subgraph $H-6$ is decomposable because it has the module $\{3,5\}$. Therefore, H is minimal for $\{2,3,4,5\}$.

(vi) Consider an integer k with $k\geq 6$. By part (iii) of Lemma 2.3, $P_k^{2,5}$ is prime.

First, we will prove that $P_k^{2,5}$ is minimal for $\{1,2,3,k\}$. For each nonempty subset B of $\{5,\dots,k-1\}$, the subgraph $P_k^{2,5}-B$ is decomposable because it is disconnected with at least three vertices. Moreover, for each subset C of $\{1,\dots,k\}-\{1,2,3,k\}$ containing 4 the subgraph $P_k^{2,5}-C$ is decomposable because it has the module $\{1,3\}$. Therefore, $P_k^{2,5}$ is minimal for $\{1,2,3,k\}$.

Second, we will prove that $P_k^{2,5}$ is minimal for $\{2,3,4,k\}$. For each nonempty subset B of $\{5,\dots,k-1\}$, the subgraph $P_k^{2,5}-B$ is decomposable because it is disconnected with at least three vertices. Moreover, for each subset C of $\{1,\dots,k\}-\{2,3,4,k\}$ containing 1 the subgraph $P_k^{2,5}-C$ is decomposable because it has the module $\{2,4\}$. Therefore, $P_k^{2,5}$ is minimal for $\{2,3,4,k\}$.

Third, we will prove that $P_k^{2,5}$ is minimal for $\{1,3,k-1,k\}$. For each nonempty subset B of $\{2\}\cup\{5,\dots,k-2\}$, the subgraph $P_k^{2,5}-B$ is decomposable because it is disconnected with at least three vertices. Moreover, for each subset C of $\{1,\ldots,k\}-\{1,3,k-1,k\}$ containing 4 the subgraph $P_k^{2,5}-C$ is decomposable because it has the module $\{1,3\}$. Therefore, $P_k^{2,5}$ is minimal for $\{1,3,k-1,k\}$.

(vii) Assume that $k\geq 6$ and $2\leq i\leq \left\lfloor\frac{n-4}{2}\right\rfloor+1$. First, we will prove that $P_k^{i,i+3}$ is prime. If $i=2$, then $P_k^{i,i+3}$ is $P_k^{2,5}$ and it is prime by part (iii) of Lemma 2.3. Now, assume that $i\neq 2$. Denote $P_k^{i,i+3}$ by H. Let $X=V(H)-\{i+1,i+2\}$. Notice that $H[X]$ is prime because it is P_{k-2}. It is not difficult to verify that for each $t\in\{i+1,i+2\}$, $t\notin\langle X\rangle$ and $t\notin X(u)$ for any element u in X. Thus, by part (i) of Proposition 2.2, $i+1\in Ext(X)$ and $i+2\in Ext(X)$. Since $i\not\sim\{i+1,i+2\}$ in H, $\{i+1,i+2\}$ is not a module of H. Therefore, H is prime by part (ii) of Proposition 2.2.

Second, we will prove that H is minimal for $\{1,i+1,i+2,k\}$. For each nonempty subset B of $V(H)-\{1,i+1,i+2,k\}$, the subgraph $H-B$ is decomposable because it is disconnected with at least three vertices. Therefore, H is minimal for $\{1,i+1,i+2,k\}$. □

Lemma 2.5. *Let A be a 4-vertex nonstable subset in a triangle-free graph G. Then*

(i) $|E(G[A])|=4$ *if and only if* $G[A]$ *is* C_4.
(ii) $|E(G[A])|=3$ *if and only if either* $G[A]$ *is* P_4 *or* $G[A]$ *is* $S_{1,1,1}$.

Proof. (i) If $G[A]$ is C_4, then $|E(G[A])|=4$. Now assume that $|E(G[A])|=4$. It is clear that the number of edges of a 2-vertex graph is at most 1 and the number of edges of a 3-vertex triangle-free graph is at most 2. Thus, $G[A]$ is connected. Hence, it suffices to prove that $d_{G[A]}(v)=2$ for each vertex v of $G[A]$.

First, to the contrary, suppose that there is a vertex u of $G[A]$ such that $d_{G[A]}(u)=1$. Since $G[A-\{u\}]$ is a 3-vertex triangle-free graph, the number of edges of $G[A-\{u\}]$ is at most 2. Thus, the number of edges of $G[A]$ is 3; which is a contradiction.

Second, to the contrary, suppose that there is a vertex u of $G[A]$ such that $d_{G[A]}(u)=3$. Since $G[A]$ is a triangle-free graph, $G[A-\{u\}]$ has no edges. Thus, the number of edges of $G[A]$ is at most 3; which is a contradiction.

(ii) If $G[A]$ is P_4 or $G[A]$ is $S_{1,1,1}$, then $|E(G[A])|=3$. Now assume that $|E(G[A])|=3$. Since $G[A]$ is a triangle-free graph, $G[A]$ is a acyclic. Thus, $G[A]$ is a tree because $G[A]$ is a 4-vertex acyclic graph with 3 edges. Therefore, $G[A]$ is P_4 or $G[A]$ is $S_{1,1,1}$. □

Remark 2.6. The sufficient condition of Theorem 1.1 is given by Lemma 2.4. In order to prove the necessary condition of Theorem 1.1, we consider a nonstable 4-vertex subset A in a triangle-free graph G and we assume that G is minimal for A. Clearly, if $G[A]$ is P_4, then G is P_4 and then G satisfies the first condition of Theorem 1.1. Therefore, using Lemma 2.5, to prove that G satisfies one of the conditions of Theorem 1.1, we will distinguish the following cases: $G[A]$ is C_4, $G[A]$ is $S_{1,1,1}$, $E(G[A])$ consists of two adjacent edges, $E(G[A])$ consists of two nonadjacent edges, and $E(G[A])$ consists of a single edge. These cases are studied in Lemmas 2.7–2.11.

In each case, we will prove that there exists a vertex subset X including A such that the induced subgraph $G[X]$ satisfies one of the conditions of Theorem 1.1; which implies that $G=G[X]$ because G is minimal for A.

Lemma 2.7. *Let A be a 4-vertex subset in a triangle-free graph G such that $G[A]$ is C_4. If G is minimal for A, then G satisfies the fourth or the fifth condition of Theorem 1.1.*

Proof. Let $A = \{x, y, z, w\}$. We may assume that (x, y, z, w) is C_4. Since $\{x, z\}$ is a module of $G[A]$ and is not a module of G, there is a vertex u outside A such that $u \not\sim \{x, z\}$ in G. We may assume that u is adjacent to x and nonadjacent to z. Since G is a triangle-free graph, u is nonadjacent to either y or w. Since $\{y, w\}$ is a module of $G[A \cup \{u\}]$ and is not a module of G, there is a vertex v outside $A \cup \{u\}$ such that $v \not\sim \{y, w\}$ in G. We may assume that v is adjacent to y and is nonadjacent to w. Since G is a triangle-free graph, v is nonadjacent to either x or z. Let H denote the subgraph of G induced by $A \cup \{u, v\}$. Thus, H is $C_6^{2,5}$ when u is adjacent to v with $A = \{2, 3, 4, 5\}$ or H is $P_6^{2,5}$ when u is adjacent to v with $A = \{2, 3, 4, 5\}$. Therefor, H satisfies the fourth condition or the fifth condition of Theorem 1.1. □

Lemma 2.8. *Let A be a 4-vertex subset in a triangle-free graph G such that $G[A]$ is $S_{1,1,1}$. If G is minimal for A, then G satisfies one of the conditions* (iii), (iv), (v) *and* (viii) *of Theorem* 1.1.

Proof. Let $A = \{x, y, z, w\}$ such that $G[A]$ is $S_{1,1,1}$. We may assume that $d_{G[A]}(x) = 3$. Since $\{y, z, w\}$ is a module of $G[A]$ and is not a module of G, there is a vertex u outside A such that $u \not\sim \{y, z, w\}$. Thus, $d_{G[A]}(u) \in \{1, 2\}$. We may assume that $d_{G[A]}(u)$ is maximum over all u outside A such that $u \not\sim \{y, z, w\}$.

To begin, assume that $d_{G[A]}(u) = 1$. We may assume that u is adjacent to y. Since G is a triangle-free graph, u is nonadjacent to x. Since $\{z, w\}$ is a module of $G[A \cup \{u\}]$ and is not a module of G, there is a vertex v outside $A \cup \{u\}$ such that $v \not\sim \{z, w\}$ in G. We may assume that v is adjacent to z and is nonadjacent to w. Since G is a triangle-free graph, v is nonadjacent to x. Since $v \not\sim \{y, z, w\}$ and $|d_{G[A]}(u)|$ is maximum over all u outside A such that $u \not\sim \{y, z, w\}$, v is nonadjacent to y. Let H denote the subgraph of G induced by $A \cup \{u, v\}$. Thus, $N_H(v) \in \{\{z\}, \{z, u\}\}$. Thus, H is $S_{1,2,2}$ with $A = \{r, a_1, b_1, c_1\}$ when $N_H(v) = \{z\}$ or H is $P_6^{1,5}$ with $A = \{1, 4, 5, 6\}$ when $N_H(v) = \{z, u\}$. Therefore, H satisfies the condition (iii) or the condition (iii) of Theorem 1.1.

Finally, assume that $d_{G[A]}(u) = 2$. We may assume that u is adjacent to y and z. Since G is a triangle-free graph, u is nonadjacent to x. Since $\{y, z\}$ is a module of $G[A \cup \{u\}]$ and is not a module of G, there is a vertex v outside $A \cup \{u\}$ such that $v \not\sim \{y, z\}$. We may assume that v is adjacent to y and is nonadjacent to z. Since G is a triangle-free graph, v is nonadjacent to either x or u. Let K denote the subgraph of G induced by $A \cup \{u, v\}$. Thus, $N_K(v) \in \{\{y\}, \{y, w\}\}$. Hence, K is $P_6^{2,5}$ with $A = \{1, 2, 3, 5\}$ when $N_K(v) = \{y\}$ or K is $C_6^{2,5}$ with $A = \{1, 2, 3, 5\}$ when $N_K(v) = \{y, w\}$. Therefore, K satisfies the condition (iv) or the condition (v) of Theorem 1.1. □

Lemma 2.9. *Let A be a 4-vertex subset in a triangle-free graph G such that $E(G[A])$ consists of two adjacent edges. If G is minimal for A, then G satisfies one of the conditions* (ii), (vi) *and* (ix) *of Theorem* 1.1.

Proof. Let $A = \{x, y, z, w\}$. We may assume that y is adjacent to x and z. Denote x by v_1, y by v_2 and z by v_3. Thus, v_1, v_2, v_3 is an x, z-path. Let $u_1, u_2, \ldots, u_q$ be a shortest w, v_i-path where $q \geq 3$ and i is the least index such that $d(w, v_i) = min\{d(w, v_j) : 1 \leq j \leq 3\}$. From the definition of i, the vertices u_j and v_t are nonadjacent for each element (j, t) of $\{1, \ldots, q-2\} \times \{1, 2, 3\}$, and the vertices u_{q-1} and v_t are nonadjacent for each t with $t < i$. Let H denote the subgraph of G induced by $\{v_1, v_2, v_3, u_1, u_2, \ldots, u_{q-1}\}$.

First, assume that $i = 2$. Since G is a triangle-free graph, u_{q-1} is adjacent neither to x nor to z. Since $\{x, z\}$ is a module of H and is not a module of G, there is a vertex b outside $V(H)$ such that $b \not\sim \{x, z\}$ in G. We may assume that b is adjacent to x and is nonadjacent to z. Since G is a triangle-free graph, b is nonadjacent to y. From the definition of i, the vertex b is nonadjacent to u_j where $j \leq q-2$. Let K denote the subgraph of G induced by $V(H) \cup \{b\}$. Thus, $N_K(b) \in \{\{x\}, \{x, u_{q-1}\}\}$.

Thus, K is $S_{1,2,q-1}$ with $A = \{r, a_1, b_1, c_{q-1}\}$ when $N_K(b) = \{x\}$ or K is $P_{q+3}^{2,5}$ with $A = \{1, 2, 3, q+3\}$ when $N_K(b) = \{x, u_{q-1}\}$. Since $q \geq 3$, K satisfies the condition (ix) or (vi) of Theorem 1.1.

Second, assume that $i = 1$. Since G is a triangle-free graph, $N_H(u_{q-1}) \in \{\{x\}, \{x, z\}\}$.

If $N_H(u_{q-1}) = \{x\}$, then H is P_{q+2} with $A = \{1, 2, 3, q+2\}$. Since $q \geq 3$, H satisfies the condition (ii) of Theorem 1.1.

Now, assume that $N_H(u_{q-1}) = \{x, z\}$. Since $\{x, z\}$ is a module of H and is not a module of G, there is a vertex a outside $V(H)$ such that $a \not\sim \{x, z\}$ in G. We may assume that a is adjacent to x and is nonadjacent to z. Since G is a triangle-free graph, a is nonadjacent to either y or u_{q-1}. Let K denote the subgraph of G induced by $V(H) \cup \{a\}$. Since $u_1, u_2, \ldots, u_q$ is a shortest w, v_1-path, a is nonadjacent to u_j where $j \leq q-3$. Thus, $N_K(a) \in \{\{x\}, \{x, u_{q-2}\}\}$.

First, assume $N_K(a) = \{x\}$. Therefore, K is $P_{q+3}^{2,5}$ such that $A = \{2, 3, 4, q+3\}$. Since $q \geq 3$, K satisfies the condition (vi) of Theorem 1.1.

Second, assume $N_K(a) = \{x, u_{q-2}\}$. We will prove that $q = 3$. To the contrary, suppose that $q \geq 4$. Thus, $K - u_{q-1}$ is P_t with $t \geq 5$. Hence, $K - u_{q-1}$ is a prime proper induced subgraph of G containing A; which contradicts the fact that G is minimal for A. Thus, K is $P_6^{2,5}$ with $A = \{2, 3, 4, 6\}$. Therefore, K satisfies the condition (vi) of Theorem 1.1.

Finally, assume that $i = 3$. Clearly, H is P_{q+2} with $A = \{1, 2, 3, q+2\}$. Since $q \geq 3$, H satisfies the condition (ii) of Theorem 1.1. □

Lemma 2.10. *Let A be a 4-vertex subset in a triangle-free graph G such that $E(G[A])$ consists of two nonadjacent edges. If G is minimal for A, then G satisfies the condition* (ii) *of Theorem* 1.1.

Proof. Let $A = \{x, y, z, w\}$. We may assume that xy and zw are the edges of $G[A]$ and $d_G(x, z) \leq d_G(y, z)$. Let $v_1, v_2, \ldots, v_k$ be a shortest z, x-path, where $k \geq 3$. Since G is triangle-free, y is nonadjacent to v_{k-1}. Also, y is nonadjacent to v_j where $j \in \{2, \ldots, k-2\}$ because $d_G(x, z) \leq d_G(y, z)$. Thus, $z, v_2 \ldots, v_{k-1}, x, y$ is a z, y-path.

To begin, assume that $v_2 = w$, then $k \geq 4$ and $G[\{z, w \ldots, v_{k-1}, x, y\}]$ is P_{k+1} with $A = \{1, 2, k, k+1\}$. Since $k \geq 4$, H satisfies the condition (ii) of Theorem 1.1.

Finally, assume that $v_2 \neq w$. Denote H the subgraph of G induced by $\{w, z, v_2, \ldots, v_{k-1}, x, y\}$. If $|N_H(w)| \geq 2$, then consider the largest index j where $j \in \{2, \ldots, k-1\}$ such that $v_j \in N_H(w)$. Hence, $H - \{v_2, \ldots, v_{j-1}\}$ is P_t with $t \geq 5$. Thus, $H - \{v_2, \ldots, v_{j-1}\}$ is a proper prime induced subgraph of G containing A; which contradicts the fact that G is minimal for A. Therefore, H is P_{k+2} with leaves contained in A. Since $k \geq 3$, H satisfies the condition (ii) of Theorem 1.1. □

Lemma 2.11. *Let A be a 4-vertex subset in a triangle-free graph G such that $E(G[A])$ consists of a single edge. If G is minimal for A, then G satisfies one of the conditions* (ii), (vi), (vii), (ix) *and* (x) *Theorem* 1.1.

Proof. Let $A = \{x, y, z, w\}$. We may assume that xy is the edge of $G[A]$ and $d_G(x, z) \leq d_G(y, z)$. Let $v_1, v_2, \ldots, v_t$ be a shortest z, x-path, where $t \geq 3$. Since G is triangle-free, y is nonadjacent to v_{t-1}. Also, y is nonadjacent to v_j where $j \in \{2, \ldots, t-2\}$ because $d_G(x, z) \leq d_G(y, z)$. Thus, $G[\{z, v_2 \ldots, v_{t-1}, x, y\}]$ is P_{t+1}.

If $w \in \{v_j : 2 \leq j \leq t-1\}$, then $G[\{z, v_2 \ldots, v_{t-1}, x, y\}]$ is P_{t+1} with leaves y and z. Thus, $G[\{z, v_2 \ldots, v_{t-1}, x, y\}]$ satisfies the condition (ii) of Theorem 1.1.

Now assume that $w \notin \{v_j : 2 \leq j \leq t-1\}$. Denote y by v_{t+1} and denote by H the subgraph of G induced by $\{w, z, v_2, \ldots, v_{t-1}, x, y\}$. Since xy is the unique edge of $G[A]$, $N_H(w) \subset \{v_2, \ldots, v_{t-1}\}$. We distinguish the following two cases depending on the neighborhood of w.

To begin, assume that $N_H(w) \cap \{v_2, \ldots, v_{t-1}\} \neq \emptyset$. We will prove that $|N_H(w)| = 1$. To the contrary, suppose that $|N_H(w)| \geq 2$. Consider the least index i and the largest index j in $\{2, \ldots, t-1\}$ such that v_i and v_j are in $N_H(w)$. Since G is triangle-free, w is nonadjacent to v_{i+1}. Thus, $j > i+1$. Hence, $H - \{v_{i+1}, \ldots, v_{j-1}\}$ is P_k with $k \geq 6$. Therefore, $H - \{v_{i+1}, \ldots, v_{j-1}\}$ is a prime proper induced subgraph of G containing A; which contradicts the fact that G is minimal for A.

First, assume that $N_H(w) \neq \{v_2\}$. Thus H is $S_{1,m,n}$ with $m \geq 2$ such that A contains the leaves. Therefore, H satisfies the condition (x) of Theorem 1.1.

Second, assume that $N_H(w) = \{v_2\}$. Since $\{z, w\}$ is a module of H and is not a module of G, there is a vertex a outside $V(H)$ such that $a \not\sim \{y, z\}$ in G. We may assume that a is adjacent to z and is nonadjacent to w. Denote by K the subgraph of G induced by $V(H) \cup \{a\}$.

If $|N_K(a)| = 1$, then K is $S_{1,2,n}$ such that $A = \{a_1, b_1, c_{n-1}, c_n\}$ where $n = t-1$. Therefore, H satisfies the condition (ix) of Theorem 1.1.

Now assume that $|N_K(a)| \geq 2$. Since G is a triangle-free graph, $v_2 \notin N_K(a)$. Let j be the largest index in $\{3, \ldots, t+1\}$ such that $v_j \in N_K(a)$.

- If $t = 3$, then $j \in \{3, 4\}$ and $N_K(a) = \{z, v_j\}$ because G is a triangle-free graph. If $j = 3$, then K is $P_6^{2,5}$ with $A = \{1, 3, 5, 6\}$, and thus K satisfies the condition (vi) of Theorem 1.1. If $j = 4$, then K is $P_6^{1,5}$ with $A = \{1, 2, 4, 6\}$, and thus K satisfies the condition (iii) of Theorem 1.1.
- If $t \geq 4$ and $j = 3$, then K is $P_{t+3}^{2,5}$ with $A = \{1, 3, t+2, t+3\}$. Therefore, H satisfies the condition (vi) of Theorem 1.1.
- If $t \geq 4$ and $4 \leq j \leq t$, then $K - \{v_3, \ldots, v_{j-1}\}$ is P_m where $m \geq 6$. Therefore, $K - \{v_3, \ldots, v_{j-1}\}$ is a prime proper induced subgraph of G containing A; which contradicts the fact that G is minimal for A.
- If $t \geq 4$ and $j = t+1$, then $x \notin N_K(a)$ because G is a triangle-free graph. Thus, $G[\{w, v_2, z, a, y, x\}]$ is P_6. Therefore, $G[\{w, v_2, z, a, y, x\}]$ is a prime proper induced subgraph of G containing A; which contradicts the fact that G is minimal for A.

Finally, assume that $N_H(w) \cap \{v_2, \ldots, v_{t-1}\} = \emptyset$.

Let $u_1, u_2, \ldots, u_q$ be a shortest w, v_i-path where $q \geq 3$ and i is the least index such that $d(w, v_i) = min\{d(w, v_j) : 1 \leq j \leq t+1\}$. Since G is triangle-free, u_{q-1} is nonadjacent to v_{i+1} when $i \leq t$. From the definition of i, the vertices u_j and v_r are nonadjacent for each element (j, r) of $\{1, \ldots, q-2\} \times \{1, \ldots, t+1\}$, and the vertices u_{q-1} and v_r are nonadjacent for each r with $r < i$. Denote by K the subgraph of G induced by $\{v_1, \ldots, v_{t+1}, u_1, u_2, \ldots, u_{q-1}\}$.

We will prove that either $N_K(u_{q-1}) = \{v_i\}$ or ($i < t$ and $N_K(u_{q-1}) = \{v_i, v_{i+2}\}$).

If $i \in \{t, t+1\}$, then $N_H(u_{q-1}) = \{v_i\}$. If $i = t-1$, then $N_K(u_{q-1}) \in \{\{v_{t-1}\}, \{v_{t-1}, v_{t+1}\}\}$. Now assume that $i \leq t-2$ and $|N_K(u_{q-1})| \geq 2$. Consider the largest index j such that $u_{q-1}v_j \in E(G)$. Hence, $v_1, \ldots, v_i, u_{q-1}, v_j, v_{j+1}, \ldots, v_t$ is a z, x-path of G. Thus, $j = i+2$ because $v_1, \ldots, v_t$ is a shortest z, x-path of G. Therefore, $N_K(u_{q-1}) = \{v_i, v_{i+2}\}$.

First, assume that $N_K(u_{q-1}) = \{v_i\}$.

If $i \in \{1, t+1\}$, then K is P_k with $k \geq 6$ such that A contains the leaves. Therefore, K satisfies the condition (ii) of Theorem 1.1.

If $i \notin \{1, t+1\}$, then K is $S_{k,m,n}$ with $m \geq 2$ such that A contains the leaves. Therefore, K satisfies the condition (x) of Theorem 1.1.

Second, assume that $i < t$ and $N_K(u_{q-1}) = \{v_i, v_{i+2}\}$. If $2 \leq i \leq t-2$, then $K - v_{i+1}$ is $S_{k,m,n}$ with $m \geq 2$. Therefore, $K - v_{i+1}$ is a prime proper induced subgraph of G containing A; which contradicts the fact that G is minimal for A. Therefore, $i \in \{1, t-1\}$.

Now assume that $i = 1$. If $q \geq 4$, then $K - v_2$ is $S_{1,m,n}$ with $m \geq 2$. Therefore, $K - v_2$ is a prime proper induced subgraph of G containing A; which contradicts the fact that G is minimal for A. Thus, $q = 3$. Therefore, K is $P_k^{2,5}$ with $k \geq 6$ such that $A = \{1, 3, k-1, k\}$ where $k = t+3$. Therefore, K satisfies the condition (vi) of Theorem 1.1.

Third, assume that $i = t-1$. Let j be the minimum of $t-1$ and $q-1$. Clearly, $2 \leq j \leq \lfloor \frac{t+q-4}{2} \rfloor + 1$ and K is $P_{t+q}^{j,j+3}$ such that $A = \{1, j+1, j+2, t+q\}$. Therefore, K satisfies the condition (vii) of Theorem 1.1. □

Proof of Corollary 1.2. It is not difficult to verify that there are no two isomorphic different graphs in the union $\{P_6^{1,5}\} \cup \{P_k : k \geq 4\} \cup \{C_6^{2,5}\} \cup \{\{P_k^{i,i+3} : i \in \{2, \ldots, \lfloor \frac{k-4}{2} \rfloor + 1\}\} : k \geq 6\} \cup \{S_{k,m,n} : m \geq 2\}$.

If $n = 4$, then the result holds because P_4 is the unique prime graph with four vertices. By Theorem 1.1, P_5 is the unique 5-vertex triangle-free prime graph which is minimal for some nonstable 4-vertex subset, and the only nonisomorphic 6-vertex triangle-free graphs which are minimal for some nonstable 4-vertex subset are P_6, $P_6^{1,5}$, $P_6^{2,5}$, $C_6^{2,5}$ and $S_{1,2,2}$. Therefore, the result holds for $n \in \{4, 5, 6\}$.

Now, assume that $n \geq 7$. By Theorem 1.1, the nonisomorphic n-vertex triangle-free graphs which are minimal for some nonstable 4-vertex subset are P_n, $P_n^{i,i+3}$ where $2 \leq i \leq \lfloor \frac{k-4}{2} \rfloor + 1$, and the family of $S_{k,m,t}$, where $k \leq m \leq t$, $m \geq 2$, and $k+m+t+1 = n$.

From the proof of Corollary 1.2 in [1], the number of nonisomorphic such $S_{k,m,t}$ equals $[\frac{(n-1)^2}{12}] - \lfloor \frac{n-4}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor - 2$.

Therefore, there are exactly $[\frac{(n-1)^2}{12}] - \lfloor \frac{n-4}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n-4}{2} \rfloor - 1$ nonisomorphic n-vertex triangle-free graphs which are minimal for some nonstable 4-vertex subset when $n \geq 7$. □

Acknowledgment

The author would like to thank Dr. Youssef Boudabbous whose many helpful comments and suggestions helped to improve the quality of the paper.

References

[1] M. Alzohairi, Y. Boudabbous, 3-minimal triangle-free graphs, Discrete Math. 331 (2014) 3–8.

[2] A. Cournier, P. Ille, Minimal indecomposable graphs, Discrete Math. 183 (1998) 61–80.

[3] A. Ehrenfeucht, G. Rozenberg, Primitivity is hereditary for 2-structures, fundamental study, Theoret. Comput. Sci. 3 (70) (1990) 343–358.
[4] R. Fraïssé, L'intervalle en théorie des relations, ses généralisations, filtre intervallaire et clôture d'une relation, in: M. Pouzet, D. Richard (Eds.), Order, Description and Roles, North-Holland, Amsterdam, 1984, pp. 313–342.
[5] T. Gallai, Transitiv orientierbare graphen, Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66.
[6] J. Spinrad, P4-trees and substitution decomposition, Discrete Appl. Math. 39 (1992) 263–291.