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Towards a Morse theory on Banach spaces via ultrafunctions

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Abstract. Morse Theory on Banach spaces would be a useful tool in nonlinear analysis but its development is hindered by many technical problems. In this paper we present an approach based on a new notion of generalized functions called "ultrafunctions" which solves some of the technical questions involved.

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1. Introduction

In this paper we start a study of Morse Theory on Banach spaces using the theory of ultrafunctions [2–5]; the **ultrafunctions** are a new notion of generalized functions based on the general ideas of Non Archimedean Mathematics (NAM) of Non Standard Analysis (NSA).

Based on our experience NAM allows to construct models of the physical world in a more elegant and simple way, in many circumstances. Contrary to common belief, the ideas behind NSA and NMA date back to the 1870s, when mathematicians such as Du Bois-Reymond, Veronese, Hilbert and Levi-Civita investigated it. Since then its development stopped, until

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the '60s when Abraham Robinson presented his Non Standard Analysis. For a historical analysis of these facts we refer to Ehrlich [21] and to Keisler [23] for a very clear exposition of NSA.

Ultrafunctions are a particular class of functions based on a superreal field $\mathbb{R}^* \supset \mathbb{R}$. More exactly, to any continuous function $f : \mathbb{R}^N \to \mathbb{R}$, we associate in a canonical way an ultrafunction $\widetilde{f} : (\mathbb{R}^*)^N \to \mathbb{R}^*$ which extends f; the ultrafunctions are many more than the functions and among them we can find solutions of functional equations which do not have any solutions among the real functions or the distributions; this allows to overcome some difficulties of Morse Theory in Banach spaces.

Many authors have been working on the adaptation of Morse Theory on Banach spaces [6–8,25,26], but many problems arise: a really important one is the difficulty in defining what a (weakly) nondegenerate critical point is and how to define its Morse index, since any critical point of a C^2 functional on a Banach space is degenerate and it is not possible to apply the generalized Morse Lemma (for a reference on the generalized Morse Lemma see [22]).

In recent times, a lot of delicate work has been done in this direction, developing extremely refined tools and techniques to study problems in nonlinear analysis [9–20,24]. Our approach is totally different, we avoid many of the difficulties involved in the definitions by using the properties of hyperfinite function spaces.

We believe that the flexibility of the ultrafunction approach can be fruitful for the development of the theory. In this paper we present a foundational basis for this theory; other articles dealing with applications are to follow.

1.1. Notation

We fix some notation. Since this paper does not deal with application, we use some function spaces as model spaces for the theory; let Ω be a subset of $\mathbb{R}^N$:

- $\mathcal{C}(\Omega)$ denotes the set of real continuous functions defined on Ω;
- $\mathcal{C}_0\left(\overline{\Omega}\right)$ denotes the set of real continuous functions on $\overline{\Omega}$ which vanish on $\partial\Omega$;
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order k;
- $\mathcal{C}_0^k\left(\overline{\Omega}\right) = \mathcal{C}^k\left(\overline{\Omega}\right) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$;
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega \subset \mathbb{R}^N$;
- $L^2(\Omega)$ denotes the set of square integrable functions on Ω.

2. Preliminary notions

In this section we present some background material necessary to follow the following part. We underline that this material is not original but we cite it in order to make the article (almost) self contained. We refer to [2–5] for a more detailed treatment.

2.1. Non Archimedean fields

Here, we recall the basic definitions and facts regarding non-Archimedean fields. In the following, $\mathbb{K}$ will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 1. Let $\mathbb{K}$ be an ordered field. Let $\xi \in \mathbb{K}$. We say that:

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ is finite if there exists $n \in \mathbb{N}$ such as $|\xi| < n$;
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

Definition 2. An ordered field $\mathbb{K}$ is called Non-Archimedean if it contains an infinitesimal $\xi \neq 0$.

It is easily seen that all infinitesimals are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

Definition 3. A superreal field is an ordered field $\mathbb{K}$ that properly extends $\mathbb{R}$.

It is easy to show, due to the completeness of $\mathbb{R}$, that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of "closeness":

Definition 4. We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.

Clearly, the relation "$\sim$" of infinite closeness is an equivalence relation.

Theorem 5. *If $\mathbb{K}$ is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the* ***shadow*** *or the* ***standard part*** *of ξ.*

Given a finite number ξ, we denote its shadow as $sh(\xi)$, and we put $sh(\xi) = +\infty$ ($sh(\xi) = -\infty$) if $\xi \in \mathbb{K}$ is a positive (negative) infinite number.

Definition 6. Let $\mathbb{K}$ be a superreal field, and $\xi \in \mathbb{K}$ a number. The monad of ξ is the set of all numbers that are infinitely close to it:

$$\mathfrak{mon}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim \zeta\},$$

and the galaxy of ξ is the set of all numbers that are finitely close to it:

$$\mathfrak{gal}(\xi) = \{\zeta \in \mathbb{K} : \xi - \zeta \text{ is finite}\}.$$

By definition, it follows that the set of infinitesimal numbers is $\mathfrak{mon}(0)$ and that the set of finite numbers is $\mathfrak{gal}(0)$.

2.2. The Λ-limit

In this section we will introduce a particular superreal field $\mathbb{K}$ and we will analyze its main properties by means of Λ-theory, in particular by means of the notion of Λ-limit (for complete proofs and for further properties of the Λ-limit, the reader is referred to [1–5]).

We recall that the superstructure on $\mathbb{R}$ is defined as follows:

$$\mathbb{U}=\bigcup_{n=0}^{\infty}\mathbb{U}_n$$

where $\mathbb{U}_n$ is defined by induction as follows:

$$\begin{aligned}\mathbb{U}_0 &= \mathbb{R};\\ \mathbb{U}_{n+1} &= \mathbb{U}_n\cup\mathcal{P}\left(\mathbb{U}_n\right).\end{aligned}$$

Here $\mathcal{P}\left(E\right)$ denotes the power set of E. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $\mathbb{U}$ contains almost every usual mathematical object. Now, if we denote by $\mathcal{P}_\omega(X)$ the collection of the finite subsets of X, we set

$$\mathfrak{L}=\mathcal{P}_\omega(\mathbb{U}),$$

and we will refer to $\mathfrak{L}$ as the "parameter space". Clearly $(\mathfrak{L},\subset)$ is a directed set[2]. We add to $\mathfrak{L}$ one point at infinity Λ and we define the following family of neighborhoods of infinity:

$$\{\Lambda\cup Q\mid Q\in\mathcal{U}\}$$

where $\mathcal{U}$ is a fine ultrafilter on $\mathfrak{L}$, namely it is a filter such that

- if $A\cup B=\mathfrak{L}$, then
$$A\in\mathcal{U}\text{ or }B\in\mathcal{U};\tag{1}$$
- $\forall\lambda_0\in\mathfrak{L}$, $\{\lambda\in\mathfrak{L}\mid\lambda_0\subset\lambda\}\in\mathcal{U}$.

A function $\varphi:D\rightarrow E$ defined on a directed set will be called a *net* (with values in E). If φ_λ is a real net, we have that

$$\lim_{\lambda\rightarrow\Lambda}\varphi_\lambda=L$$

if and only if

$$\forall\varepsilon>0,\quad\exists Q\in\mathcal{U}\quad\text{such that,}\,\forall\lambda\in Q,\ \left|\varphi_\lambda-L\right|<\varepsilon.\tag{2}$$

We will refer to the sets in Q as **qualified sets**.

Notice that this topology on $\mathfrak{L}\cup\{\Lambda\}$ satisfies this interesting property:

Proposition 7. *If the net φ_λ has a converging subnet, then it is a **converging** net.*

Proof. Suppose that the net φ_λ has a converging subnet to $L\in\mathbb{R}$. We fix $\varepsilon>0$ arbitrarily and we have to prove that $Q_\varepsilon\in\mathcal{U}$ where

$$Q_\varepsilon=\left\{\lambda\in\mathfrak{L}\mid\left|\varphi_\lambda-L\right|<\varepsilon\right\}.$$

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[2] We recall that a directed set is a partially ordered set $(D,\prec)$ such that, $\forall a,b\in D,\ \exists c\in D$ such that

$$a\prec c\quad\text{and}\quad b\prec c.$$

We argue indirectly and we assume that

$$Q_\varepsilon \notin \mathcal{U}.$$

Then, by (1), $N = \mathfrak{L} \backslash (Q_\varepsilon \cap E) \in \mathcal{U}$ and hence

$$\forall \lambda \in N, |\varphi_\lambda - L| \geq \varepsilon.$$

This contradicts the fact that φ_λ has a subnet which converges to L. □

We have the following result:

Theorem 8. *There exists a superreal field $\mathbb{K} \supset \mathbb{R}$ and a Hausdorff topology on the space $(\mathfrak{L} \times \mathbb{R}) \cup \mathbb{K}$ such that*

1. *Every net $\varphi : \mathfrak{L} \times \mathbb{R} \to \mathbb{R}$ has a unique limit*
$$L = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)).$$
Moreover we assume that every $\xi \in \mathbb{K}$ is the limit of some net $\varphi : \mathfrak{L} \times \mathbb{R} \to \mathbb{R}$.
2. *If $r \in \mathbb{R}$*
$$\lim_{\lambda \to \Lambda} (\lambda, r) = r.$$
3. *For all $\varphi, \psi : \mathfrak{L} \to \mathbb{R}$:*
$$\begin{aligned} \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) + \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) &= \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) + \psi(\lambda)); \\ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) &= \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \psi(\lambda)). \end{aligned}$$

Idea of the proof. The proof of this theorem is in [5]. We will now sketch it for the sake of the reader. We set

$$I = \{\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \mid \varphi(x) = 0 \text{ in a qualified set}\}.$$

It is not difficult to prove that I is a maximal ideal in $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$; then

$$\mathbb{K} := \frac{\mathfrak{F}(\mathfrak{L}, \mathbb{R})}{I}$$

is a field. In the following, we shall identify a real number $c \in \mathbb{R}$ with the equivalence class of the constant net $[c]_I$.

Now, we equip $(\mathfrak{L} \times \mathbb{R}) \cup \mathbb{K}$ with the following topology τ. A basis of neighborhoods of $[\varphi]_I$ is given by

$$N_{\varphi,Q} := \{(\lambda, \varphi(\lambda)) \mid \lambda \in Q\} \cup \{[\varphi]_I\}, \quad Q \in \mathcal{U}. \quad \square$$

From now on, in order to simplify the notation we will write

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)),$$

and we call it a Λ-limit.

2.3. *Natural extension of sets and functions*

The notion of a Λ-limit can be extended to sets and functions in the following way:

Definition 9. Let E_λ, $\lambda \in \mathfrak{L}$, be a family of sets in $\mathbb{R}^N$ and let

$$\lim_{\lambda\uparrow\Lambda}\ E_\lambda := \left\{\lim_{\lambda\uparrow\Lambda}\psi(\lambda) \mid \psi(\lambda) \in E_\lambda\right\}.$$

A set which is a Λ-limit is called **internal**. In particular, if $\forall\lambda \in \mathfrak{L}$, $E_\lambda = E$, we set $\lim_{\lambda\uparrow\Lambda}\ E_\lambda = E^*$, namely

$$E^* := \left\{\lim_{\lambda\uparrow\Lambda}\psi(\lambda) \mid \psi(\lambda) \in E\right\}.$$

E^* is called the **natural extension** of E.

Notice that, while the Λ-limit of a sequence of numbers with constant value $r \in \mathbb{R}$ is r, the Λ-limit of a constant sequence of sets with value $E \subseteq \mathbb{R}$ gives a larger set, namely E^*. In general, the inclusion $E \subseteq E^*$ is proper.

This definition, combined with axiom (Λ-1), entails that

$$\mathbb{K} = \mathbb{R}^*.$$

Given any set E, we can associate to it two sets: its natural extension E^* and the set E^σ, where

$$E^\sigma = \{x^* \mid x \in E\}\,. \tag{3}$$

Clearly E^σ is a copy of E; however it might be different as a set since, in general, $x^* \neq x$. Moreover $E^\sigma \subset E^*$ since every element of E^σ can be regarded as the Λ-limit of a constant sequence.

Definition 10. Let

$$f_\lambda :\ E_\lambda \to \mathbb{R}, \quad \lambda \in \mathfrak{L},$$

be a family of functions. We define a function

$$f : \left(\lim_{\lambda\uparrow\Lambda}\ E_\lambda\right) \to \mathbb{R}^*$$

as follows: for every $\xi \in (\lim_{\lambda\uparrow\Lambda}\ E_\lambda)$ let

$$f(\xi) := \lim_{\lambda\uparrow\Lambda}\ f_\lambda(\psi(\lambda)),$$

where $\psi(\lambda)$ is a net of numbers such that

$$\psi(\lambda) \in E_\lambda \quad \text{and} \quad \lim_{\lambda\uparrow\Lambda}\psi(\lambda) = \xi.$$

A function which is a Λ-limit is called **internal**. In particular if, $\forall\lambda \in \mathfrak{L}$,

$$f_\lambda = f, \quad f : E \to \mathbb{R},$$

we set

$$f^* = \lim_{\lambda\uparrow\Lambda} f_\lambda.$$

$f^* : E^* \to \mathbb{R}^*$ is called the **natural extension** of f.

The Λ-limit can be extended to a larger family of nets; let us consider a net

$$\varphi : \mathfrak{L} \to \mathbb{U}_n. \tag{4}$$

We will define $\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)$ by induction on n. For $n = 0$, $\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)$ is defined by Theorem 8; so by induction we may assume that the limit is defined for $n-1$ and we define it for the net (4) as follows:

$$\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) = \left\{ \lim_{\lambda\uparrow\Lambda} \psi(\lambda) \mid \psi : \mathfrak{L} \to \mathcal{U}_{n-1} \text{ and } \forall\lambda \in \mathfrak{L},\ \psi(\lambda) \in \varphi(\lambda) \right\}. \tag{5}$$

Definition 11. A mathematical entity (number, set, function or relation) which is the Λ-limit of a net is called **internal**.

Let us note that, if (f_λ), (E_λ) are, respectively, a net of functions and a net of sets, the Λ-limit of these nets defined by (5) coincides with the Λ-limit given by Definitions 9 and 10. The following theorem is a fundamental tool in using the Λ-limit:

Theorem 12. ***(Leibniz Principle)*** *Let $\mathcal{R}$ be a relation in $\mathbb{U}_n$ for some $n \geq 0$ and let $\varphi, \psi : \mathfrak{L} \to \mathbb{U}_n$. If*

$$\forall\lambda \in \mathfrak{L}, \quad \varphi(\lambda)\mathcal{R}\psi(\lambda)$$

then

$$\left(\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)\right) \mathcal{R}^* \left(\lim_{\lambda\uparrow\Lambda} \psi(\lambda)\right).$$

When $\mathcal{R}$ is $\in$ or $=$ we will not use the symbol $*$ to denote their extensions, since their meaning is unaltered in universes constructed over $\mathbb{R}^*$. To give an example of how Leibniz Principle can be used to prove facts about internal entities, let us prove that if $K \subseteq \mathbb{R}$ is a compact set and (f_λ) is a net of continuous functions then $f = \lim_{\lambda\uparrow\Lambda} f_\lambda$ has a maximum on K^*. For every λ let ξ_λ be the maximum value attained by f_λ on K, and let $x_\lambda \in K$ be such that $f_\lambda(x_\lambda) = \xi_\lambda$. For every λ, for every $y_\lambda \in K$ we have that $f_\lambda(y_\lambda) \leq f_\lambda(x_\lambda)$. By Leibniz Principle, setting

$$x = \lim_{\lambda\uparrow\Lambda} x_\lambda$$

we have that

$$\forall y \in K \quad f(y) \leq f(x),$$

so $\xi = \lim_{\lambda \uparrow \Lambda} \xi_\lambda$ is the maximum of f on K and it is attained at x.

2.4. Ultrafunction theory

Let Ω be a set in $\mathbb{R}^N$ and let $V(\Omega)$ be a (real or complex) vector space such that $\mathcal{D}(\overline{\Omega}) \subseteq V(\Omega) \subseteq L^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

Definition 13. Given the function space $V(\Omega)$ we set

$$V_\Lambda(\Omega) := \lim_{\lambda \uparrow \Lambda} V_\lambda(\Omega),$$

where

$$V_\lambda(\Omega) = Span(V(\Omega) \cap \lambda).$$

$V_\Lambda(\Omega)$ will be called the **space of ultrafunctions** generated by $V(\Omega)$.

Using the above definition, if $V(\Omega)$, $\Omega \subset \mathbb{R}^N$, is a real function space then we can associate to it three function spaces of hyperreal functions, namely $V(\Omega)^\sigma$, $V_\Lambda(\Omega)$ and $V(\Omega)^*$:

$$V(\Omega)^\sigma = \{f^* \mid f \in V(\Omega)\}, \tag{6}$$

$$V_\Lambda(\Omega) = \left\{ \lim_{\lambda \uparrow \Lambda} f_\lambda \mid f_\lambda \in V_\lambda(\Omega) \right\}, \tag{7}$$

$$V(\Omega)^* = \left\{ \lim_{\lambda \uparrow \Lambda} f_\lambda \mid f_\lambda \in V(\Omega) \right\}. \tag{8}$$

Clearly we have

$$V(\Omega)^\sigma \subset V_\Lambda(\Omega) \subset V(\Omega)^*.$$

Let us see the relations of the space of ultrafunctions $V_\Lambda(\Omega)$ with the space of “standard functions” $V(\Omega)^\sigma$ and the space of internal functions $V(\Omega)^*$. Given any vector space of functions $V(\Omega)$, the space of ultrafunction generated by $V(\Omega)$ is a vector space of hyperfinite dimension that includes $V(\Omega)^\sigma$, and the ultrafunctions are Λ-limits of functions in V_λ. Hence the ultrafunctions are particular internal functions

$$u : (\mathbb{R}^*)^N \to \mathbb{C}^*.$$

Since $V_\Lambda(\Omega) \subset \left[L^2(\mathbb{R})\right]^*$, it can be equipped with the following scalar product

$$(u, v) = \int^* u(x)\overline{v(x)}\, dx,$$

where $\int^*$ is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1(\Omega) \to \mathbb{C}.$$

Notice that the Euclidean structure of $V_\Lambda(\Omega)$ is the Λ-limit of the Euclidean structure of every V_λ given by the usual L^2 scalar product. The norm of an ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

2.5. *Morse theory*

Let $\mathfrak{M}$ be a finite dimensional Riemannian manifold and let

$$J : \mathfrak{M} \to \mathbb{R}$$

be a functional of class C^2.

A point $u \in \mathfrak{M}$, is called a critical point of J if $dJ(u) = 0$. A number $c \in \mathbb{R}$ is a called a critical value of J if there is a critical point $u \in \mathfrak{M}$ such that $J(u) = c$. A critical point is called nondegenerate if $H_J(u)$ is nonsingular, namely if

$$[\forall \varphi \in T_u\mathfrak{M},\ H_J(u)\,[\psi, \varphi] = 0] \Rightarrow \psi = 0.$$

If $a, b \in \mathbb{R}$, we set

$$\begin{aligned} J^b &= \{u \in \mathfrak{M} | J(u) \leq b\} \\ J_a^b &= J^b \backslash J^a = \{u \in \mathfrak{M} | a < J(u) \leq b\} \\ K_a^b &= \left\{u \in J_a^b | dJ(u) = 0\right\}. \end{aligned}$$

The Morse index of a quadratic form $a\,[\varphi]$ is the number of negative eigenvalues of any matrix representation of $a\,[\varphi]$. The Morse index of a critical point u, denoted by $m(u)$, is the Morse index of the Hessian quadratic form $H_J(u)\,[\varphi]$. If u is a nondegenerate critical point, we define the **polynomial Morse index** of u as follows

$$i_t(u) = t^{m(u)}.$$

We have introduced the notion of polynomial Morse index because this notion allows to define the index of any isolated critical point, even if it is degenerate; the definition is the following:

$$i_t(u) = \sum_{k=0}^{N} \dim\left[H^k(J^c, J^c \backslash \{u\})\right]\, t^k, \quad c = J(u)$$

where N is the dimension of the manifold $\mathfrak{M}$, $H^k(A, B)$ is the kth Alexander–Spanier cohomology group of the couple (A, B) with real coefficients and we denote by $\dim\left[H^k(A, B)\right]$ the dimension of $H^k(A, B)$ regarded as real vector space. It is a well known fact of Morse theory that, if u is a nondegenerate critical point, the two definitions of $i_t(u)$ agree.

We define the Morse polynomial of J_a^b as follows:

$$M_t(J_a^b) = \sum_{u \in K_a^b} i_t(u).$$

Thus $M(t)$ is a polynomial with coefficients in $\mathbb{N} \cup \{+\infty\}$. If all the critical points in K_a^b are not degenerate, $M(1)$ is the cardinality of K_a^b, namely the number of the critical points of J

in J_a^b. If some critical point is degenerate, then $M(1)$ is the number of critical points counted with their multiplicity where the multiplicity of a critical point u is given by $i_1(u)$.

The Betti (or Poincaré) polynomial of J_a^b is a topological invariant defined as follows:

$$P_t(J_a^b) = \sum_{k=0}^{N} \dim \left[H^k(J^b, J^a) \right] t^k.$$

The integer $\dim \left[H^k(J^b, J^a) \right]$ is called the kth Betti number of J_a^b.

In the rest of the paper, we shall use the following important result in Morse theory.

Theorem 14. *Let us assume that*
- $\overline{J_a^b}$ *is compact (or more generally* J *satisfies the Palais–Smale conditions in* $[a, b]$*),*
- K_a^b *is a finite set.*

Then both $M_t(J_a^b)$ *and* $P_t(J_a^b)$ *are finite and there exists a polynomial* Q *with coefficients in* $\mathbb{N}$ *such that*

$$M_t(J_a^b) = P_t(J_a^b) + (1 + t)Q(t).$$

We will define our own generalized version of the Palais–Smale conditions in the present paper; for a classical definition refer to [7].

3. Morse theory for ultrafunctions

3.1. Basic results

Let $V \subset C^1(\Omega)$ be a Banach space and let

$$J : V \to \mathbb{R}$$

be a functional of class C^2. In the applications, we will assume that J has the following structure:

$$J(u) = \int F(x, u, \nabla u)\, dx. \tag{9}$$

As we emphasized in the introduction, the main difficulty for the development of Morse Theory in Banach spaces is to define the right concept of nondegeneracy and of Morse index for a critical point.

We will be interested in Morse theory for the functional

$$J_\Lambda : V_\Lambda \to \mathbb{R}^*$$

where V_Λ is a space of ultrafunctions and J_Λ is the restriction of J^* to V_Λ. For example, a suitable space for the functional (9) is $V_\Lambda(\Omega) := [C^2(\Omega) \cap C_0^1(\overline{\Omega})]_\Lambda$.

Now let us describe the main objects of Morse theory in the ultrafunctions framework.

Definition 15. An ultrafunction $u \in V_\Lambda$ is called a critical point of $J_\Lambda : V_\Lambda \to \mathbb{R}^*$ if

$$\forall \varphi \in V_\Lambda, \quad dJ_\Lambda(u)[\varphi] = 0$$

where dJ is the differential of J.

In particular, if J is the functional (9), we have that $u \in V_\Lambda = [C^2(\Omega) \cap C^1_0(\overline{\Omega})]_\Lambda$ is a critical point if

$$\forall \varphi \in V_\Lambda(\Omega), \qquad \int \left[\frac{\partial F}{\partial (\nabla u)} \cdot \nabla \varphi + \frac{\partial F}{\partial u} \varphi \right] dx = 0.$$

Here $\frac{\partial F}{\partial (\nabla u)}$ denotes the vector $\left(\frac{\partial F}{\partial u_{x_1}}, \ldots, \frac{\partial F}{\partial u_{x_N}} \right)$.

The Hessian quadratic form $H_{J^*}(u)$ of J^* is defined on $V^* \times V^*$; we will denote by $H_{J_\Lambda}(u)$ its restriction to $V_\Lambda \times V_\Lambda$. A critical point of J_Λ is called nondegenerate if

$$\forall \varphi \in V_\Lambda, \quad H_{J_\Lambda}(u)\,[\psi, \varphi] = 0 \Rightarrow \psi = 0.$$

Since $H_{J_\Lambda}(u)$ is a quadratic form defined on a hyperfinite space V_Λ, its Morse index is well defined and hence also the Morse index $m_\Lambda(u)$ of u is well defined.

Given two hyperreal numbers $a < b$, we set

$$\begin{aligned} J^b_\Lambda &= \{u \in V_\Lambda | J_\Lambda(u) \le b\} \\ [J^b_a]_\Lambda &= J^b_\Lambda \backslash J^a_\Lambda = \{u \in V_\Lambda | a < J_\Lambda(u) \le b\} \\ [K^b_a]_\Lambda &= \left\{u \in J^b_a | dJ_\Lambda(u) = 0\right\}. \end{aligned}$$

Next we must define the Morse index, the Morse polynomial and the Betti polynomial in the frame of ultrafunctions. We could define them intrinsically as we have done for the above notions. However it seems easier to define them by means of a Λ-limit.

We set

$$M_t([J^b_a]_\Lambda) = \lim_{\lambda \uparrow \Lambda} \, M_t(J^{b_\lambda}_{a_\lambda} \cap V_\lambda)$$

where a_λ and b_λ are two real nets such that

$$\lim_{\lambda \uparrow \Lambda} \, a_\lambda = a, \qquad \lim_{\lambda \uparrow \Lambda} \, b_\lambda = b. \tag{10}$$

Analogously, we define the "generalized" Betti polynomial as follows:

$$P_t([J^b_a]_\Lambda) = \lim_{\lambda \uparrow \Lambda} \, P_t(J^{b_\lambda}_{a_\lambda} \cap V_\lambda).$$

Now it is possible to state an abstract theorem for Morse theory in the framework of ultrafunctions:

Theorem 16. *Let*

$$J : V \to \mathbb{R}$$

be a C^2-functional and

$$J_\Lambda : V_\Lambda \to \mathbb{R}^*$$

be the restriction of J^* to V_Λ. Let $a, b \in \mathbb{R}^*$ satisfy (10) and assume that

- *for almost every $\lambda \in \mathfrak{L}$, $\overline{J^{b_\lambda}_{a_\lambda}}$ is compact (or J satisfies (PS) in $[a_\lambda, b_\lambda]$),*
- *for almost every $\lambda \in \mathfrak{L}$, $K^{b_\lambda}_{a_\lambda}$ is finite.*

Then $M_t([J_a^b]_\Lambda), P_t([J_a^b]_\Lambda) \in \mathfrak{pol}(\mathbb{N})^*$ *where*

$$\mathfrak{pol}(\mathbb{N}) = \{\text{polynomials with coefficients in } \mathbb{N}\}$$

and there exists a polynomial $Q \in \mathfrak{pol}(\mathbb{N})^*$ *such that*

$$M_t([J_a^b]_\Lambda) = P_t([J_a^b]_\Lambda) + (1+t)Q(t).$$

Proof. For almost every $\lambda \in \mathfrak{L}$, $\overline{J_{a_\lambda}^{b_\lambda}}$ is compact and $K_{a_\lambda}^{b_\lambda}$ is finite; then by Theorem 14, $M_t(J_{a_\lambda}^{b_\lambda})$ and $P_t(J_{a_\lambda}^{b_\lambda}) \in \mathfrak{pol}(\mathbb{N})$ and there exists a polynomial $Q_\lambda \in \mathfrak{pol}(\mathbb{N})$ such that

$$M_t(J_{a_\lambda}^{b_\lambda}) = P_t(J_{a_\lambda}^{b_\lambda}) + (1+t)Q_\lambda(t).$$

The theorem follows by taking the Λ-limit. □

3.2. *Ultrafunctions versus Sobolev spaces*

Usually, the critical points of functional of type (9) are studied in the Sobolev space $W_0^{1,p}(\Omega)$ provided that the functional J can be extended to $W_0^{1,p}(\Omega)$ as a C^1 functional. In this section, we will investigate some relation between the ultrafunction and the Sobolev space approach.

So we will assume that J can be extended to a C^1-functional in a Banach space $W \subset L^1(\Omega)$ (with some abuse of notation we will denote this extension by the same letter J):

$$J : W \to \mathbb{R}.$$

So

$$V^\sigma \subset W^\sigma \subset V_\Lambda.$$

In the following, to simplify the notation, we will identify V^σ and V as well as W^σ and W.

The next theorems will establish some relations between the critical points of J_Λ in V_Λ and the critical points of J in W.

The first result in this direction is (almost) trivial:

Theorem 17. *Under the same framework and the same assumptions of Theorem 16 every critical point of* J *in* W *is a critical point of* J_Λ *in* V_Λ.

Proof. Let $u \in W$ be a critical point of J; we will use the fact that $V(\Omega)^\sigma \subset V_\Lambda(\Omega)$ to prove the thesis.

Let u_λ be the constant net $u_\lambda = u$; then

$$\lim_{\lambda \uparrow \Lambda} u_\lambda = u^* \in V(\Omega)^\sigma \subset V_\Lambda(\Omega),$$

and let J_λ be the constant net $J_\lambda = J$; then

$$dJ_\lambda(u_\lambda)[\phi_\lambda] = 0$$

for every $\phi_\lambda \in V_\lambda(\Omega)$; therefore, taking the Λ-limit of a constant net we have the thesis. □

The above theorem cannot be inverted in the sense that it is false that every critical point of J_Λ is a critical point of J in W. However, there are conditions which insure the existence of critical point of J in W. More precisely the next theorem states that, under suitable conditions, "infinitely close" to any critical point of J_Λ there is a critical point of J.

This theorem exploits a compactness condition which is a variant of the usual Palais–Smale condition (PS) [7]. We recall the Palais–Smale condition is a basic tool for Morse theory in infinite dimensional manifolds (see e.g. [7]). Here it is used only to relate some critical points of J_Λ with the critical points of J.

Definition 18 (*Palais–Smale condition for ultrafunctions (PSU).*)**.** We say that the functional

$$J : W \to \mathbb{R}$$

satisfies (PSU) in the interval $[a,b] \subset \mathbb{R}$ if for every net $\{u_\lambda\}_{\lambda\in\mathfrak{L}}$ such that

(A) $\forall\lambda \in \mathfrak{L}, J(u_\lambda) \in [a,b]$
(B) $\forall\lambda \in \mathfrak{L}, \forall v \in V_\lambda,\ dJ(u_\lambda)[v] = 0$

there is a converging subnet $\{u_\lambda\}_{\lambda\in\mathfrak{D}}$ ($\mathfrak{D} \subset \mathfrak{L}$) in the topology of W, such that

$$\lim_{\lambda\to\Lambda} u_\lambda \in W.$$

Remark 19. Notice that, by Proposition 7, the sequence $\{u_\lambda\}_{\lambda\in\mathfrak{L}}$ itself is converging.

Theorem 20. *Let us assume that W is a Banach space and that $V \subset W \subset V_\Lambda$. Let*

$$J : W \to \mathbb{R}$$

be a C^1-functional which satisfies (PSU) in the interval $[a,b]$. Then, if $\bar{u}$ is a critical point of

$$J_\Lambda : V_\Lambda \to \mathbb{R}^*$$

with $J_\Lambda(\bar{u}) \in [a,b]^*$, there exists $w \in K_a^b$ such that

$$\|\bar{u} - w^*\|_{W^*} \sim 0.$$

Remark 21. Notice that in the above theorem, it is possible that $\bar{u} = w^*$. Obviously, this fact always occurs if W is a Hilbert space and all the critical values of J in $[a,b]$ are not degenerate.

Proof of Theorem 20. Let

$$\bar{u} = \lim_{\lambda\uparrow\Lambda} u_\lambda;$$

Then, since (PSU) holds, there is a function $w \in W$ and a subnet of u_λ such that

$$\|u_\lambda - w\|_W \to 0.$$

By Proposition 7, $\|u_\lambda - w\|$ is a converging net, and hence, for every $\varepsilon > 0$, exists $Q \in \mathcal{U}$ such that $\forall \lambda \in Q$,

$$\|u_\lambda - w\|_W \leq \varepsilon.$$

If you take the Λ-limit of the above inequality, you get that

$$\|\bar{u} - w^*\|_{W^*} \leq \varepsilon.$$

Since ε is arbitrary, we conclude that

$$\|\bar{u} - w^*\|_{W^*} \sim 0. \quad \square$$

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