Arab Journal of Mathematical Sciences

Arab J Math Sci 22 (2016) 242–249

Total graph of a module with respect to singular submodule

Jituparna Goswami[a,*], Kukil Kalpa Rajkhowa[b], Helen K. Saikia[c]

[a] Department of Applied Sciences, Gauhati University Institute of Science and Technology, Guwahati-781014, India

[b] Department of Mathematics, Cotton College State University, Guwahati-781001, India

[c] Department of Mathematics, Gauhati University, Guwahati-781014, India

Received 26 May 2015; received in revised form 13 October 2015; accepted 20 October 2015
Available online 10 November 2015

Abstract. Let R be a commutative ring with unity and M be an R-module. We introduce the total graph of a module M with respect to singular submodule $Z(M)$ of M as an undirected graph $T(\Gamma(M))$ with vertex set as M and any two distinct vertices x and y are adjacent if and only if $x + y \in Z(M)$. We investigate some properties of the total graph $T(\Gamma(M))$ and its induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$. In some aspects, we have noticed some sort of finiteness.

2010 Mathematics Subject Classification: 05; C; 25

Keywords: Commutative ring; Module; Singular submodule; Total graph

1. Introduction

In 1988, Istvan Beck [10] opened up the fascinating insight which relates a graph with the algebraic structure ring. He introduced the zero divisor graph of a commutative ring, and later on, this introduction was slightly modified by D.D. Anderson and M. Naseer in [7]. Further modification to the concept of the zero-divisor graph was made in [6]. Many authors studied the zero-divisor graph in the sense of Anderson–Livingston as in [6]. Since then, the concept of the zero divisor graph of ring has been playing a vital rule in its expansion. Motivating from this well expanded idea of Beck, lots of correspondences of a graph with algebraic structures have been introduced with a variety of applications. Some of them are

* Corresponding author.
E-mail addresses: jituparnagoswami18@gmail.com (J. Goswami), kukilrajkhowa@yahoo.com (K.K. Rajkhowa), hsaikia@yahoo.com (H.K. Saikia).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.10.002

the comaximal graph of a commutative ring by Sharma and Bhatwadekar [16], the total graph of commutative ring by Anderson and Badawi [4], the intersection graph of ideals of a ring by Chakrabarty et al. [11], etc.

In 2008, Anderson and Badawi [4] defined the total graph of a commutative ring R, which is an undirected graph with vertex set as R with any two vertices are adjacent if and only if its ring sum is a zero divisor of R. In that paper, they discussed the characteristics of total graph and its two induced subgraphs by considering two cases, namely, the set of zero divisors $Z(R)$ of R is an ideal of R and $Z(R)$ is not an ideal of R. Thereafter, Akbari et al. [3] continued this concept of total graph of commutative rings. Ahmad Abbasi and Shokoofe Habibi [1] discussed the total graph of a commutative ring with respect to the proper ideals. Anderson and Badawi [5] interpreted the total graph of a commutative ring without zero element. In [17], M.H. Shekarriz et al. observed some basic graph theoretic properties of the total graph of a finite commutative ring. The prospect for total graph of modules is also observed in recent times. A. Abbasi and S. Habibi [2] investigated the total graph of a commutative ring with respect to the proper submodules of a module. The total torsion element graph of a module over a commutative ring was introduced by S. Atani and S. Habibi [8]. The above module based concepts of total graph extend the work of Anderson and Badawi [4].

In this article, we introduce the notion of singularity of a module over a ring and define the total graph of a module M with respect to singular submodule $Z(M)$. Before going to our discussion we recall the following.

Let R be a commutative ring. An element x of R is called a zero-divisor of R if there exists a non-zero element y of R with $xy = 0$. The collection of all zero-divisors of R is denoted by $Z(R)$, and henceforth, we use it. An ideal I of R is an essential ideal if its intersection with any non-zero ideal of R is non-zero. For the R-modules M and N, a mapping $f : M \to N$ is said to be a module homomorphism if $f(x + y) = f(x) + f(y)$ and $f(rx) = rf(x)$ for all $x, y \in M$ and $r \in R$. If f is also one-one, then it is said to be a module monomorphism. A one-one and onto module homomorphism is called a module isomorphism.

Throughout this discussion, all graphs are undirected. Let G be an undirected graph with the vertex set $V(G)$, unless otherwise mentioned. If G contains n vertices then we write $|V(G)| = n$. Two graphs G and H are isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. A subgraph of G is a graph having all of its vertices and edges in G. A spanning subgraph of G contains all vertices of it. For any set S of vertices of G, the induced subgraph $\langle S\rangle$ is the maximal subgraph of G with vertex set S. Thus two points of S are adjacent in $\langle S\rangle$ if and only if they are adjacent in G. The degree of a vertex v in a graph G is the number of edges incident with v. The degree of a vertex v is denoted by $deg(v)$. The vertex v is $isolated$ if $deg(v) = 0$. A walk in G is an alternating sequence of vertices and edges, $v_0x_1v_1...x_nv_n$ in which each edge x_i is $v_{i-1}v_i$. The length of such a walk is n, the number of occurrences of edge in it. A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a cycle or circuit is a closed walk with all points distinct (except the first and last). A cycle of length 3 is called a triangle. An acyclic graph does not contain a cycle. G is connected if there is a path between every two distinct vertices. A graph which is not connected is called a disconnected graph. A totally disconnected graph does not contain any edges. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x, y) = \infty$. The eccentricity $e(v)$ of a vertex v in a connected graph G is $max\ d(u, v)$ for all u in $V\ (G)$. A vertex with minimum eccentricity is called a center of G. The maximum eccentricity of G is called the diameter of G. If in a graph any two vertices are adjacent, it is called a complete graph, denoted by K^{α} where α is the number of vertices of the graph. A complete subgraph of G is called a clique. A maximum clique of G is a clique with largest number of vertices and the number of vertices of a maximum clique is called the clique number of G, denoted by $\omega(G)$. G is said to be a bipartite graph or bigraph if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 with every edge of G joining V_1 and V_2. If $|V_1| = \alpha$ and $|V_2| = \beta$ and every vertex of V_1 is adjacent to every vertex of V_2, G is called a complete bipartite graph, denoted by $K^{\alpha,\beta}$. We say that two (induced) subgraphs G_1 and G_2 of G are disjoint if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_2 (respectively, G_1). A Hamiltonian cycle is a spanning cycle in a graph. G is called Hamiltonian if it has a Hamiltonian cycle. Also $\kappa(G)$ is the smallest number of vertices removal of which makes G disconnected. The cartesian product of graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(a, b), (a', b') \in V(G) \times V(H)$ are adjacent if and only if $(i)a = a'$ and b is adjacent to b', or $(ii)b = b'$ and a is adjacent to a'. Any undefined terminology can be found in [9,12–15].

2. Total graph of a module M with respect to singular submodule $Z(M)$

Let R be a commutative ring with unity and M be an R-module. Let $Z(M)$ be the set of those $x \in M$ for which the ideal $\{r \in R | xr = 0\}$ is essential in R, i.e. $Z(M) = \{x \in M | xI = 0, \textit{for some essential ideal } I \textit{ of } R\}$. Then $Z(M)$ is a submodule of M, called the singular submodule of M. Let $\overline{Z}(M) = M - Z(M)$.

We introduce and investigate the total graph of M with respect to $Z(M)$, denoted by $T(\Gamma(M))$, as the (undirected) graph with all elements as vertices, and for distinct $x, y \in M$, the vertices x and y are adjacent, written as $x \ adj \ y$ if and only if $x + y \in Z(M)$. Let $Z(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$, with vertices $Z(M)$, and let $\overline{Z}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $\overline{Z}(M)$.

Example 1. Let $M = \mathbb{Z}_4$ be the module of integers modulo 4 and $R = \mathbb{Z}_8$ be the ring of integers modulo 8. Then the essential ideals of R are $I = \{0, 2, 4, 6\}$ and R itself. We have $Z(M) = \{0, 2\}$ and therefore $\overline{Z}(M) = \{1, 3\}$.

Let us now observe the graph $T(\Gamma(M))$ and its induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$ from Fig. 1. It is very easy to conclude that $Z(\Gamma(M))$ is complete and also disjoint from $\overline{Z}(\Gamma(M))$.

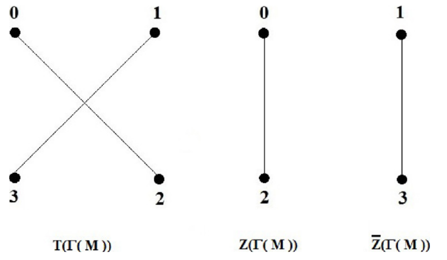

Fig. 1. The total graph $T(\Gamma(M))$ and its induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$.

We start this section with the monomorphic character of module which depicts the corresponding graphical character. We observe that the monomorphic character of module carries the graphical character.

Lemma 2.1. *Let $f : M_1 \to M_2$ be a module monomorphism. If* x adj y *then* f(x) adj f(y), *for $x, y \in M_1$.*

Proof. Let $x\ adj\ y$. Then there exists an essential ideal I of R such that $(x + y)I = 0$. Then it is easy see that $(f(x) + f(y))I = 0$. This completes the proof. □

Theorem 2.1. *Let $f : M_1 \to M_2$ be a module monomorphism. If $T(\Gamma(M_1))$ is a complete graph, then so is $T(\Gamma(f(M_1)))$.*

Proof. Suppose that $T(\Gamma(M_1))$ is a complete graph. To show $T(\Gamma(f(M_1)))$ is also a complete graph. For this, we assume $y_1, y_2 \in f(M_1)$. So, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for the elements x_1 and x_2 in M_1 respectively. As $T(\Gamma(M_1))$ is a complete graph, therefore $x_1\ adj\ x_2$. Then from the above lemma we get, $y_1\ adj\ y_2$. Thus $T(\Gamma(f(M_1)))$ is also a complete graph. □

Theorem 2.2. *Let $f : M_1 \to M_2$ be a module isomorphism. Then f is also an isomorphism from $T(\Gamma(M_1))$ onto $T(\Gamma(M_2))$.*

Proof. We need only to show that adjacency relation is preserved. For this, we assume that $x\ adj\ y$, for $x, y \in M_1$. Then there exists an essential ideal I of R such that $(x + y)I = 0$. It can be easily obtained that $f(x)\ adj\ f(y)$. Hence the result. □

Theorem 2.3. *For any $x, y \in \overline{Z}(M)$, $x\ adj\ y$ if and only if every element of $x + Z(M)$ is adjacent to every element of $y + Z(M)$.*

Proof. Let $a = x+z_1 \in x+Z(M)$, $b = y+z_2 \in y+Z(M)$. If $x\ adj\ y$, then $x+y \in Z(M)$. This gives $((a - z_1) + (b - z_2)) \in Z(M)$ i.e. $(a + b) - (z_1 + z_2) \in Z(M)$. As $Z(M)$ is a submodule of M, so $a + b \in Z(M)$. From this $a\ adj\ b$. Conversely, if $a\ adj\ b$ then $a + b \in Z(M)$. From this $(x + z_1) + (y + z_2) \in Z(M)$. Therefore $x + y \in Z(M)$. Hence $x\ adj\ y$. □

Theorem 2.4. *The following holds:*

(1) *$Z(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$ and $Z(\Gamma(M))$ is disjoint from $\overline{Z}(\Gamma(M))$.*
(2) *If N is a submodule of M, then $T(\Gamma(N))$ is the (induced) subgraph of $T(\Gamma(M))$.*

Theorem 2.5. *The following holds:*

(1) *Assume that G is an induced subgraph of $\overline{Z}(\Gamma(M))$ and let x and y be two distinct vertices of G that are connected by a path in G. Then there exists a path in G of length 2 between x and y. In particular, if $\overline{Z}(\Gamma(M))$ is connected, then $diam(\overline{Z}(\Gamma(M))) \leq 2$.*
(2) *Let x and y be distinct elements of $\overline{Z}(\Gamma(M))$ that are connected by a path. If $x + y \notin Z(M)$, then $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $\overline{Z}(\Gamma(M))$.*

Proof. (1) It is enough to show that if x_1, x_2, x_3, and x_4 are distinct vertices of G and there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 to x_4, then x_1 and x_4 are adjacent. So $x_1+x_2, x_2+x_3, x_3+x_4 \in Z(M)$ gives $x_1+x_4 = (x_1+x_2)-(x_2+x_3)+(x_3+x_4) \in Z(M)$, since $Z(M)$ is a submodule of M. Thus x_1 adj x_4. So, if $\overline{Z}(\Gamma(M))$ is connected, then $diam(\overline{Z}(\Gamma(M))) \leq 2$.

(2) Since $x + y \in \overline{Z}(\Gamma(M))$ and $x + y \notin Z(M)$, there exists $z \in \overline{Z}(\Gamma(M))$ such that $x - z - y$ is a path of length 2 by part (1) above. Thus $x + z, z + y \in Z(M)$, and hence $x - y = (x + z) - (z + y) \in Z(M)$. Also, since $x + y \notin Z(M)$, we must have $x \neq -x$ and $y \neq -x$. Thus $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $\overline{Z}(\Gamma(M))$. □

Theorem 2.6. *The following statements are equivalent.*

(1) $\overline{Z}(\Gamma(M))$ *is connected.*
(2) *Either* $x + y \in Z(M)$ *or* $x - y \in Z(M)$ *for all* $x, y \in \overline{Z}(M)$.
(3) *Either* $x + y \in Z(M)$ *or* $x + 2y \in Z(M)$ *for all* $x, y \in \overline{Z}(M)$. *In particular, either* $2x \in Z(M)$ *or* $3x \in Z(M)$ *(but not both) for all* $x \in \overline{Z}(M)$.

Proof. (1) ⇒ (2) Let $x, y \in \overline{Z}(M)$ be such that $x+y \notin Z(M)$. If $x = y$, then $x, y \in \overline{Z}(M)$. Otherwise, $x-(-y)-y$ is a path from x and y by Theorem 2.5(2), and hence $x-y \in Z(M)$.

(2) ⇒ (3) Let $x, y \in \overline{Z}(M)$, and suppose that $x + y \notin Z(M)$. By assumption, since $(x + y) - y = x \notin Z(M)$, we conclude that $x + 2y = (x + y) + y \in Z(M)$. In particular, if $x \in \overline{Z}(M)$, then either $2x \in Z(M)$ or $3x \in Z(M)$. Both $2x$ and $3x$ cannot be in $Z(M)$ since then $x = 3x - 2x \in Z(M)$, a contradiction.

(3) ⇒ (1) Let $x, y \in \overline{Z}(M)$ be distinct elements of M such that $x + y \notin Z(M)$. By hypothesis, since $x + 2y \in Z(M)$, we get $2y \notin Z(M)$. Thus $3y \in Z(M)$ by hypothesis. Since $x + y \notin Z(M)$ and $3y \in Z(M)$, we conclude $x \neq 2y$, and hence $x - 2y - y$ is a path from x to y in $\overline{Z}(M)$. □

Example 2. Let $R = Z_4$ denote the ring of integers modulo 4 and $M = Z_8$ be the ring of integers modulo 8. Then M is an R-module with the usual operations, and $Z(M) = \{0, 2, 4, 6\}$. Thus $\overline{Z}(M) = \{1, 3, 5, 7\}$. By Theorem 2.6, we conclude that $\overline{Z}(\Gamma(M))$ is connected which can be observed from Fig. 2.

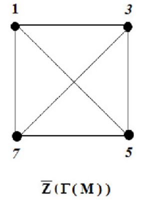

Fig. 2. The induced subgraph $\overline{Z}(\Gamma(M))$.

Theorem 2.7. *Let* $|Z(M)| = \alpha$ *and* $|M/Z(M)| = \beta$.

(1) *If* $2 \in Z(R)$ *then* $\overline{Z}(\Gamma(M))$ *is the union* $\beta - 1$ *disjoint* K^{α}*'s.*
(2) *If* $2 \notin Z(R)$ *then* $\overline{Z}(\Gamma(M))$ *is the union of* $(\beta - 1)/2$ *disjoint* $K^{\alpha,\alpha}$*'s.*

Proof. (1) It is obvious that $x + Z(M) \subseteq \overline{Z}(M)$ for every $x \notin Z(M)$. Let $x + x_1, x + x_2 \in x+Z(M)$, where $x_1, x_2 \in Z(M)$. Since $Z(M)$ is a submodule of M, so $(x+x_1)+(x+x_2) = 2x + x_1 + x_2 \in Z(M)$. Thus the coset $x + Z(M)$ is a complete subgraph of $\overline{Z}(M)$. Again any two distinct cosets form disjoint subgraphs of $\overline{Z}(M)$. If not, suppose $x + x_1$ is adjacent to $y + x_2$ for some $x, y \in \overline{Z}(M)$ and $x_1, x_2 \in Z(M)$ then $x - y = (x + y) - 2y \in Z(M)$ since $Z(M)$ is submodule of M and $2y \in Z(M)$. From this we get $x + Z(M) = y + Z(M)$, a contradiction. Hence $\overline{Z}(\Gamma(M))$ is a union of $\beta - 1$ disjoint (induced) subgraphs $x + Z(M)$, each of which is a K^{α}, where $\alpha = |Z(M)| = |x + Z(M)|$.

(2) Let $x \in \overline{Z}(M)$ and $2 \notin Z(R)$. Then no two distinct elements of $x + Z(M)$ are adjacent, because, if $x + x_1$ is adjacent to $x + x_2$, $x_1, x_2 \in Z(M)$; $2x \in Z(M)$. This implies that for some essential ideal I of R we have $2xI = 0$. Now, we have for every non-zero ideal K of R, $I \cap K \neq 0$, i.e. there exists a non-zero $x \in R$ with $x \in I \cap K$. From this we get $x + x = 2x \in I$ and $2x \in K$. But $2 \notin Z(R)$, therefore $2x \neq 0$. Thus $2x$ is a non-zero element with $2x \in 2I \cap K$ leading onto $2I$ is an essential ideal of R. This will imply that $x \in Z(M)$, as $x(2I) = 0$, which is a contradiction. Also, since $2x \notin Z(M)$, two cosets $x + Z(M)$ and $-x + Z(M)$ are disjoint. Moreover, it is easy to observe that every element of $x + Z(M)$ is adjacent to every element of $-x + Z(M)$. Thus $(x + Z(M)) \cup (-x + Z(M))$ is a complete bipartite (induced) subgraph of $\overline{Z}(\Gamma(M))$. Again, if $x + x_1$ is adjacent to $y + x_2$ for some $x, y \in \overline{Z}(M)$ and $x_1, x_2 \in Z(M)$, then $x + y \in Z(M) - 0$, and so $x + Z(M) = -y + Z(M)$. Hence $\overline{Z}(\Gamma)$ is the union of $(\beta - 1)/2$ disjoint (induced) subgraphs $(x + Z(M)) = (-y + Z(M))$, each of which is a $K^{\alpha,\alpha}$, where $\alpha = |Z(M)| = |x + Z(M)|$. □

Theorem 2.8. *Let $M - Z(M) \neq \phi$.*

(1) *If $\overline{Z}(\Gamma(M))$ is complete then either $|M/Z(M)| = 2$ or $|M/Z(M)| = |M| = 3$.*
(2) *If $\overline{Z}(\Gamma(M))$ is connected then either $|M/Z(M)| = 2$ or $|M/Z(M)| = 3$.*
(3) *If $\overline{Z}(\Gamma(M))$ (and hence $Z(\Gamma(M))$ and $T(\Gamma(M))$) is totally disconnected then either $Z(M) = 0$ or $2 \in Z(R)$.*

Proof. Suppose that $|M/Z(M)| = \beta$ and $|Z(M)| = \alpha$.

(1) First we assume $\overline{Z}(\Gamma(M))$ is complete. This implies that $\overline{Z}(\Gamma(M))$ is a single K^{α} or $K^{1,1}$, by Theorem 2.7. If $2 \in Z(R)$, then $\beta - 1 = 1$ i.e. $\beta = 2$ and thus $|M/Z(M)| = 2$. Again, if $2 \notin Z(R)$ then $\alpha = 1$ and $(\beta - 1)/2 = 1$. Hence $Z(M) = 0$ and $\beta = 3$; thus $3 = \beta = |M/Z(M)| = |M|$.

(2) Suppose that $\overline{Z}(\Gamma(M))$ is connected. This implies that $\overline{Z}(\Gamma(M))$ is a single K^{α} or $K^{\alpha,\alpha}$, by Theorem 2.7. If $2 \in Z(R)$, then $\beta - 1 = 1$ i.e. $\beta = 2$ and thus $|M/Z(M)| = 2$. Again, if $2 \notin Z(R)$ then $(\beta - 1)/2 = 1$ i.e. $\beta = 3$ and thus $|M/Z(M)| = 3$.

(3) $\overline{Z}(\Gamma(M))$ is totally disconnected if and only if it is a disjoint union of K^1's. Thus by Theorem 2.7 we have $|Z(M)| = 1$ and $|M/Z(M)| = 1$, and hence the result. □

Theorem 2.9. *Let x be a vertex of the graph $T(\Gamma(M))$. Then*

$$deg(x) = \begin{cases} |Z(M)| - 1, & \text{if } 2 \in Z(R) \text{ and } x \in Z(M) \\ |Z(M)|, & \text{otherwise.} \end{cases}$$

Proof. If $x_i \in Z(M)$, the vertex $x \in M$ is adjacent to vertices $x_i - x$. Then $deg(x) = |Z(M)| - 1$ if and only if $x = x_i - x$ for some $x_i \in Z(M)$ i.e. if and only if $2x \in Z(M)$. If $2x \notin Z(M)$, then $deg(x) = |Z(M)|$. If $2 \in Z(R)$, then $2x \in Z(M)$ for all $x \in M$, thus $deg(x) = |Z(M)| - 1$ i.e. all vertices of the graph $T(\Gamma(M))$ are of degree $|Z(M)| - 1$. Again, if $2 \notin Z(R)$, then two cases arise.

Case-1—If $x \in Z(M)$, then $deg(x) = |Z(M)| - 1$.

Case-2—If $x \notin Z(M)$, then $deg(x) = |Z(M)|$.

It follows that $deg(x) = \begin{cases} |Z(M)| - 1, & \text{if } 2 \in Z(R) \text{ and } x \in Z(M) \\ |Z(M)|, & \text{otherwise.} \end{cases}$ □

Theorem 2.10. *Let M_1 and M_2 be two finite modules over a finite ring R. Then the following holds.*

(1) *If $T(\Gamma(M_1))$ is a Hamiltonian graph, then so is $T(\Gamma(M_1 \times M_2))$.*
(2) *If $\overline{Z}(\Gamma(M_1))$ is a Hamiltonian graph, then so is $\overline{Z}(\Gamma(M_1 \times M_2))$.*

Proof. (i) Let $M_1 = \{m_1, m_2, \ldots, m_s\}$ and $M_2 = \{m'_1, m'_2, \ldots, m'_t\}$ be such that the sequence $m_1, m_2, \ldots, m_s$ is a Hamiltonian cycle. Then $m_1 + m_s \in Z(M_1)$. Thus we get the Hamiltonian cycle in $T(\Gamma(M_1 \times M_2))$ as

$(m_1, m'_1), (m_2, m'_1), \ldots, (m_s, m'_1), (m_1, m'_2), \ldots, (m_s, m'_2), \ldots, (m_1, m'_t), \ldots, (m_s, m'_t)$.

(ii) Suppose that $\overline{Z}(M_1) = \{m_1, m_2, \ldots, m_s\}$ and $\overline{Z}(M_2) = \{m'_1, m'_2, \ldots, m'_t\}$. The above Hamiltonian cycle is also a Hamiltonian cycle for $\overline{Z}(\Gamma(M_1 \times M_2))$. □

Theorem 2.11. *Let $M = M_1 \times M_2$ be finite module. Then $\kappa(T(\Gamma(M))) \geq |M_1| + |M_2| - 4$.*

Proof. Let (x, y) and (x', y') be two distinct elements of M. If $x \neq x', y' \neq \pm y, \lambda \notin \{y, -y, y', -y'\}$, then consider the paths $(x, y), (-x, \lambda), (-x', -\lambda), (-x', -y')$ for $\lambda \in M_2$. If $\eta \in M_1$ and $(x, y) \neq (\eta, -y)$ and $(x', y') \neq (-\eta, -y')$, then consider the paths $(x, y), (\eta, -y), (-\eta, -y'), (x', y')$. If $(x, y) \neq (\eta, -y)$ and $(x', y') = (-\eta, -y')$, then consider the paths $(x, y), (\eta, -y), (x', y')$. If $(x, y) = (\eta, -y)$ and $(x', y') \neq (-\eta, -y')$, then consider the paths $(x, y), (-\eta, -y'), (x', y')$. If $(x, y) = (\eta, -y)$ and $(x', y') = (-\eta, -y')$ for some η, then consider the paths $(x, y), (x', y')$ and $(x, y), (\eta, -y), (x', y')$ for some $\eta \neq x$. So there are at least $|M_1| + |M_2| - 4$ disjoint paths from (x, y) to (x', y').

Let $x \neq x', y' \neq y$ and $y' = -y$. Then the paths $(x, y), (-x, \lambda), (-x', -\lambda), (x', -y)$ for $\lambda \in M_2 - \{\pm b\}$ and the paths $(x, y), (\eta, -y), (-\eta, y), (x', -y)$ for $\eta \in M_1 - \{-x, x'\}$ are $|M_1| + |M_2| - 4$ disjoint paths. Let $x \neq x', y' = y$. Consider the paths $(x, y), (-x, \lambda), (-x', -\lambda), (x', -y)$ for $\lambda \in M_2 - \{y, -y\}$ and the paths $(x, y), (\eta, -y)$, (x', y) for $\eta \in M_1 - \{x, x'\}$. If $x = x'$, since (x, y) and (x', y') are distinct, then $y \neq y'$ and the proof is the same as the case $x \neq x'$ and $y = y'$. □

Acknowledgments

The authors would like to thank the referees for their careful reading of the article and valuable suggestions to improve the article.

References

[1] A. Abbasi, S. Habibi, On the total graph of a commutative ring with respect to proper ideals, J. Korean Math. Soc. 49 (2012) 85–98. 1.

[2] A. Abbasi, S. Habibi, On the total graph of a module over a commutative ring with respect to proper submodules, J. Algebra Appl. 11 (3) (2012).
[3] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, The total graph and regular graph of a commutative ring, J. Pure Appl. Algebr. 213 (2009) 2224–2228.
[4] D.F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008) 2706–2719.
[5] D.F. Anderson, A. Badawi, On the total graph of a commutative ring without the zero element, J. Algebra Appl. 11 (4) (2012) 1250074. (18 pages). http://dx.doi.org/10.1142/S0219498812500740.
[6] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999) 434–447.
[7] D.D. Anderson, M. Naseer, Beck's coloring of a commutative ring, J. Algebra 159 (1993) 500–514.
[8] S.E. Atani, S. Habibi, The total torsion element graph of a module over a commutative ring, An. St. Univ. Ovidius Constant 19 (1) (2011) 23–34.
[9] R. Balkrishnan, K. Ranganathan, A Text Book of Graph Theory, Springer-verlag New York, Inc., 2008, Reprint.
[10] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988) 208–226.
[11] I. Chakrabarty, S. Ghosh, T.K. Mukherjee, M.K. Sen, Intersection graphs of ideals of rings, Discrete Math. 309 (17) (2009) 5381–5392.
[12] K.R. Goodearl, Ring Theory, Marcel Dekker, 1976.
[13] F. Harary, Graph Theory, Addison-Wesley Publishing Company, Inc., Reading, Mass, 1969.
[14] I. Kaplansky, Commutative Rings, Revised ed., University of Chicago Press, Chicago, 1974.
[15] J. Lambeck, Lectures on Rings and Modules, Blaisdell Publishing Company, Waltham, Toronto, London, 1966.
[16] P.K. Sharma, S.M. Bhatwadekar, A note on graphical representation of rings, J. Algebra 176 (1995) 124–127.
[17] M.H. Shekarriz, M.H.S. Haghighi, H. Sharif, On the total graph of a finite commutative ring, Comm. Algebra 40 (2012) 2798–2807.