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Topics in differential geometry associated with position vector fields on Euclidean submanifolds

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Abstract. The position vector field is the most elementary and natural geometric object on a Euclidean submanifold. The purpose of this article is to survey six research topics in differential geometry in which the position vector field plays very important roles. In this article we also explain the relationship between position vector fields and mechanics, dynamics, and D'Arcy Thompson's law of natural growth in biology.

Keywords: Position vector field; Rectifying curve; Rectifying submanifold; Finite type submanifold; Ricci soliton; Biharmonic submanifold; Constant-ratio submanifold; Self-shrinker; Thompson's law of natural growth

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1. Introduction

For an n-dimensional submanifold M in the Euclidean m-space $\mathbb{E}^m$, the most elementary and natural geometric object is the position vector field $\mathbf{x}$ of M. The position vector, also known as *location vector* or *radius vector*, is a Euclidean vector $\mathbf{x} = \overrightarrow{OP}$ that represents the position of a point $P \in M$ in relation to an arbitrary reference origin O.

Among extrinsic invariants of a submanifold, the most natural and important one is the mean curvature vector H. In physics, the mean curvature vector field is the tension field imposed on the submanifold arising from the ambient space. In materials science, surface tension is used for either surface stress or surface free energy. It is well-known that surface tension is responsible for the shape of liquid droplets.

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The well-known formula of E. Beltrami (1835–1900) provides a simple relationship between the position vector field $\mathbf{x}$ and the mean curvature vector field H of M, namely,

$$\Delta\mathbf{x} = -nH, \tag{1.1}$$

where Δ denotes the Laplacian of M with respect to its induced metric on M (cf. [12,18,29, 33,89]).

A Euclidean submanifold is called a *minimal submanifold* if its mean curvature vector vanishes identically. The history of minimal submanifolds goes back to J.L. Lagrange (1736–1813) who initiated in 1760 the study of minimal surfaces in Euclidean 3-space (cf. [63]). Since then the theory of minimal surfaces has attracted many mathematicians for more than two centuries. In particular, minimal surfaces and minimal submanifolds in Riemannian manifolds of constant curvature have been investigated very extensively since then (see, e.g. [22,82,84]).

The position vector field also plays important roles in physics, in particular, in mechanics. In any equation of motion, the position vector $\mathbf{x}(t)$ is usually the most sought-after quantity because the position vector field defines the motion of a particle (i.e. a point mass)—its location relative to a given coordinate system at some time variable t. The first and the second derivatives of the position vector field with respect to time t give the velocity and acceleration of the particle.

The main purpose of this article is to survey six research topics in differential geometry in which the position vector field plays important roles. In this survey article we also explain the relationship between position vector fields and mechanics, dynamics, and D'Arcy Thompson's law of natural growth in biology.

2. Rectifying curves

In elementary differential geometry, most geometers describe a curve as a unit speed curve $\mathbf{x} = \mathbf{x}(s)$ whose position vector field is expressed in terms of an arc-length parameter s. In order to define curvature and torsion of a space curve, one needs the well-known Frenet formulas which can be obtained as follows:

Consider a unit-speed curve $\mathbf{x}: I \to \mathbb{E}^3$, defined on a real interval $I = (\alpha, \beta)$, that has at least four continuous derivatives. Put $\mathbf{t} = \mathbf{x}'(s)$. In general, it is possible that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can introduce a unique vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature* of the given curve $\mathbf{x}(t)$. Since $\mathbf{t}$ is a constant length vector field, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ which is a unit vector field orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ of the curve by the equation $\mathbf{b}' = -\tau\mathbf{n}$.

The famous *Frenet formulas* are given by

$$\begin{cases} \mathbf{t}' = \kappa\mathbf{n}, \\ \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b}' = -\tau\mathbf{n}. \end{cases} \tag{2.1}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$ and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively. A curve in $\mathbb{E}^3$ is called a *twisted curve* if it has nonzero curvature and nonzero torsion. A *helix* (or *curve of* *constant slope*) is defined by the property that the tangent makes a constant angle with a fixed line.

It is well-known in elementary differential geometry that a curve in $\mathbb{E}^3$ lies in a plane if its position vector lies in its osculating plane at each point; and it lies on a sphere if its position vector lies in its normal plane at each point. In view of these basic facts, the author asked the following very simple natural geometric question in [17]:

Question. *When does the position vector of a space curve $\mathbf{x}{:}I \to \mathbb{E}^3$ always lie in its rectifying plane?*

For simplicity, we call such a curve a *rectifying curve* in [17]. Clearly, the position vector field $\mathbf{x}$ of a rectifying curve $\mathbf{x}{:}I \to \mathbb{E}^3$ satisfies

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s) \tag{2.2}$$

for some functions λ and μ.

A well known theorem of Lancret [65] states that a twisted curve in $\mathbb{E}^3$ is a helix if and only if the ratio $\tau{:}\kappa$ is a nonzero constant.

On the other hand, we have the following result for rectifying curves.

Theorem 2.1 *([17]). A twisted curve $\mathbf{x}{:}I \to \mathbb{E}^3$ is a rectifying curve if and only if the ratio $\tau{:}\kappa$ is a nonconstant linear function in arc-length function s.*

The Frenet equations can be interpreted *kinematically* as follows: If a moving point traverses the curve in such a way that s is the time parameter, then the moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ moves in accordance with the Frenet formulas (2.1). It is known in mechanics that this motion contains, apart from an instantaneous translation, an instantaneous rotation with angular velocity vector given by the *Darboux rotation vector*

$$\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}. \tag{2.3}$$

The direction of the Darboux vector is that of the *instantaneous axis of rotation*, and its length $\sqrt{\kappa^2 + \tau^2}$ is called the *angular speed*.

By applying (2.2), we know that *rectifying curves are exactly space curves whose axis of instantaneous rotation always passes through the origin of $\mathbb{E}^3$.*

The fundamental theorem for curves in $\mathbb{E}^3$ states that, up to rigid motions, a curve is uniquely determined by its curvature and torsion given as functions of its arc-length. To determine the curve, it invokes solving the Frenet equations. A result of S. Lie (1984–1899) and J.-G. Darboux (1842–1917) shows that solving the Frenet equations is equivalent to solving the complex Riccati equation:

$$\frac{dw}{ds} = \mathrm{i}\left(\frac{\tau}{2}w^2 - \frac{\tau}{2} - \kappa w\right). \tag{2.4}$$

In practice, for a space curve with prescribed curvature $\kappa(s)$ and torsion $\tau(s)$, the solutions of Eq. (2.4) are often impossible to find explicitly. Fortunately, the author is able to determine explicitly all rectifying curves in $\mathbb{E}^3$.

Theorem 2.2 *([17]). A twisted curve $\mathbf{x}{:}I \to \mathbb{E}^3$ is a rectifying curve if and only if it is given by*

$$\mathbf{x}(t) = a\sec(t + b)\mathbf{y}(t), \tag{2.5}$$

where a, b are real numbers with $a \neq 0$ and $\mathbf{y} = \mathbf{y}(t)$ is a unit-speed curve in the unit sphere $S_o^2(1)$ centered at the origin $o \in \mathbb{E}^3$.

For a unit speed curve $\mathbf{y} = \mathbf{y}(t)$ lying on $S_o^2(1)$, let $C_{\mathbf{y}}$ denote the *cone with vertex at $O \in \mathbb{E}^3$ over the spherical curve $\mathbf{y}$*.

We may parametrize the cone $C_{\mathbf{y}}$ as

$$C_{\mathbf{y}}(t, u) = u\mathbf{y}(t), \quad u \in \mathbf{R}^+. \tag{2.6}$$

A well-known result in differential geometry states that a helix is a geodesic on the cylinder containing the helix in $\mathbb{E}^3$. On the other hand, we have the following result for rectifying curves.

Theorem 2.3. *Each rectifying curve given by (2.5) is a geodesic on the cone $C_{\mathbf{y}}$.*

Proof. For a given positive function $\rho = \rho(t)$ on an interval I, we put

$$\mathbf{z}(t) = \rho(t)\mathbf{y}(t). \tag{2.7}$$

Then $\mathbf{z}$ is a regular curve lying on the cone $C_{\mathbf{y}}$. Consider the integral functional with fixed end points:

$$L(\mathbf{z}, \rho) = \int_{t_0}^{t_1} \sqrt{\rho^2 + \rho'^2}\, dt, \tag{2.8}$$

with the energy function $f(t, \rho, \rho') = \sqrt{\rho^2 + \rho'^2}$.

A fundamental result from calculus of variations states that the Euler–Lagrange equation of the functional (2.8) is given by

$$\frac{\partial f}{\partial \rho} - \frac{d}{dt}\left(\frac{\partial f}{\partial \rho'}\right) = 0. \tag{2.9}$$

Therefore, by using $f = \sqrt{\rho^2 + \rho'^2}$, we derive from (2.9) that the Euler–Lagrange equation of (2.8) is the following differential equation:

$$\rho\rho'' - 2\rho'^2 - \rho^2 = 0. \tag{2.10}$$

After solving (2.10), we find

$$\rho = a \sec(t + b) \tag{2.11}$$

for some real numbers $a \neq 0$ and b. Consequently, by applying Theorem 2.2 we obtain the theorem. □

For a regular curve $\mathbf{x}(s)$ in $\mathbb{E}^3$ with positive curvature, the curve given by the *Darboux vector*

$$\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b} \tag{2.12}$$

is called the *centrode* of $\mathbf{x}$. The centrodes play some important roles in mechanics and joint kinematics (see, for instance, [54,55,83,91,93]).

The following result provides a link between centrodes and rectifying curves.

Theorem 2.4 ([40]). *The centrode of a unit speed curve in $\mathbb{E}^3$ with constant curvature $\kappa \neq 0$ and non-constant torsion τ is a rectifying curve.*

Conversely, every rectifying curve in $\mathbb{E}^3$ is the centrode of some unit speed curve with constant curvature $\kappa \neq 0$ and with non-constant torsion.

Similarly, we also have the following.

Theorem 2.5 ([40]). *The centrode of a unit speed curve in $\mathbb{E}^3$ with non-constant curvature κ and constant torsion $\tau \neq 0$ is a rectifying curve.*

Conversely, every rectifying curve in $\mathbb{E}^3$ is the centrode of some unit speed curve with nonconstant curvature and nonzero constant torsion.

Remark 2.1. The centrode of a curve with nonzero constant curvature and nonzero constant torsion is a point.

Remark 2.2. Theorems 2.4 and 2.5 imply that the curves in $\mathbb{E}^3$ with nonzero constant curvature and the curves with nonzero constant torsion can be related via rectifying curves.

Remark 2.3. Rectifying curves in $\mathbb{E}^3$ have many other nice properties, see [17,40]. After [17,40], there are many articles published which investigate rectifying curves in various ambient spaces; and many new results in this respect have been obtained (see [59,60,71, 94] among many others).

Remark 2.4. In a recent article [35], the author introduces the notion of rectifying submanifolds; extending the notion of rectifying curves in a very natural way. Some basic properties and the complete classification of rectifying submanifolds are obtained in [35]. In particular, the author proved in [35] that a Euclidean submanifold is a rectifying submanifold if and only if the tangential component of its position vector field is a concurrent vector field. Furthermore, the author defines rectifying submanifolds in arbitrary Riemannian manifolds in [36]. For results on rectifying submanifolds in this respect, see [36].

3. Finite type submanifolds

The theory of finite type submanifolds began in the late 1970s through the author's attempts to find the best possible estimates of the total mean curvature of a compact submanifold of Euclidean space, and to find a notion of "degree" for submanifolds of Euclidean space (cf. [13–15]). The theory of finite type submanifolds is another research subject in which the position vector field plays an essential role.

Let $x{:}M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into the Euclidean m-space $\mathbb{E}^m$. Denote by Δ the Laplace operator of M. The immersion x is said to be *of finite type* if the position vector field $\mathbf{x}$ of M in $\mathbb{E}^m$ admits a finite spectral decomposition:

$$\mathbf{x} = c + \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k \tag{3.1}$$

where c is a constant vector in $\mathbb{E}^m$ and $\mathbf{x}_1, \dots, \mathbf{x}_k$ are non-constant maps satisfying

$$\Delta \mathbf{x}_i = \lambda_i \mathbf{x}_i, \; i = 1, \dots, k, \tag{3.2}$$

for some eigenvalues $\lambda_1, \dots, \lambda_k$ of Δ.

If all of the eigenvalues $\lambda_1, \ldots, \lambda_k$ in (3.2) associated with the spectral decomposition (3.1) of $\mathbf{x}$ are mutually different, then the immersion x (or the submanifold M) is said to be of *k-type*. A submanifold is said to be *of infinite type* if it is not of finite type.

The family of submanifolds of finite type is huge, it contains many important families of submanifolds, including all minimal submanifolds of Euclidean space, all minimal submanifolds of hyperspheres as well as parallel submanifolds and equivariantly immersed compact homogeneous submanifolds. Furthermore, just like minimal submanifolds, submanifolds of finite type are characterized by a spectral variation principle, namely critical points of directional deformations (see [41] for details).

On one hand, the notion of finite type submanifolds provides a very natural way to connect the theory of spectral geometry with the theory of submanifolds. On the other hand, one can apply the theory of finite type submanifolds to study the spectral geometry of submanifolds.

The first results on finite type submanifolds as well as on finite type maps were collected in the author's books [15,16]. In 1991, a list of twelve open problems and three conjectures on finite type submanifolds was published in [19]. Also, a detailed survey of the results on this subject up to 1996 was given in [21]. For the most up-to-date survey on this subject, see the author's 2015 book [33].

Two main conjectures on finite type submanifolds are the following (cf. [16,21]).

Conjecture A. *The only compact hypersurfaces of finite type in Euclidean space are ordinary hyperspheres.*

Conjecture B. *The only finite type surfaces in $\mathbb{E}^3$ are minimal surfaces, open portions of spheres, and open portions of circular cylinders.*

Although there are many articles providing affirmative partial supports to these two conjectures, these two conjectures remain open since 1985.

The theory of finite type submanifolds and its applications remain an active research subject in recent years. For more important open problems and conjectures on this subject, we refer to [19,21,32,33].

4. Biharmonic submanifolds

A Euclidean submanifold is minimal if and only if its position vector field $\mathbf{x}$ is harmonic, i.e. $\Delta\mathbf{x} = 0$ by (1.1). A submanifold M of $\mathbb{E}^m$ is called *bi-harmonic* if the position vector field $\mathbf{x}$ satisfies

$$\Delta^2\mathbf{x} = 0. \tag{4.1}$$

Obviously, every minimal submanifold in $\mathbb{E}^m$ is trivial biharmonic. Hence the real question is "when a biharmonic submanifold is minimal". It follows from (4.1) and Hopf's lemma that biharmonic submanifolds in a Euclidean space are always non-compact.

The study of biharmonic submanifolds was initiated by the author in the middle of the 1980s in his program of understanding submanifolds of finite type. The author showed in 1985 that biharmonic surfaces in $\mathbb{E}^3$ are minimal (vein independently by Jiang [61] in his study of Euler–Lagrange's equation of bi-energy functional). This result was the starting point of Ivko Dimitrić's work on his doctoral thesis [47] at Michigan State University. In fact, I. Dimitrić extended the author's result on biharmonic surfaces in $\mathbb{E}^3$ to biharmonic hypersurfaces in $\mathbb{E}^{n+1}$, $n \geq 3$, with at most two distinct principal curvatures [38,48]. He also proved that every biharmonic submanifold of finite type in any Euclidean space is minimal. Moreover, he proved that pseudo-umbilical biharmonic Euclidean submanifolds are minimal. Also, the author pointed out in [19] that spherical biharmonic submanifolds in Euclidean spaces are always minimal as well.

Nowadays, the study of biharmonic submanifolds is a very active research subject. In particular, biharmonic submanifolds have received growing attention with much progress done since the beginning of this century.

The following conjecture was proposed by the author about 25 years ago.

Biharmonic Conjecture (*[19]*). *The only biharmonic submanifolds of Euclidean space are the minimal ones.*

In order to state major results of this conjecture, we give some definitions.

Definition 4.1. An immersed submanifold M of a Riemannian manifold $\tilde{M}$ is called *properly immersed* if the immersion of M is a proper map, i.e., the preimage of each compact set in $\tilde{M}$ is compact in M. A hypersurface of a Euclidean space is called *weakly convex* if it has non-negative principal curvatures.

Definition 4.2. Let M be a submanifold of a Riemannian manifold with inner product $\langle\ ,\ \rangle$. Then M is called *ϵ-superbiharmonic* if its mean curvature vector H satisfies

$$\langle \Delta H, H\rangle \geq (\epsilon - 1)|\nabla H|^2, \tag{4.2}$$

where $\epsilon \in [0, 1]$ is a constant.

Definition 4.3. For a complete Riemannian manifold (M, g) and $\alpha \geq 0$, if the sectional curvature K of M satisfies

$$K \geq -L(1 + dist_M(\,\cdot\,, q_0)^2)^{\alpha/2}$$

for some $L > 0$ and $q_0 \in M$, then we say that K has a polynomial growth bound of order α from below.

Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of T_pM, the scalar curvature τ at p is

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a r-subspace of T_pM with $r \geq 2$ and let $\{e_1, \ldots, e_r\}$ be an orthonormal basis of L. The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{4.3}$$

For given integers $n \geq 3$, $k \geq 1$, we denote by $S(n, k)$ the finite set consisting of k-tuples $(n_1, \ldots, n_k)$ of integers satisfying $2 \leq n_1, \ldots, n_k < n$ and $\sum_{j=1}^{k} n_i \leq n$.

Put $S(n) = \cup_{k\geq 1} S(n,k)$. For each k-tuple $(n_1,\ldots,n_k) \in S(n)$, the author introduced in the 1990s the Riemannian invariant $\delta(n_1,\ldots,n_k)$ by

$$\delta(n_1,\ldots,n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \quad p \in M,$$

where $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1,\ldots,k$ (cf. [20,23,29] for details).

For an n-dimensional submanifold of $\mathbb{E}^m$ and for a k-tuple $(n_1,\ldots,n_k) \in S(n)$, the author proved the following general sharp inequality [23,29]:

$$\delta(n_1,\ldots,n_k) \leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}\|H\|^2, \tag{4.4}$$

where $\|H\|^2 = \langle H,H\rangle$ denotes the squared mean curvature of M.

Definition 4.4. A submanifold M of $\mathbb{E}^m$ is called $\delta(n_1,\ldots,n_k)$-ideal if it satisfies the equality case of (2.9) identically.

Roughly speaking, ideal submanifolds are submanifolds which receive the least possible tension from their ambient space. For the most recent survey on δ-invariants and ideal immersions, see [29,30] for details.

The following results provide strong supports to the biharmonic conjecture.

(1) Biharmonic hypersurface in $\mathbb{E}^4$ [57].
(2) Biharmonic hypersurface with 3 distinct principal curvatures in $\mathbb{E}^{n+1}$ [52].
(3) $\delta(2)$-ideal and $\delta(3)$-ideal biharmonic hypersurfaces in $\mathbb{E}^{n+1}$ [44].
(4) Properly immersed biharmonic submanifolds [72].
(5) Biharmonic submanifolds which are complete and proper [2].
(6) Weakly convex biharmonic submanifolds [70].
(7) Biharmonic submanifolds satisfying the decay condition at infinity

$$\lim_{\rho\to\infty} \frac{1}{\rho^2}\int_{f^{-1}(B_\rho)} \|H\|^2 dv = 0,$$

where f is the immersion, B_ρ is a geodesic ball of N with radius ρ [92].

Remark 4.1. Y.-L. Ou showed in [85] that the Biharmonic Conjecture cannot be generalized to biharmonic conformal submanifolds in Euclidean spaces.

Remark 4.2. The Biharmonic Conjecture is false if the ambient Euclidean space is replaced by a pseudo-Euclidean space [42,43].

Remark 4.3. The Biharmonic Conjecture remains open after 25 years.

The following is an extension of the Biharmonic Conjecture made in [9] by R. Caddeo, S. Montaldo and C. Oniciuc.

Generalized Chen's Conjecture. *Every biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.*

Y.-L. Ou and L. Tang [87] proved that the Generalized Chen Conjecture *is false in general* by constructing foliations of proper biharmonic hyperplanes in some conformally flat 5-manifolds with negative sectional curvature (see, also [69]).

On the other hand, there are many partial results since the early 2000s which support the Generalized Chen Conjecture under some additional conditions on the ambient spaces (see, for instance, [3,7,9,74,73,77–79,86,92]).

5. Constant-ratio submanifolds

D'Arcy Thompson (1860–1948) was a pioneer of mathematical biology. His most famous work is his book "On Growth and Form" published in 1917 (see [90]). Many other editions were published since 1917. The *theory of growth and form* of Thompson provides a very nice link between biology and differential geometry of position vector fields.

The central theme of Thompson's book is that biologists of his time overemphasized evolution as the fundamental determinant of the form and structure of living organisms, and underemphasized the roles of physical laws and mechanics. Hence, he advocated *structuralism* as an alternative to survival of the fittest in governing the form of species.

On the concept of *allometry*, the study of the relationship of body size and shape, Thompson wrote: "An organism is so complex a thing, and growth so complex a phenomenon, that for growth to be so uniform and constant in all the parts as to keep the whole shape unchanged would indeed be an unlikely and an unusual circumstance. Rates vary, proportions change, and the whole configuration alters accordingly."

In the section '*The Equiangular Spiral in its Dynamical Aspect*' of Thompson's book, he wrote: "In mechanical structures, *curvature* is essentially a mechanical phenomenon. It is found in flexible structures as a result of bending, or it may be introduced into construction for the purpose of resisting such a bending-moment. But neither shell nor tooth nor claw are flexible structures; they have not been bent into their peculiar curvature, they have grown into it.

We may for a moment, however, regard the equiangular or logarithmic spiral of our shell from the dynamic point of view, by looking at growth itself as the force concerned. In the growing structure, let growth at a point P be resolved into a force F acting along the line joining P to a pole O, and a force T acting in a direction perpendicular to OP; and let the magnitude of these forces (or of these rates of growth) remain constant. It follows that the resultant of the forces F and T (as PQ) makes a constant angle with the radius vector [position vector]. But a constant angle between tangent and radius vector [position vector] is a fundamental property of the "equiangular" spiral: the very property with which Descartes started his investigation, and that which gives its alternative name to the curve.

In such a spiral, radial growth and growth in the direction of the curve bear a constant ratio to one another. For, if we consider a consecutive radius vector OP', whose increment as compared with OP is dr, while ds is the small arc PP', then $dr/ds = \cos\alpha = constant$.

In the growth of a shell, we can conceive no simpler law than this, that it shall widen and lengthen in the same unvarying proportions: and this simplest of laws is that which Nature tends to follow. The shell, like the creature within it, grows in size but does not change its shape; and the existence of this constant relativity of growth, or constant similarity of form, is of the essence, and may be made the basis of a definition, of the equiangular spiral."

Accidentally or not, *Thompson's law of natural growth* has a natural link[1] to the author's *constant-ratio submanifolds* in his study of position vector fields done in [24].

Let $x: M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into the Euclidean m-space. Let us denote by $\mathbf{x}$ the position vector field of M as before.

For each given submanifold M in $\mathbb{E}^m$, there is an orthogonal decomposition of the position vector $\mathbf{x}$ at each point on M:

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^\perp, \tag{5.1}$$

where $\mathbf{x}^T$ and $\mathbf{x}^\perp$ denote the tangential and normal components of $\mathbf{x}$ at the point, respectively. Let $\|\mathbf{x}^T\|$ and $\|\mathbf{x}^\perp\|$ be the length of $\mathbf{x}^T$ and $\mathbf{x}^\perp$, respectively.

Definition 5.1 (*[24,26]*)**.** A submanifold M of a Euclidean space (or more generally, of a pseudo-Euclidean space) is called a *constant-ratio submanifold* if the ratio $\|\mathbf{x}^T\|:\|\mathbf{x}^\perp\|$ is a constant on M.

Remark 5.1. Constant-ratio curves in a plane are exactly the equiangular curves in the sense of Thompson. Hence, constant-ratio submanifolds can be regarded as a higher dimensional version of Thompson's equiangular curves. For this reason, constant-ratio submanifolds are also known in some literature as *equiangular submanifold* (see [56]; see also [52,76]).

Remark 5.2. Constant-ratio curves also relate to the motion in a central force field which obeys the inverse-cube law. In fact, the *trajectory of a mass particle subject to a central force of attraction located at the origin which obeys the inverse-cube law is a curve of constant-ratio*.

Remark 5.3. The inverse-cube law originated with Sir Isaac Newton (1642–1727) in his letter sent on December 13, 1679 to Robert Hooke (1635–1703). This letter is of great historical importance since it reveals the state of Newton's development of dynamics at that time (see, for instance, [8], [64, pages 266–271], [80] and [81, Book I, Section II, Proposition IX]).

Let ρ denote the distance function of M in $\mathbb{E}^m$, i.e., $\rho = \|\mathbf{x}\|$. It was proved in [28] that the Euclidean submanifold M is of constant-ratio if and only if the gradient of the distance function ρ has constant length.

Constant ratio hypersurfaces in a Euclidean space were completely classified as follows.

Theorem 5.1 (*[24]*)**.** *Let $x: M \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion of a Riemannian n-manifold into a Euclidean $(n+1)$-space. Then M is of constant-ratio if and only if one of the following three cases occurs:*

(a) *M is an open portion of a hypersphere $S^n(r)$ of $\mathbb{E}^{n+1}$ centered at the origin.*
(b) *M is an open portion of a cone with vertex at the origin.*

[1] The author thanks professor Leopold Verstraelen who pointed out this nice link to the author several years after the appearance of [24].

(c) *There exists a local coordinate system* $\{s, u_2, \dots, u_n\}$ *on* M *such that the immersion* x *is given by*

$$x(s, u_2, \dots, u_n) = (cs)Y(s, u_2, \dots, u_n),$$

where $Y(s, u_2, \dots, u_n)$ *is a parametrization of the unit hypersphere* $S^n(1)$ *centered at the origin which satisfies the following two conditions:*
(c.1) Y_s *is perpendicular to* $Y_{u_2}, \dots, Y_{u_n}$ *and*
(c.2) $|Y_s| = \sqrt{1-c^2}/(cs)$.

A submanifold M of the complex Euclidean m-space $\mathbf{C}^m$ is called a *totally real submanifold* if the complex structure J of $\mathbf{C}^n$ maps each tangent space T_pM, $p \in M$, into the corresponding normal space $T_p^{\perp}M$, i.e., $J(T_pM) \subset T_p^{\perp}M$.

Totally real constant-ratio submanifolds in $\mathbf{C}^m$ were classified as follows.

Theorem 5.2 (*[31]*). *Let* $x{:}M \to \mathbf{C}^m$ *be a totally real immersion of a Riemannian* n*-manifold* M *into* $\mathbf{C}^m$*. Then* M *is of constant-ratio if and only if one of the following four statements holds:*

(1) M *is an open portion of a totally real cone with vertex at the origin.*
(2) *Up to a suitable dilation,* $x{:}M \to \mathbf{C}^m$ *is given by*

$$x(t, u_2, \dots, u_n) = e^{\mathrm{i}t}\phi(u_2, \dots, u_n),$$

where ϕ *is an* $(n-1)$*-dimensional* C*-totally real submanifold of the Sasakian* $S^{2m-1}(1)$.
(3) *Up to a suitable dilation,* M *is an anti-invariant submanifold of the Sasakian* $S^{2m-1}(1)$ *with* $\xi \notin TM$*, where* ξ *is the Reeb vector field.*
(4) *Up to a suitable dilation,* $x{:}M \to \mathbf{C}^m$ *is given by*

$$x(s, u_2, \dots, u_n) = bs\psi(s, u_2, \dots, u_n), \quad s \neq 0,$$

where b *is a positive number* < 1 *and* $\psi{:}M \to S^{2m-1}(1)$ *is an immersion satisfying*
(4.a) $\langle \psi_s, \psi_s \rangle = (1-b^2)/(b^2s^2)$,
(4.b) $\langle \psi, \mathrm{i}\psi_{u_i} \rangle = -s\,\langle \psi_s, \mathrm{i}\psi_{u_i} \rangle$*, and*
(4.c) $\left\langle \psi_{u_i}, \mathrm{i}\psi_{u_j} \right\rangle = \left\langle \psi_s, \psi_{u_j} \right\rangle = 0$,
for $i, j = 2, \dots, n$.

Remark 5.4. Constant ratio submanifolds in pseudo-Euclidean space with arbitrary codimension were classified in [27].

Remark 5.5. Constant ratio submanifolds are related to the notion of *convolution manifolds* introduced by the author in [25,28] as well.

Remark 5.6. It was known in [24] that the tangential component $\mathbf{x}^T$ of the position vector field $\mathbf{x}$ of a constant-ratio hypersurface in $\mathbb{E}^{n+1}$ defines a principal direction. In [52], Y. Fu and M.I. Munteanu call a surface in $\mathbb{E}^3$ satisfying this property on $\mathbf{x}^T$ a *generalized constant-ratio surface*. They proved in [53] that a generalized constant-ratio surface in $\mathbb{E}^3$ can be parametrized as

$$x(s,t) = s(\cos u(s) f(t) + \sin u(s) f(t) \times f'(t)),$$

where $f(t)$ is a unit speed curve on the unit 2-sphere centered at the origin and $u(s) = \int^s t^{-1}\cot\theta(t)dt$ for a function $\theta(s) \in (0, \frac{\pi}{2})$.

6. RICCI SOLITONS

A vector field η on a Riemannian manifold (M, g) is said to define a *Ricci soliton* if it satisfies the Ricci soliton equation:

$$\frac{1}{2}\mathcal{L}_\eta g + Ric = \lambda g, \tag{6.1}$$

where $\mathcal{L}_\eta g$ denotes the Lie-derivative of the metric tensor g with respect to ξ, Ric is the Ricci tensor of (M, g) and λ is a constant (see, e.g., [34,38,46]).

We denote a Ricci soliton by (M, g, η, λ). The vector field η is called the *potential field*. A Ricci soliton (M, g, η, λ) is said to be *shrinking, steady* or *expanding* according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. A *trivial Ricci soliton* is one for which the potential field ξ is zero or Killing, in which case the metric is Einsteinian.

Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Further, Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions (cf. [75]).

A Ricci soliton (M, g, η, λ) is called *gradient* if its potential field η is the gradient of some function f on M. For a gradient Ricci soliton the soliton equation can be expressed as

$$Ric_f = \lambda g, \tag{6.2}$$

where

$$Ric_f := Ric + Hess(f) \tag{6.3}$$

is known as the Bakry–Émery curvature, where $Hess(f)$ denotes the Hessian of f. Hence a gradient Ricci soliton has constant Bakry–Émery curvature, a similar role as an Einstein manifold.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincaré conjecture.

Clearly, the most natural tangent vector field on a Euclidean submanifold M is the tangential component $\mathbf{x}^T$ of the position vector field $\mathbf{x}$ of M. In this section, we discuss a *Ricci soliton whose potential field is the tangential component* $\mathbf{x}^T$ *of the position vector field*.

In this respect, we have the following results of the author and S. Deshmukh.

Proposition 6.1. *([39]) If $(M^n, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$, then M^n has at most two distinct principal curvatures given by*

$$\kappa_1, \kappa_2 = \frac{n\alpha + \rho \pm \sqrt{(n\alpha + \rho)^2 + 4 - 4\lambda}}{2}, \tag{6.4}$$

where α is the mean curvature and ρ is the support function of M^n, i.e., $\rho = \langle \mathbf{x}, N\rangle$ and $H = \alpha N$ with N being a unit normal vector field.

Theorem 6.1 (*[39]*). *Let $(M^n, g, \mathbf{x}^T, \lambda)$ be a shrinking Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$ with $\lambda = 1$. Then M^n is an open portion of one of the following hypersurfaces of $\mathbb{E}^{n+1}$:*

(1) *A hyperplane through the origin.*
(2) *A hypersphere centered at the origin.*
(3) *A flat hypersurface generated by lines through the origin of $\mathbb{E}^{n+1}$.*
(4) *A spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}$, $2 \leq k \leq n-1$.*

By applying 6.1 the next theorem was proved.

Theorem 6.2 (*[37]*). *Let $(M^n, g, \mathbf{x}^T, \lambda)$ be a Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$. Then M^n is one of the following hypersurfaces of $\mathbb{E}^{n+1}$:*

(1) *A hyperplane through the origin.*
(2) *A hypersphere centered at the origin.*
(3) *An open part of a flat hypersurface generated by lines through the origin.*
(4) *An open part of a circular hypercylinder $S^1(r) \times \mathbb{E}^{n-1}$, $r > 0$.*
(5) *An open part of a spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}$, $2 \leq k \leq n-1$.*

For further results in this respect, see [4].

7. Self-shrinkers in mean curvature flow

Finally, we discuss self-shrinkers in mean curvature flow. Self-shrinkers are also closely related with the position vector field of a Euclidean submanifold. The study of self-shrinkers has become a very active research topic in recent years.

Let us consider the mean curvature flow for an isometric immersion $x{:}M \to \mathbb{E}^m$, i.e., consider a one-parameter family $x_t = x(\,\cdot\,, t)$ of immersions $x_t{:}M \to \mathbb{E}^m$ such that

$$\frac{d}{dt}x(p,t) = H(p,t), \quad x(p,0) = x(p), \quad p \in M, \tag{7.1}$$

is satisfied, where $H(p,t)$ is the mean curvature vector of M_t in $\mathbb{E}^m$ at $x(p,t)$.

An important class of solutions to the mean curvature flow equations is *self-similar shrinkers* which satisfy a system of quasi-linear elliptic PDEs of the second order, namely,

$$H = -\mathbf{x}^N,$$

where $\mathbf{x}^N$ is the normal component of the position vector field of $x{:}M \to \mathbb{E}^m$ as before. Self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a given singularity of a mean curvature flow.

The following are some known results on self-shrinkers.

U. Abresch and J. Langer classified in [1] all smooth closed self-shrinker curves in $\mathbb{E}^2$. They showed that circles are the only embedded self-shrinkers in $\mathbb{E}^2$. G. Huisken studied compact self-shrinkers in [58]. He proved that if a compact self-shrinker hypersurface in $\mathbb{E}^{n+1}$ has non-negative mean curvature $\|H\|$, then it is a hypersphere $S^n(\sqrt{n})$ with radius $\sqrt{n}$.

Compact embedded self-shrinker $S^1 \times S^{n-1}(\sqrt{n-1})$ in $\mathbb{E}^{n+1}$ was constructed by S.B. Angenent in [5]. A. Kleene and N.M. Moller proved in [62] that a complete embedded self-shrinking hypersurface of revolution in $\mathbb{E}^{n+1}$ is isometric to $\mathbb{E}^n$, $S^n(\sqrt{n})$, $\mathbf{R} \times S^{n-1}(\sqrt{n-1})$, or $S^1 \times S^{n-1}(\sqrt{n-1})$.

Let M be a complete Riemannian manifold and $p \in M$. Then M is said to have at most *polynomial volume growth* if there exists a nonnegative integer s such that $\mathrm{vol}(B_\rho(p)) \leq C\rho^s$, where $B_\rho(p)$ is the geodesic ball centered at p with radius ρ and C is a positive constant independent of ρ.

N.Q. Le and N. Sesum proved in [66] that if M is a complete embedded self-shrinker hypersurface in $\mathbb{E}^{n+1}$ with polynomial volume growth and $\|h\| < 1$, then $h = 0$; and thus M is isometric to the hyperplane, where h denotes the second fundamental form.

H.-D. Cao and H. Li proved in [10] that if a complete self-shrinker hypersurface in $\mathbb{E}^{n+1}$ has polynomial volume growth and $\|h\| < 1$, then it is isometric to either the hyperplane $\mathbb{E}^n$, the hypersphere $S^n(\sqrt{n})$, or a hypercylinder $S^k(\sqrt{k}) \times \mathbb{E}^{n-k}$ with $1 \leq k \leq n-1$. In [45], Q.-M. Cheng and G. Wei improved Cao and Li's result by showing that the same result also holds when the condition $\|h\| < 1$ is replaced by $\|h\|^2 \leq 1 + \frac{3}{7}$.

In recent years, there also exist many articles studying self-shrinkers in the mean curvature flow in arbitrary codimension (see, for instance [6,10,11,49–51,67,68,88] among others).

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