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Tight wave packet frames for $L^2(\mathbb{R})$ and $\mathcal{H}^2(\mathbb{R})$

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Abstract. In this paper we establish a characterization of tight wave packet frames for $L^2(\mathbb{R})$ and we also prove that it is possible to construct frames in $\mathcal{H}^2(\mathbb{R})$ which are given by dilation, translation and modulation of a single function ψ, where ψ, as well as $\hat{\psi}$, belongs to the Schwartz class.

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1. Introduction

Wave packet systems are countable collections of dilations, translations and modulations of a single function $\psi \in L^2(\mathbb{R})$. In [4], Cordoba and Fefferman introduced this form of wave packet system. Wave packet systems have been considered and extended by several authors, see [1,3,5,7–9]. Czaja, Kutyniok and Speegle proved that certain geometric conditions on the set of parameters in a wave packet system are necessary in order for the system to form a frame.

The classical Hardy space $\mathcal{H}^2(\mathbb{R})$ is the collection of all square integrable functions whose Fourier transform is supported in $\mathbb{R}^+ = (0, \infty)$:

$$\mathcal{H}^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi \leqslant 0\},$$

where $\hat{f}$ is the Fourier transform of f defined by

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$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx.$$

In the present paper, we consider wave packet systems as special cases of generalized shift-invariant systems, a concept studied by Ron and Shen in [10]. We established a complete characterization of tight wave packet frames for $L^2(\mathbb{R})$ and we also proved that it is possible to construct wave packet frames in $\mathcal{H}^2(\mathbb{R})$ [6].

2. Preliminaries

For $b \in \mathbb{R}$, the translation operator T_b on $L^2(\mathbb{R})$ (or $\mathcal{H}^2(\mathbb{R})$) is defined by

$$(T_bf)(x) = f(x-b),\ \ x \in \mathbb{R}.$$

For $c \in \mathbb{R}$, the modulation operator E_c is defined by

$$(E_cf)(x) = e^{2\pi icx}f(x),\ \ x \in \mathbb{R}.$$

The dilation operator associated with a non-negative $a > 0$ is

$$(D_af)(x) = \sqrt{a}f(ax),\ \ x \in \mathbb{R}.$$

The Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx$$

and the inverse Fourier transform is

$$\check{f}(x) = (\mathcal{F}^{-1}f)(x) = \int_{\mathbb{R}} f(\xi)e^{2\pi ix\xi}d\xi.$$

The Plancherel theorem asserts that

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle.$$

For $a > 0, b, c \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ (or $\mathcal{H}^2(\mathbb{R})$)

$$(D_{a^j}f) = D_{a^{-j}}\hat{f}$$

$$(E_bf) = T_b\hat{f}$$

$$(T_cf) = E_{-c}\hat{f}$$

and

$$(D_{a^j}T_{kb}E_{mc}f) = D_{a^{-j}}E_{-kb}T_{mc}\hat{f}.$$

The inner product of functions $f, g \in L^2(\mathbb{R})$ is

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \overline{f(x)}g(x)dx$$

where

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2dx\right)^{\frac{1}{2}}$$

Definition 2.1 [2]. A countable family $\{e_\alpha : \alpha \in \mathcal{A}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a frame if there exist constants A,B with $0 < A \leqslant B < \infty$ satisfying

$$A\|v\|^2 \leqslant \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \leqslant B\|v\|^2. \tag{2.1}$$

for all $v \in \mathcal{H}$. If only the right hand side inequality holds, we say that $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a Bessel system with constant B. A frame is a tight frame if A and B can be chosen so that $A = B$ and is a normalized tight frame (NTF) if $A = B = 1$. Thus, if $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a NTF in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2. \tag{2.2}$$

The concept of generalized shift-invariant system was introduced by Ron and Shen in [10].

Definition 2.2. A generalized shift-invariant system is a system of the form $\{T_{c_j k}\psi_j\}_{j\in\tau, k\in\mathbb{Z}^d}$, where τ is a countable collection of indices, $\{\psi_j\}_{j\in\mathbb{Z}} \subset L^2(\mathbb{R})$ and $\{c_j\}_{j\in\tau}$ is a collection of non-negative numbers.

Definition 2.3. Let $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+, b \neq 0, \{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ and $\psi \in L^2(\mathbb{R})$. A system of the form $\{D_{a_j} T_{bk} E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is called an irregular wave packet system.

The wave packet system $\{D_{a_j} T_{bk} E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is said to be the frame of $L^2(\mathbb{R})$ if there exist two positive constants A and B with $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, D_{a_j} T_{bk} E_{c_m}\psi\rangle|^2 \leqslant B\|f\|^2, \quad f \in L^2(\mathbb{R}). \tag{2.3}$$

The wave packet system $\{D_{a_j} T_{bk} E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is said to be a tight frame of $L^2(\mathbb{R})$ with frame bound 1, if

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, D_{a_j} T_{bk} E_{c_m}\psi\rangle|^2 = \|f\|^2, \quad f \in L^2(\mathbb{R}). \tag{2.4}$$

3. Tight wave packet frames for $L^2(R)$

Let $\psi, \tilde{\psi} \in L^2(\mathbb{R})$. For the wave packet systems $\{D_{a_j} T_{bk} E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j} T_{bk} E_{c_m}\tilde{\psi}\}_{j,k,m\in\mathbb{Z}}$, we consider the bilinear functional

$$P(f, g) = \sum_{j,m,k\in\mathbb{Z}} \langle f, D_{a_j} T_{bk} E_{c_m}\psi\rangle \langle D_{a_j} T_{bk} E_{c_m}\tilde{\psi}, g\rangle, \quad f, g \in L^2(\mathbb{R}) \tag{3.1}$$

on $L^2 \times L^2$, where the weak convergence of the bi-infinite series is defined by

$$\lim_{m_1,m_2\to\infty}\lim_{j_1,j_2\to\infty}\lim_{k_1,k_2\to\infty} \sum_{m=-m_1}^{m_2}\sum_{j=-j_1}^{j_2}\sum_{k=-k_1}^{k_2} \langle f, D_{a_j} T_{bk} E_{c_m}\psi\rangle \langle D_{a_j} T_{bk} E_{c_m}\tilde{\psi}, g\rangle. \tag{3.2}$$

Then, $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j}T_{bk}E_{c_m}\tilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ forms a dual pair if the series in (3.1) converges in the sense of (3.2) and satisfies

$$P(f,g) = \langle f,g\rangle, \quad f,g \in L^2(\mathbb{R}). \tag{3.3}$$

If both $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a_j}T_{bk}E_{c_m}\tilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ are Bessel sequences, then it is clear that the series (3.1) is absolutely convergent. We need the following notations

$$L_\psi(\xi) = \sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\right|^2 \quad \text{and} \quad L_{\tilde{\psi}}(\xi) = \sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left|\hat{\tilde{\psi}}\left(a_j^{-1}\xi - c_m\right)\right|^2. \tag{3.4}$$

Let

$$\mathcal{D} = \{f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}) \text{ and supp } \hat{f} \text{ is compact in } \mathbb{R}\}.$$

For a generalized shift-invariant system $\{T_{c_jk}g_j : j \in \tau, k \in \mathbb{Z}\}$, if

$$L(f) = \sum_{j\in\tau}\sum_{m\in\mathbb{Z}}\int_{\text{supp}\hat{f}}\left|\hat{f}\left(\xi + c_j^{-1}m\right)\right|^2\frac{1}{c_j}|\hat{g}_j(\xi)|^2 d\xi < \infty, \quad \text{for all } f \in \mathcal{D},$$

we say that $\{T_{c_jk}g_j : j \in \tau, k \in \mathbb{Z}\}$ satisfies the Local Integrability Condition (LIC).

$L^\infty(\mathbb{R})$ is the space of essentially bounded measurable functions $f : \mathbb{R} \to \mathbb{C}$, equipped with the supremums-norm.

Lemma 3.1. 1*Let* $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+, b > 0, \{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ *and* $\psi, \tilde{\psi} \in L^2(\mathbb{R})$. *Also assume that the wave packet systems* $\{D_{a_j}T_{kb}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a_j}T_{kb}E_{c_m}\tilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ *are Bessel sequences and*

$$L(f) = \frac{1}{b}\sum_{j,m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\text{supp}\hat{f}}\left|\hat{f}\left(\xi + \frac{a_j}{b}n\right)\right|^2\left|\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi < \infty,$$

and

$$L'(f) = \frac{1}{b}\sum_{j,m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\text{supp}\hat{f}}\left|\hat{f}\left(\xi + \frac{a_j}{b}n\right)\right|^2\left|\hat{\tilde{\psi}}\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi < \infty,$$

for all $f \in \mathcal{D}$. *Then,* $\{D_{a_j}T_{kb}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a_j}T_{kb}E_{c_m}\tilde{\psi}\}_{j,k,m\in\mathbb{Z}}$ *form a dual pair if and only if*

$$\frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\in\tau_\alpha}\overline{\hat{\psi}\left(a_j^{-1}\xi - c_m\right)}\hat{\tilde{\psi}}\left(a_j^{-1}(\xi + \alpha) - c_m\right) = \delta_{\alpha,0} \tag{3.5}$$

for a.e. $\xi \in \mathbb{R}, \alpha \in \Lambda$, *where*

$$\Lambda = \cup_{j\in\mathbb{Z}}a_jb^{-1}\mathbb{Z}, \quad \tau_\alpha = \{j \in \mathbb{Z} : a_j^{-1}b\alpha \in \mathbb{Z}\}.$$

Theorem 3.2. *Let* $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+, b > 0, \{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ *and* $\psi \in L^2(\mathbb{R})$. *Assume the LIC*

$$L(f) = \frac{1}{b}\sum_{j,m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\text{supp}\hat{f}}\left|\hat{f}\left(\xi + \frac{a_j}{b}n\right)\right|^2\left|\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi < \infty,$$

for all $f \in \mathcal{D}$. Then, the wave packet systems $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ are a tight frame of $L^2(\mathbb{R})$ if and only if ψ satisfies

$$\frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\in\tau_\alpha}\overline{\hat{\psi}\left(a_j^{-1}\xi - c_m\right)}\hat{\psi}\left(a_j^{-1}(\xi+\alpha) - c_m\right) = \delta_{\alpha,0} \tag{3.6}$$

for a.e. $\xi \in \mathbb{R}, \alpha \in \Lambda$, where

$$\Lambda = \cup_{j\in\mathbb{Z}} a_j b^{-1}\mathbb{Z}, \quad \tau_\alpha = \{j \in \mathbb{Z} : a_j^{-1} b\alpha \in \mathbb{Z}\}.$$

Proof. Let $\tilde{\psi} = \psi$. If $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$, then from (3.1), we have

$$P(f,f) = \sum_{j,m,k\in\mathbb{Z}}\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle\langle D_{a_j}T_{bk}E_{c_m}\tilde{\psi}, f\rangle, \quad \forall f \in L^2(\mathbb{R})$$

Now using $\tilde{\psi} = \psi$ and Eq. (2.4); we have

$$\begin{aligned} P(f,f) &= \sum_{j,m,k\in\mathbb{Z}}\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle\langle D_{a_j}T_{bk}E_{c_m}\psi, f\rangle, \\ &= \sum_{j,m,k\in\mathbb{Z}}\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle\overline{\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle}, = \sum_{j,m,k\in\mathbb{Z}}\left|\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle\right|^2 = \|f\|^2, \end{aligned}$$

for all $f \in L^2(\mathbb{R})$. Consequently, it follows that $L_\psi \in L^\infty$ and $P(f,g) = \langle f,g\rangle$ for all $f, g \in L^2(\mathbb{R})$. Thus, by Lemma 3.1, Eq. (3.5) holds for $\tilde{\psi} = \psi$, i.e. Eq. (3.6) holds.

To establish the converse, we first observe that

$$P(f,f) = \frac{1}{b}\sum_{m,j,s\in\mathbb{Z}}\int_{-\infty}^{\infty}\overline{\hat{f}(\xi)}\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\hat{f}\left(\xi + \frac{a_j}{b}s\right)\overline{\hat{\psi}\left(a_j^{-1}\xi - c_m + \frac{s}{b}\right)}d\xi,$$

where the series certainly converges for $f \in \mathcal{D}$, provided that $L_\psi \in L^\infty$. Indeed, for $\operatorname{supp}\hat{f} \subset [-H, H]$, there exists an $J \in \mathbb{Z}$, such that

$$\begin{aligned} P(f,f) &= \frac{1}{b}\left|\sum_{m\in\mathbb{Z}}\sum_{j\leqslant J}\sum_{|s|\leqslant S_j}\int_{-H}^{H}\overline{\hat{f}(\xi)}\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\hat{f}\left(\xi + \frac{a_j}{b}s\right) \times \overline{\hat{\psi}\left(a_j^{-1}\xi - c_m + \frac{s}{b}\right)}d\xi\right| \\ &\leqslant \frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\leqslant J}\sum_{|s|\leqslant S_j}\left(\int_{-H}^{H}\left|\overline{\hat{f}(\xi)}\hat{f}\left(\xi + \frac{a_j}{b}s\right)\overline{\hat{\psi}\left(a_j^{-1}\xi - c_m\right)}\right|^2 d\xi\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{-H(1+b)}^{H(1+b)}\left|\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi\right)^{\frac{1}{2}} \\ &\leqslant \text{const}\sum_{m\in\mathbb{Z}}\sum_{j\leqslant J}a_j^{-1}\|\hat{f}\|_\infty\left(\int_{-H(1+b)}^{H(1+b)}\left|\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi\right)^{\frac{1}{2}} \\ &\leqslant \text{const}\|\hat{f}\|_\infty\|\hat{\psi}\|, \quad \text{where } S_j \leqslant \frac{bH}{a_j}. \end{aligned}$$

Let

$$\Lambda = \cup_{j\in\mathbb{Z}} a_j b^{-1}\mathbb{Z}, \quad \tau_\alpha = \{j \in \mathbb{Z} : a_j^{-1} b\alpha \in \mathbb{Z}\}.$$

Using this notation, we can rewrite $P(f,f)$ as

$$P(f,f) = \frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{\alpha\in\Lambda}\int_{-\infty}^{\infty}\overline{\hat{f}(\xi)}\hat{f}(\xi+\alpha) \times \left[\sum_{j\in\tau_\alpha}\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\hat{\psi}\left(a_j^{-1}(\xi+\alpha) - c_m\right)\right]d\xi. \tag{3.7}$$

Now, suppose that Eq. (3.6) is valid. Then, on the one hand, we have

$$\frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}\left(a_j^{-1}\xi - c_m\right)}\hat{\psi}\left(a_j^{-1}\xi - c_m\right) = \frac{1}{b}\sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(a_j^{-1}\xi - c_m\right)\right|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$. Now, using Eq. (3.4), we have $L_\psi(\xi) = b$ a.e. $\xi \in \mathbb{R}$, so that $L_\psi \in L^\infty$; and on the other hand, Eq. (3.7) can be simplified to

$$P(f,f) = \int_{-\infty}^{\infty}\overline{\hat{f}(\xi)}\hat{f}(\xi)d\xi = \|f\|^2 \tag{3.8}$$

for all $f \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $L^2(\mathbb{R})$, Eq. (3.8) holds for all $L^2(\mathbb{R})$. That is $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$. This completes the proof of Theorem 3.2. □

4. Tight wave packet frames for $\mathcal{H}^2(\boldsymbol{R})$

For $\varepsilon > 0$, let s_ε be a smooth function (say $C^r, r = 0, 1, 2, \ldots,$ or C^∞) such that $s_\varepsilon(x) = 0$ if $x < \varepsilon$ and

$$s_\varepsilon^2(x) + d_\varepsilon^2(x) = 1, \tag{4.1}$$

where $d_\varepsilon(x) = s_\varepsilon(-x)$. For $0 < \varepsilon \leqslant \frac{1}{3}\pi$, let

$$b_\varepsilon(x) = s_\varepsilon(x-\pi)d_{2\varepsilon}(x-2\pi) \tag{4.2}$$

be a bell function associated with the interval $[\pi, 2\pi]$. Using Eqs. (4.1) and (4.2); we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|b_\varepsilon\left(a_j^{-1}\xi - c_m\right)\right|^2 = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|s_\varepsilon^2\left(a_j^{-1}\xi-\pi\right)d_{2\varepsilon}^2\left(a_j^{-1}\xi - 2\pi\right)\right| = \begin{cases} 0 & \text{if } a_j^{-1}\xi - \pi < \varepsilon \Rightarrow \xi < (\pi+\varepsilon)a_j \\ 0 & \text{if } a_j^{-1}\xi - 2\pi > -2\varepsilon \Rightarrow \xi > (2\pi - 2\varepsilon)a_j \\ 1 & \text{if } (\pi+\varepsilon)a_j < \xi < (2\pi-2\varepsilon)a_j. \end{cases} \tag{4.3}$$

Define ψ^ε by

$$\hat{\psi}^\varepsilon(\xi) = b_\varepsilon(\xi), \quad \xi \in \mathbb{R}. \tag{4.4}$$

It is clear that ψ^ε belongs to $\mathcal{H}^2(\mathbb{R})$. Moreover, using Eq. (4.3), we deduce

$$\sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^\varepsilon\left(a_j^{-1}\xi - c_m\right)\right|^2 = \chi_{(0,\infty)}(\xi), \quad \xi \in \mathbb{R}. \tag{4.5}$$

Theorem 4.1. *For every* ε *such that* $0 < \varepsilon \leqslant \frac{1}{3}\pi$, *the wave packet system* $\{D_{a_j}T_{bk}E_{c_m}\psi^\varepsilon\}_{j,m,k\in\mathbb{Z}}$, *where* ψ^ε *is given by* Eq. (4.4), *is a tight frame for* $\mathcal{H}^2(\mathbb{R})$ *with frame bound 1. Moreover,* ψ^ε *is in the Schwartz class if we choose* $s_\varepsilon \in C^\infty(\mathbb{R})$.

Proof. Assume that $f \in \mathcal{H}^2(\mathbb{R})$. Using Plancherel's theorem, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle f, D_{a_j}T_{bk}E_{c_m}\psi^\varepsilon\rangle|^2 &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle \mathcal{F}f, \mathcal{F}D_{a_j}T_{bk}E_{c_m}\psi^\varepsilon\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\left\langle \hat{f}, D_{a_j^{-1}}E_{-bk}T_{c_m}\hat{\psi}^\varepsilon\right\rangle\right|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle \hat{f}, E_{-kba_j}D_{a_j^{-1}}T_{c_m}\hat{\psi}^\varepsilon\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}a_j^{-1}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\int_0^\infty \hat{f}(\xi)\overline{\hat{\psi}^\varepsilon\left(a_j^{-1}\xi - c_m\right)}e^{2\pi ikba_j\xi}d\xi\right|^2 \\ &= \sum_{j\in\mathbb{Z}}a_j - 1\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\int_{a_j(\pi+\varepsilon)}^{a_j(2\pi-2\varepsilon)} \hat{f}(\xi)b_\varepsilon\left(a_j^{-1}\xi - c_m\right)e^{2\pi ikba_j\xi}d\xi\right|^2\end{aligned} \tag{4.6}$$

The sum over k in this last expression is the sum of the squares of the Fourier coefficients of the function $a_j\hat{f}(a_j\cdot + c_m)b_\varepsilon(\cdot)$ over the interval $\left[\frac{2\pi}{3}, \frac{8\pi}{3}\right]$. Thus, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\left\langle f, D_{a_j}T_{bk}E_{c_m}\psi^\varepsilon\right\rangle\right|^2 &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_{a_j(\pi+\varepsilon)}^{a_j(2\pi-2\varepsilon)}|\hat{f}(\xi)|^2\left|b_\varepsilon\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi \\ &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_0^\infty |\hat{f}(\xi)|^2\left|(\psi^\varepsilon)\left(a_j^{-1}\xi - c_m\right)\right|^2 d\xi.\end{aligned} \tag{4.7}$$

Using Eq. (4.5), we obtain

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\left\langle f, D_{a_j}T_{bk}E_{c_m}\psi^\varepsilon\right\rangle\right|^2 = \int_0^\infty |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{\mathcal{H}^2(\mathbb{R})}. \quad \square$$

Remark 4.2. Since $\|\psi^\varepsilon\|^2_{\mathcal{H}^2(\mathbb{R})} = \|(\psi^\varepsilon)\|_2^2 = \pi$. Thus, if $0 < \varepsilon \leqslant \frac{1}{3}\pi, \|\psi^\varepsilon\|_{\mathcal{H}^2(\mathbb{R})} = \sqrt{\pi} > 1$, and hence we have a family of frames in $\mathcal{H}^2(\mathbb{R})$ that do not form an orthonormal basis.

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