Arab Journal of Mathematical Sciences

Arab J Math Sci 21 (2015) 209–224

Original article

Tian invariant on generalized Calabi manifold

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Received 28 June 2013; received in revised form 25 January 2014; accepted 2 April 2014
Available online 20 June 2014

Abstract. In this article we calculate the Tian invariant on some Fano manifolds. These manifolds generalize those introduced by the first author in collaboration with Pascal Cherrier, in Ben Abdesselem and Cherrier (2009 [1]). The method used is to determine explicitly the lower bound for almost psh functions.

Keywords: Tian invariant; Calabi manifold; Positive first Chern class

2010 Mathematics Subject Classification: 53D45; 14J32

1. Introduction and statement of results

We give an explicit lower bound (as in [1] and [5]) for almost psh functions on Y, the sub-manifold of $\mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C}$ (where $m \geq 2$ and $n > 0$), consisting of the points

$$([Z], [z_m, z_{m+1}Z^{a_1}, \ldots, z_{m+n}Z^{a_n}]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C},$$

where the a_i are positive integers for all $i \in \{1, \ldots, n\}$ (verify $m - \sum_{i=1}^{n} a_i > 0$), $Z = [z_0, z_1, \ldots, z_{m-1}] \in \mathbb{P}_{m-1}\mathbb{C}$, $[z_m, z_{m+1}, \ldots, z_{m+n}] \in \mathbb{P}_n\mathbb{C}$ and $Z^{a_i} = [z_0^{a_i}, z_1^{a_i}, \ldots, z_{m-1}^{a_i}]$. Note that dimension of Y is $m + n - 1$, and that, in the above description, the point $[z_m, z_{m+1}, \ldots, z_{m+n}]$ of $\mathbb{P}_n\mathbb{C}$ depends on the choice of the coordinates $(z_0, z_1, \ldots, z_{m-1})$ of the base point $[Z]$. Indeed, the map $Y \longrightarrow P_n\mathbb{C}$, given by

$$([Z], [z_m, z_{m+1}Z^{a_1}, \ldots, z_{m+n}Z^{a_n}]) \longrightarrow [z_m, z_{m+1}, \ldots, z_{m+n}],$$

is not intrinsically defined, even if it can be defined locally after the choice of an affine chart.

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2014.04.001

An equivalent description is the following:

$$\begin{aligned}Y = \{&([z_0, z_1, \ldots, z_{m-1}], [z_m; z_{m+1}, \ldots, z_{2m}; \ldots;\\ &z_{nm+1}, \ldots, z_{(n+1)m}]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C} \text{ s.t. } \forall j \in \{1, ., n\},\\ &(z_{jm+1}, \ldots, z_{(j+1)m}) \text{ and } (z_0^{a_j}, z_1^{a_j}, \ldots, z_{m-1}^{a_j}) \text{ are collinear in } \mathbb{C}^m\}.\end{aligned}$$

Now we introduce two other coordinate systems, which will be more convenient for our later computations. Denote S the first one when all components are not zero. In this coordinate system, we have $z_0 = 1$ and S is given by

$$\begin{aligned}&([z_1, \ldots, z_m], [1; z_1^{a_1}, \ldots, z_m^{a_1}, z_{m+1}(z_1^{a_2}, \ldots, z_m^{a_2}); \ldots;\\ &\quad z_{m+n-1}(z_1^{a_n}, \ldots, z_m^{a_n})]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C}.\end{aligned}$$

The second coordinate system, which we denote S', is given, in the local chart $\{z_0 \neq 0,\ z_m \neq 0\}$, by

$$([1, z_1, \ldots, z_{m-1}], [1; z_{m+1}(1, z_1^{a_1}, \ldots, z_{m-1}^{a_1}), \ldots, z_{m+n}(1, z_1^{a_n}, \ldots, z_{m-1}^{a_n})]) \in Y$$

when we use the description

$$\begin{aligned}&([z_0, z_1, \ldots, z_{m-1}], [z_m; z_{m+1}(z_0^{a_1}, z_1^{a_1}, \ldots, z_{m-1}^{a_1}); \ldots;\\ &\quad z_{m+n}(z_0^{a_n}, z_1^{a_n}, \ldots, z_{m-1}^{a_n})]) \in Y.\end{aligned}$$

Thus, in order to make our proofs more readable, sometimes we shall work in S and sometimes in S'.

Now, we endow $\mathbb{P}_k\mathbb{C}$ with the Fubini Study metric g_k whose components, in the chart $\{[z_0, z_1, \ldots, z_k] \in \mathbb{P}_k\mathbb{C} \text{ s.t. } z_0 \neq 0\}$, are given by

$$g_{\lambda\overline{\mu}} = \partial_{\lambda\bar{\mu}} \ln(1 + x_1 + \cdots + x_k)$$

where $x_i = |z_i|^2$ and $\partial_{\lambda\bar{\mu}} = \frac{\partial^2}{\partial z_\lambda \partial \bar{z}_\mu}$. Thus, we consider the projections π_1 and π_2 of Y respectively on $\mathbb{P}_{m-1}\mathbb{C}$ and $\mathbb{P}_{mn}\mathbb{C}$, and define the metric g on Y by

$$g = \alpha\pi_1^* g_{m-1} + \beta\pi_2^* g_{mn}$$

whose components in the local chart S' are given by:

$$\begin{aligned}g_{\lambda\overline{\mu}} = &\alpha\partial_{\lambda\bar{\mu}} \ln(1 + x_1 + \cdots + x_{m-1}) + \beta\partial_{\lambda\bar{\mu}} \ln\{1 + x_{m+1}(1 + x_1^{a_1} + \cdots + x_{m-1}^{a_1})\\ &+ \cdots + x_{m+n}(1 + x_1^{a_n} + \cdots + x_{m-1}^{a_n})\},\end{aligned}$$

where $x_i = |z_i|^2$ and $\lambda, \mu = 1, \ldots, m-1, m+1, \ldots, m+n$. In the coordinate system S, its components are given by

$$\begin{aligned}g_{\lambda\overline{\mu}} = &\alpha\partial_{\lambda\bar{\mu}} \ln(x_1 + \cdots + x_m) + \beta\partial_{\lambda\bar{\mu}} \ln\{1 + (x_1^{a_1} + \cdots + x_m^{a_1})\\ &+ x_{m+1}(x_1^{a_2} + \cdots + x_m^{a_2}) + \cdots + x_{m+n}(x_1^{a_n} + \cdots + x_m^{a_n})\}.\end{aligned}$$

We prove that for $\alpha = m - \sum_{i=1}^n a_i$ and $\beta = n + 1$, the metric g belongs to the first Chern class $C_1(Y)$ and consequently Y is Fano. In fact, Our goal is to find a condition on α and β such that the quantity

$$\begin{aligned}F_{0,m} = &(1 + |z_1|^2 + \cdots + |z_{m-1}|^2)^\alpha\{1 + |z_{m+1}|^2(|z_1|^{2a_1} + \cdots + |z_{m-1}|^{2a_1})\\ &+ \cdots + |z_{m+n}|^2(|z_1|^{2a_n} + \cdots + |z_{m-1}|^{2a_n})\}^\beta,\end{aligned}$$

written in the local chart $\{z_0 \neq 0,\ z_m \neq 0\}$, is a metric on the line bundle $\Lambda^{m+n-1}T^*Y$. Then, its Ricci will be exactly the metric g and will, by definition, belong to $c_1(Y)$ so that Y will be Fano.

The first change of charts we consider is

$$\begin{aligned}&\varphi_1(z_1, \ldots, z_{m-1}; z_{m+1}, \ldots, z_{m+n})\\ &\quad = \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{m-1}}{z_1}; z_{m+1}z_1^{a_1}, \ldots, z_{m+1}z_1^{a_1}; \ldots; z_{m+n}z_1^{a_n}, \ldots, z_{m+n}z_1^{a_n}\right)\end{aligned}$$

its Jacobian J_1 verifies

$$|J_1|^2 = \frac{1}{|z_1|^{2(m-(a_1+\cdots+a_n))}}.$$

In the new chart, the expression of $F_{0,m}$ becomes

$$F_{1,m} = \frac{1}{|z_1|^{2\alpha}} F_{0,m},$$

and the condition $\alpha = m - (a_1 + \cdots + a_n)$ is proven.

Now, let us consider the change of charts

$$\varphi_2 : (z_1, \ldots, z_{m-1}; z_{m+1}, \ldots, z_{m+n}) = \left(z_1, \ldots, z_{m-1}; \frac{1}{z_{m+1}}, \frac{z_{m+2}}{z_{m+1}}, \ldots, \frac{z_{m+n}}{z_{m+1}}\right).$$

Its Jacobian J_2 verifies

$$|J_2|^2 = \frac{1}{|z_{m+1}|^{2(n+1)}}$$

and $F_{0,m+1}$ becomes

$$F_{0,m+1} = \frac{1}{|z_{m+1}|^{2\beta}} F_{0,m}.$$

which proves the second condition $\beta = n + 1$.

Now we consider the automorphisms group G_{m-1} on $\mathbb{P}_{m-1}\mathbb{C}$ spanned by the automorphisms $\sigma_{i,j}$ and $\tau_{l,\theta}$ defined $\forall i, j \in \{0, 1, \ldots, m-1\}$, $l \in \{0, \ldots, m-1\}$ and $\theta \in [0, 2\pi]$ by

$$\sigma_{i,j}([z_0, \ldots, z_i, \ldots, z_j, \ldots, z_{m-1}]) = [z_0, \ldots, z_j, \ldots, z_i, \ldots, z_{m-1}]$$

and

$$\tau_{l,\theta}([z_0, \ldots, z_l, \ldots, z_{m-1}]) = [z_0, \ldots, z_l e^{i\theta}, \ldots, z_{m-1}].$$

On $\mathbb{P}_{mn}\mathbb{C}$, we define another automorphisms group G_{nm} spanned by

1. for $\theta \in [0, 2\pi]$,and $l \in \{0, \ldots, n\}$,
$$\begin{aligned}&\tau'_{l,\theta}([z_m, z_{m+1}Z^{a_1}, \ldots, z_{m+l}Z^{a_l}, \ldots, z_{m+n}Z^{a_n}])\\ &\quad = ([z_m, z_{m+1}Z^{a_1}, \ldots, z_{m+l}e^{i\theta}Z^{a_l}, \ldots, z_{m+n}Z^{a_n}]),\end{aligned}$$
where $Z^{a_i} = (z_0^{a_i}, \ldots, z_{m-1}^{a_i}) \in \mathbb{C}^m$.
2. The action of the above defined automorphisms $\sigma_{i,j}$ and $\tau_{l,\theta}$ of G_{m-1} on $Z = (z_0, \ldots, z_{m-1}) \in \mathbb{C}^m$. Z is given by the description:
$$([Z], [z_m, z_{m+1}Z^{a_1}, \ldots, z_{m+n}Z^{a_n}]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C}.$$

The groups G_{m-1} and G_{nm} generate a natural automorphisms group G on Y, which we use later on.

Let us consider the functions

$$\psi_0 = \ln\left\{ \frac{(|z_0^{(0)}|\cdots|z_{m-1}^{(0)}|)^{\frac{2(m-(a_1+\cdots+a_n))}{m}}}{(|z_0^{(0)}|^2+\cdots+|z_{m-1}^{(0)}|^2)^{m-(a_1+\cdots+a_n)}} \right. \\ \left. \times \frac{|z|^{2(n+1)}}{[|z|^2+(|z_1^{(1)}|^2+\cdots+|z_m^{(1)}|^2)+\cdots+(|z_1^{(n)}|^2+\cdots+|z_m^{(n)}|^2)]^{(n+1)}} \right\}$$

and $\forall k \in \{1,\ldots,n\}$

$$\psi_k = \ln\left\{ \frac{(|z_0^{(0)}|\cdots|z_{m-1}^{(0)}|)^{\frac{2(m-(a_1+\cdots+a_n))}{m}}}{(|z_0^{(0)}|^2+\cdots+|z_{m-1}^{(0)}|^2)^{m-(a_1+\cdots+a_n)}} \right. \\ \left. \times \frac{[|z_1^{(k)}|\cdots|z_m^{(k)}|]^{2(n+1)/m}}{[|z|^2+(|z_1^{(1)}|^2+\cdots+|z_m^{(1)}|^2)+\cdots+(|z_1^{(n)}|^2+\cdots+|z_m^{(n)}|^2)]^{(n+1)}} \right\}$$

ψ_0 and the ψ_k are functions defined on

$$\left(\mathbb{C}^m \setminus \bigcup_i \{z_i^{(0)}=0\}\right) \times \left(\mathbb{C}^{nm+1} \setminus \bigcup_{j,k} \{z_j^{(k)}=0\}\right)$$

where $z \in \mathbb{C}$ and $(z_i^{(k)})_{0\le i\le m-1}$ are the coordinates on $\mathbb{C}^m$. They are homogeneous of degree zero in the variables of $\mathbb{C}^m$ and $\mathbb{C}^{nm+1}$ separately. Thus, they define $(n+1)$ functions on $\mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C}$, and, by restriction on Y, $(n+1)$ functions on Y given by:

$$\psi_0 = \ln\left\{ \frac{(x_0\cdots x_{m-1})^{\frac{(m-(a_1+\cdots+a_n))}{m}}}{(x_0+\cdots+x_{m-1})^{m-(a_1+\cdots+a_n)}} \right. \\ \left. \times \frac{x_m^{n+1}}{[x_m+x_{m+1}(x_0^{a_1}+\cdots+x_{m-1}^{a_1})+\cdots+x_{m+n}(x_0^{a_n}+\cdots+x_{m-1}^{a_n})]^{(n+1)}} \right\} \tag{1}$$

and, $\forall k \in \{1,\ldots,n\}$

$$\psi_k = \ln\left\{ \frac{(x_0\ldots x_{m-1})^{\frac{(m-(a_1+\cdots+a_n))}{m}}}{(x_0+\cdots+x_{m-1})^{m-(a_1+\cdots+a_n)}} \right. \\ \left. \times \frac{(x_{m+k}x_0^{a_k}\cdots x_{m+k}x_{m-1}^{a_k})^{(n+1)/m}}{[x_m+x_{m+1}(x_0^{a_1}+\cdots+x_{m-1}^{a_1})+\cdots+x_{m+n}(x_0^{a_n}+\cdots+x_{m-1}^{a_n})]^{(n+1)}} \right\}, \tag{2}$$

where $x_i = |z_i|^2$ and the points of Y are described by their homogeneous coordinates, that is:

$$([z_0,\ldots,z_{m-1}],[z_m; z_{m+1}z_0^{a_1},\ldots,z_{m+1}z_{m-1}^{a_1};\ldots;z_{m+n}z_0^{a_n},\ldots,z_{m+n}z_{m-1}^{a_n}]).$$

$\psi = \inf(\psi_0,\psi_1,\ldots,\psi_n)$ is then an extremal function, in the sense of the following result

Theorem 1. *Let $\varphi \in C^\infty(Y)$ be a g-admissible function and G-invariant satisfying* $\sup \varphi = 0$ *on Y, then $\varphi \geq \psi$.*

Let us recall that φ is said to be g-admissible, when the matrix of terms $g_{\lambda\overline{\mu}} + \frac{\partial^2 \varphi}{\partial z^\lambda \partial \overline{z}^\mu}$ is positive definite.

Corollary 1. $\forall \alpha < \frac{1}{n+1}$, $\exists C > 0$ *such that* $\forall \varphi \in \mathcal{C}^\infty(Y)$, *$g$-admissible, G-invariant, satisfying* $\sup \varphi = 0$ *on Y we have:*

$$\int_Y \exp(-\alpha\varphi) dv \leq C.$$

(dv is the volume element on Y with respect to the metric g). See [9,2] *and* [8].

2. Proof of the results

2.1. Proof of Theorem 1

The proof requires two lemmas. In each step, we use the G-invariance of the functions

$$\varphi([z_0, \dots, z_{m-1}], [z_m; z_{m+1}(z_0^{a_1}, \dots, z_{m-1}^{a_1}); \dots; z_{m+n}(z_0^{a_n}, \dots, z_{m-1}^{a_n})]),$$

who can be written with $x_i = |z_i| > 0$ as

$$\varphi([x_0, \dots, x_{m-1}], [x_m; x_{m+1}(x_0^{a_1}, \dots, x_{m-1}^{a_1}); \dots; x_{m+n}(x_0^{a_n}, \dots, x_{m-1}^{a_n})]).$$

Then, in S, we can write the function φ as

$$\varphi([x_1, \dots, x_m], [1; (x_1^{a_1}, \dots, x_m^{a_1}), x_{m+1}(x_1^{a_2}, \dots, x_m^{a_2}); \dots; x_{m+n-1}(x_1^{a_n}, \dots, x_m^{a_n})]).$$

Lemma 1. *Let $\varphi \in C^\infty(Y)$, be a g-admissible G-invariant function. Then, for all $x_i = |z_i| > 0$, we have*

$$\begin{aligned}
&(\varphi - \psi)([x_1, \dots, x_m], [1; (x_1^{a_1}, \dots, x_m^{a_1}); x_{m+1}(x_1^{a_2}, \dots, x_m^{a_2}); \dots; \\
&\quad x_{m+n-1}(x_1^{a_n}, \dots, x_m^{a_n})]) \qquad (3)\\
&\geq (\varphi - \psi)[1^{[m]}], [1; \zeta_1^{[m]}; x_{m+1}\zeta_2^{[m]}; \dots; x_{m+n-1}\zeta_n^{[m]}], \qquad (4)
\end{aligned}$$

where $h^{[m]} = (h, \dots, h) \in \mathbb{C}^m$ et $\zeta_i = (x_1 \dots x_m)^{a_i/m}$.

Proof. We proceed by induction. Assume that, for $1 \leq j < m$ and for all $(x_1, \dots, x_m) \in \mathbb{R}^m$ with $x_i > 0$,

$$\begin{aligned}
&(\varphi - \psi)([x_1, \dots, x_m], [1; (x_1^{a_1}, \dots, x_m^{a_1}); x_{m+1}(x_1^{a_2}, \dots, x_m^{a_2}); \dots; \\
&\qquad x_{m+n-1}(x_1^{a_n}, \dots, x_m^{a_n})]) \\
&\quad \geq (\varphi - \psi)([(x_1 \dots x_j)^{1/j}, \dots, (x_1 \dots x_j)^{1/j}, x_{j+1}, \dots, x_m], \\
&\qquad [1; ((x_1 \dots x_j)^{a_1/j}, \dots, (x_1 \dots x_j)^{a_1/j}, x_{j+1}^{a_1}, \dots, x_m^{a_1}) \\
&\qquad x_{m+1}((x_1 \dots x_j)^{a_2/j}, \dots, (x_1 \dots x_j)^{a_2/j}, x_{j+1}^{a_2}, \dots, x_m^{a_2}), \dots, \\
&\qquad x_{m+n-1}((x_1 \dots x_j)^{a_n/j}, \dots, (x_1 \dots x_j)^{a_n/j}, x_{j+1}^{a_n}, \dots, x_m^{a_n})]) \qquad (5)
\end{aligned}$$

which is obviously verified for $j = 1$. Now, assume that inequality (5) did not hold for $j + 1$, then, there would be a point $(u_1, \ldots, u_m) \in \mathbb{R}^m$, with $u_i > 0$ for all i, such that

$$\begin{aligned}
&(\varphi - \psi)([u_1, \ldots, u_m], [1; (u_1^{a_1}, \ldots, u_m^{a_1}); u_{m+1}(u_1^{a_2}, \ldots, u_m^{a_2}); \ldots;\\
&\quad u_{m+n-1}(u_1^{a_n}, \ldots, u_m^{a_n})])\\
&\quad < (\varphi - \psi)[(u_1 \ldots u_{j+1})^{1/j+1}, \ldots, (u_1 \ldots u_{j+1})^{1/j+1}, u_{j+2}, \ldots, u_m],\\
&\quad [1; ((u_1 \ldots u_{j+1})^{a_1/j+1}, \ldots, (u_1 \ldots u_{j+1})^{a_1/j+1};\\
&\quad u_{m+1}((u_1 \ldots u_{j+1})^{a_2/j+1}, \ldots, (u_1 \ldots u_{j+1})^{a_2/j+1}, u_{j+2}^{a_2}, \ldots, u_m^{a_2}), \ldots,\\
&\quad u_{m+n-1}((u_1 \ldots u_{j+1})^{a_n/j+1}, \ldots, (u_1 \ldots u_{j+1})^{a_n/j+1}, u_{j+2}^{a_n}, \ldots, u_m^{a_n})]). \qquad (6)
\end{aligned}$$

Using the G-invariance of φ,we can assume that $u_1 \le \cdots \le u_m$. On the other hand, taking into account the G-invariance of φ and the induction assumption (5) at the points

$$\begin{aligned}
&([u_1, \ldots, u_j, u_{j+1}, \ldots, u_m], [1; (u_1^{a_1}, \ldots, u_j^{a_1}, u_{j+1}^{a_1}, \ldots, u_m^{a_1});\\
&\quad u_{m+1}(u_1^{a_2}, \ldots, u_j^{a_2}, u_{j+1}^{a_2}, \ldots, u_m^{a_2}), \ldots,\\
&\quad u_{m+n-1}(u_1^{a_n}, \ldots, u_j^{a_n}, u_{j+1}^{a_n}, \ldots, u_m^{a_n})])
\end{aligned}$$

and

$$\begin{aligned}
&([u_2, \ldots, u_{j+1}, u_1, u_{j+2}, \ldots, u_m], [1; (u_2^{a_1}, \ldots, u_{j+1}^{a_1}, u_1^{a_1}, u_{j+2}^{a_1}, \ldots, u_m^{a_1});\\
&\quad u_{m+1}(u_2^{a_2}, \ldots, u_{j+1}^{a_2}, u_1^{a_2}, u_{j+2}^{a_2}, \ldots, u_m^{a_2}), \ldots,\\
&\quad u_{m+n-1}(u_2^{a_n}, \ldots, u_{j+1}^{a_n}, u_1^{a_n}, u_{j+2}^{a_n}, \ldots, u_m^{a_n})])
\end{aligned}$$

of Y, we can write

$$\begin{aligned}
&(\varphi - \psi)([u_1, \ldots, u_j, u_{j+1}, \ldots, u_m], [1; (u_1^{a_1}, \ldots, u_j^{a_1}, u_{j+1}^{a_1}, \ldots, u_m^{a_1});\\
&\quad u_{m+1}(u_1^{a_2}, \ldots, u_j^{a_2}, u_{j+1}^{a_2}, \ldots, u_m^{a_2}); \ldots; u_{m+n-1}(u_1^{a_n}, \ldots, u_j^{a_n}, u_{j+1}^{a_n}, \ldots, u_m^{a_n})])\\
&\quad \ge (\varphi - \psi)([(u_1 \ldots u_j)^{1/j}, \ldots, (u_1 \ldots u_j)^{1/j}, u_{j+1}, \ldots, u_m],\\
&\quad [1; ((u_1 \ldots u_j)^{a_1/j}, \ldots, (u_1 \ldots u_j)^{a_1/j}, u_{j+1}^{a_1}, \ldots, u_m^{a_1})\\
&\quad u_{m+1}((u_1 \ldots u_j)^{a_2/j}, \ldots, (u_1 \ldots u_j)^{a_2/j}, u_{j+1}^{a_2}, \ldots, u_m^{a_2})\\
&\quad , \ldots, u_{m+n-1}((u_1 \ldots u_j)^{a_n/j}, \ldots, (u_1 \ldots u_j)^{a_n/j}, u_{j+1}^{a_n}, \ldots, u_m^{a_n})]), \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
&(\varphi - \psi)([u_2, \ldots, u_{j+1}, u_1, u_{j+2}, \ldots, u_m], [1; (u_2^{a_1}, \ldots, u_{j+1}^{a_1}, u_1^{a_1}, u_{j+2}^{a_1}, \ldots, u_m^{a_1});\\
&\quad u_{m+1}(u_2^{a_2}, \ldots, u_{j+1}^{a_2}, u_1^{a_2}, u_{j+2}^{a_2}, \ldots, u_m^{a_2}); \ldots;\\
&\quad u_{m+n-1}(u_2^{a_n}, \ldots, u_{j+1}^{a_n}, u_1^{a_n}, u_{j+2}^{a_n}, \ldots, u_m^{a_n})])\\
&\quad \ge (\varphi - \psi)([(u_2 \ldots u_{j+1})^{1/j}, \ldots, (u_2 \ldots u_{j+1})^{1/j}, u_1, u_{j+2}, \ldots, u_m],\\
&\quad [1; ((u_2 \ldots u_{j+1})^{a_1/j}, \ldots, (u_2 \ldots u_{j+1})^{a_1/j}, u_1^{a_1}, u_{j+2}^{a_1}, \ldots, u_m^{a_1})\\
&\quad u_{m+1}((u_2 \ldots u_{j+1})^{a_2/j}, \ldots, (u_2 \ldots u_{j+1})^{a_2/j}, u_1^{a_2}, u_{j+2}^{a_2}, \ldots, u_m^{a_2})\\
&\quad u_{m+n-1}((u_2 \ldots u_{j+1})^{a_n/j}, \ldots, (u_2 \ldots u_{j+1})^{a_n/j}, u_1^{a_n}, u_{j+2}^{a_n}, \ldots, u_m^{a_n})]). \qquad (8)
\end{aligned}$$

Let us consider the curve C, of equation $t^j x_{j+1} = u_1 \dots . u_{j+1}$, in the real plane

$$\begin{aligned}&\{([t, \dots, t, x_{j+1}, u_{j+2}, \dots, u_m], [1; (t^{a_1}, \dots, t^{a_1}, x_{j+1}^{a_1}, u_{j+2}^{a_1}, \dots, u_m^{a_1})\\&\quad u_{m+1}(t^{a_2}, \dots, t^{a_2}, x_{j+1}^{a_2}, u_{j+2}^{a_2}, \dots, u_m^{a_2}), \dots,\\&\quad u_{m+n-1}(t^{a_n}, \dots, t^{a_n}, x_{j+1}^{a_n}, u_{j+2}^{a_n}, \dots, u_m^{a_n})])\},\end{aligned}$$

where t and x_{j+1} are variables. The points

$$\begin{aligned}P_1 = {}&[(u_1 \dots u_j)^{1/j}, \dots, (u_1 \dots u_j)^{1/j}, u_{j+1}, \dots, u_m],\\&[1; ((u_1 \dots u_j)^{a_1/j}, \dots, (u_1 \dots u_j)^{a_1/j}, u_{j+1}^{a_1}, \dots, u_m^{a_1}),\\&u_{m+1}(((u_1 \dots u_j)^{a_2/j}, \dots, (u_1 \dots u_j)^{a_2/j}, u_{j+1}^{a_2}, \dots, u_m^{a_2}), \dots\\&u_{m+n-1}((u_1 \dots u_j)^{a_n/j}, \dots, (u_1 \dots u_j)^{a_n/j}, u_{j+1}^{a_n}, \dots, u_m^{a_n})])\end{aligned}$$

and

$$\begin{aligned}P_2 = {}&([(u_2 \dots u_{j+1})^{1/j}, \dots, (u_2 \dots u_{j+1})^{1/j}, u_1, u_{j+2}, \dots, u_m],\\&[1; ((u_2 \dots u_{j+1})^{a_1/j}, \dots, (u_2 \dots u_{j+1})^{a_1/j}, u_1^{a_1}, u_{j+2}^{a_1}, \dots, u_m^{a_1});\\&u_{m+1}((u_2 \dots u_{j+1})^{a_2/j}, \dots, (u_2 \dots u_{j+1})^{a_2/j}, u_1^{a_2}, u_{j+2}^{a_2}, \dots, u_m^{a_2}); \dots;\\&u_{m+n-1}((u_2 \dots u_{j+1})^{a_n/j}, \dots, (u_2 \dots u_{j+1})^{a_n/j}, u_1^{a_n}, u_{j+2}^{a_n}, \dots, u_m^{a_n})]),\end{aligned}$$

belong to this curve C. Note that we cannot have $u_1 = \cdots = u_{j+1}$, for, otherwise, (6) would be an equality.

Taking into account that we have chosen $u_1 \le \cdots \le u_{j+1}$, the points P_1 and P_2 are on different sides of the diagonal $t = x_{j+1}$ of the plane described above.

Note that the curve C intersects this diagonal at the point

$$\begin{aligned}P_3 = {}&([(u_1 \dots u_{j+1})^{1/j+1}, \dots, (u_1 \dots u_{j+1})^{1/j+1}, u_{j+2}, \dots, u_m],\\&[1; ((u_1 \dots u_{j+1})^{a_1/j+1}, \dots, (u_1 \dots u_{j+1})^{a_1/j+1}, u_{j+2}^{a_1}, , \dots, u_m^{a_1})\\&u_{m+1}((u_1 \dots u_{j+1})^{a_2/j+1}, \dots, (u_1 \dots u_{j+1})^{a_2/j+1}, u_{j+2}^{a_2}, , \dots, u_m^{a_2}), \dots,\\&; \dots; u_{m+n-1}((u_1 \dots u_{j+1})^{a_n/j+1}, \dots, (u_1 \dots u_{j+1})^{a_n/j+1}, u_{j+2}^{a_n}, \dots, u_m^{a_n})]), \quad (9)\end{aligned}$$

which appears in inequality (6). On the other hand, using relations (6)–(8) we obtain that

$$(\varphi - \psi)(P_3) > (\varphi - \psi)(P_1) \text{ et } (\varphi - \psi)(P_3) > (\varphi - \psi)(P_2),$$

which proves that the function $(\varphi - \psi)$ reaches a local maximum on the curve C. Consequently, the restriction of the G-invariant function $(\varphi - \psi)$ to the holomorphic curve (that we denote again by C) $\xi^p z = u_1 \dots . u_{j+1}$ of the complex dimensional 2-plane

$$\begin{aligned}&\{([\xi, \dots, \xi, z, u_{j+2}, \dots, u_m], [1; (\xi^{a_1}, \dots, \xi^{a_1}, z^{a_1}, u_{j+2}^{a_1}, \dots, u_m^{a_1});\\&\quad u_{m+1}(\xi^{a_2}, \dots, \xi^{a_2}, z^{a_2}, u_{j+2}^{a_2}, \dots, u_m^{a_2}),\\&\quad \dots, u_{m+n-1}(\xi^{a_n}, \dots, \xi^{a_n}, z^{a_n}, u_{j+2}^{a_n}, \dots, u_m^{a_n})])\},\end{aligned}$$

reaches a local maximum at a point $P = C(\zeta)$. Let us set

$$\begin{aligned} C(\zeta) &= ([1, C^1(\zeta), \dots, C^{m-1}(\zeta)], [1, C^{m+1}(\zeta)(C^1(\zeta)^{a_2}, \dots, C^{m-1}(\zeta)^{a_2}), \dots, \\ &\quad C^{m+n}(\zeta)(C^1(\zeta)^{a_n}, \dots, C^{m-1}(\zeta)^{a_n})]), \\ \dot{C}^\lambda(\xi) &= \frac{dC^\lambda}{d\xi}(\xi) \quad \text{and} \quad \dot{C}^{\overline{\mu}}(\xi) = \overline{\dot{C}^\mu(\xi)}. \end{aligned}$$

Note that, by the continuity of $(\varphi - \psi)$, we can always choose the point

$$([u_1, \dots, u_m], [1; (u_1^{a_1}, \dots, u_m^{a_1}); u_{m+1}(u_1^{a_2}, \dots, u_m^{a_2}); \dots; u_{m+n-1}(u_1^{a_n}, \dots, u_m^{a_n})])$$

in inequality (6) such that

$$(u_1 \dots u_m)^{a_i/m}(u_{m+1} \dots u_{m+n-1})^{1/n} \neq 1, \quad \forall i \in \{1, \dots, n\}.$$

Thus, the equation of C as well as the definition of ψ_0 and the ψ_i, shows that every point of the curve C and for all $i \in \{1, \dots, n\}$ satisfies

$$\begin{aligned} &\psi_0([\xi, \dots, \xi, z, u_{j+2}, \dots, u_m], [1; (\xi^{a_1}, \dots, \xi^{a_1}, z^{a_1}, u_{j+2}^{a_1}, \dots, u_m^{a_1}); \\ &\qquad u_{m+1}(\xi^{a_2}, \dots, \xi^{a_2}, z^{a_2}, u_{j+2}^{a_2}, \dots, u_m^{a_2}); \dots; \\ &\qquad u_{m+n-1}(\xi^{a_n}, \dots, \xi^{a_n}, z^{a_n}, u_{j+2}^{a_n}, \dots, u_m^{a_n})]) \\ &\quad \neq \psi_i([\xi, \dots, \xi, z, u_{j+2}, \dots, u_m], [1; (\xi^{a_1}, \dots, \xi^{a_1}, z^{a_1}, u_{j+2}^{a_1}, \dots, u_m^{a_1}); \\ &\qquad u_{m+1}(\xi^{a_2}, \dots, \xi^{a_2}, z^{a_2}, u_{j+2}^{a_2}, \dots, u_m^{a_2}); \dots; \\ &\qquad u_{m+n-1}(\xi^{a_n}, \dots, \xi^{a_n}, z^{a_n}, u_{j+2}^{a_n}, \dots, u_m^{a_n})]). \end{aligned} \tag{10}$$

Consequently, we can assume that $\psi = \psi_0$ in a neighborhood of P, the proof being exactly the same if we assume $\psi = \psi_i$ (for $i \in \{1, \dots, n\}$) in a neighborhood of P. Therefore

$$\frac{\partial^2}{\partial\xi\partial\overline{\xi}}\{(\varphi - \psi_0)(C(\zeta))\} = \frac{\partial^2(\varphi - \psi_0)}{\partial z_\lambda \partial \overline{z}_\mu}(C(\zeta))\dot{C}^\lambda(\zeta)\dot{C}^{\overline{\mu}}(\zeta) \leq 0.$$

Since

$$-\frac{\partial^2 \psi_0}{\partial z_\lambda \partial \overline{z}_\mu} = g_{\lambda\overline{\mu}},$$

the previous inequality expresses the fact that the Hermitian form of the matrix:

$$\left(g_{\lambda\overline{\mu}} + \frac{\partial^2 \varphi}{\partial z_\lambda \partial \overline{z}_\mu}\right)_{\lambda,\mu} = \left(\frac{\partial^2(\varphi - \psi_0)}{\partial z_\lambda \partial \overline{z}_\mu}\right)_{\lambda,\mu}$$

is negative at $P = C(\zeta)$. This contradicts the g-admissibility of φ at P. So that inequality (5) holds for $j+1$ and Lemma 1 is proven.

In the next lemma, it is more convenient, for our computations, to use the chart given by $\{z_0 \neq 0\}$ and $\{z_m \neq 0\}$ in the parametrization

$$[z_0, z_1, \dots, z_{m-1}], [z_m; z_{m+1}(z_0^{a_1}, z_1^{a_1}, \dots, z_{m-1}^{a_1}); \dots; z_{m+n}(z_0^{a_n}, z_1^{a_n}, \dots, z_{m-1}^{a_n})].$$

Lemma 2. *Let $\varphi \in C^\infty(Y)$ be a g-admissible, G-invariant function, verifying* $\sup\varphi = 0$ *on* Y*. Then for all* $\mu_i > 0$*, we have:*

$$(\varphi - \psi)([1^{[m]}], [1; \mu_1^{[m]}; \dots; \mu_n^{[m]}]) \geq 0. \tag{11}$$

Proof. Consider the point $R_0 \in Y$ where φ reaches its maximum. Using the G-invariance of φ, we can write

$$R_0 = ([v_0, \dots, v_{m-1}], [v_m; v_{m+1}(v_0^{a_1}, \dots, v_{m-1}^{a_1}); \dots; v_{m+n}(v_0^{a_n}, \dots, v_{m-1}^{a_n})]),$$

where the positive reals v_i verify $v_0 \geq v_1 \geq \cdots \geq v_{m-1}$.

We have two separate cases, according to whether $v_m \neq 0$, or $v_m = 0$.

Case A: $v_m \neq 0$. In this case, we use the coordinates system given in $\{v_0 \neq 0, v_m \neq 0\}$ by fixing $v_0 = 1$ and $v_m = 1$, we can write R_0 as

$$R_0 = ([1, u_1, \dots, u_{m-1}], [1; u_{m+1}(1, u_1^{a_1}, \dots, u_{m-1}^{a_1}); \dots; u_{m+n}(u_0^{a_n}, \dots, u_{m-1}^{a_n})]),$$

where the positive reals u_i satisfy $1 \geq u_1 \geq \cdots \geq u_{m-1}$.

Proceeding by contradiction, assume there is a point

$$R_1 = ([1^{[m]}], [1; \zeta_1^{[m]}; \dots; \zeta_n^{[m]}]),$$

such that $\zeta_k > 0$ for $k \in \{1, \dots, n\}$, with

$$(\varphi - \psi)(R_1) < 0. \tag{12}$$

We consider two sub-cases:

• $u_{m+k} \leq \zeta_k, \forall k \in \{1, \dots, n\}$.

We introduce the auxiliary function

$$\psi_m^0 = \ln \frac{x_0^{m-(a_1+\cdots+a_n)}}{(x_0 + \cdots + x_{m-1})^{m-(a_1+\cdots+a_n)}} \times \frac{x_m^{n+1}}{[x_m + (x_{m+1}x_0^{a_1} + \cdots + x_{m+1}x_{m-1}^{a_1}) + \cdots + (x_{m+n}x_0^{a_n} + \cdots + x_{n+m}x_{m-1}^{a_n})]^{(n+1)}}.$$

Since $\varphi \leq 0$, we have

$$(\varphi - \psi_m^0)([1, 0^{[m-1]}], [1; 0^{[mn]}]) = \varphi([1, 0^{[m-1]}], [1; 0^{[mn]}]) \leq 0. \tag{13}$$

On the other hand, the identities $\varphi(R_0) = 0$ and $\psi_m^0 \leq 0$, give us

$$(\varphi - \psi_m^0)(R_0) \geq 0. \tag{14}$$

If $R_0 \neq ([1, 0^{[m-1]}], [1; 0^{[mn]}])$, then $\psi_m^0(R_0) < 0$ and inequality (14) is strict. If $R_0 = ([1, 0^{[m-1]}], [1; 0^{[mn]}])$, we can choose another point R in the neighborhood of R_0 such that $(\varphi - \psi_m^0)(R) > 0$. Indeed, if in a neighborhood of R_0 we have $(\varphi - \psi_m^0) \leq 0$, then, since $(\varphi - \psi_m^0)(R_0) = 0$, $(\varphi - \psi_m^0)$ reaches a local maximum local at R_0, and this contradicts the admissibility of φ at this point, (recall that

$$\partial_{\lambda\overline{\mu}}(\varphi - \psi_m^0)(R_0) = (g_{\lambda\overline{\mu}} + \partial_{\lambda\overline{\mu}}\varphi)(R_0)).$$

In all cases, we deduce that there exists a point

$$R'_0 = ([1, b_1 \dots, b_{m-1}], [1; b_{m+1}(1, b_1^{a_1}, \dots, b_{m-1}^{a_1}); \dots; b_{m+n}(1, b_1^{a_n}, \dots, b_{m-1}^{a_n})]),$$

satisfying

$$(\varphi - \psi_m^0)(R'_0) > 0. \tag{15}$$

By the continuity and the G-invariance of φ, we can assume $1 \geq b_1 \geq \dots \geq b_{m-1} > 0$. On the other hand, the inequality (12) as well as the definitions of R_1, ψ_m^0, ψ_0 and ψ implies that

$$(\varphi - \psi_m^0)(R_1) = (\varphi - \psi_0)(R_1) \leq (\varphi - \psi)(R_1) < 0. \tag{16}$$

Now, we consider the curve:

$$[0,1] \ni t \to c(t) = \Bigg([1, t, t^{(\ln b_2)/(\ln b_1)}, \dots, t^{(\ln b_{m-1})/(\ln b_1)}],$$
$$\Bigg[1; \zeta_1 t^{\frac{\ln(b_{m+1}/\zeta_1)}{\ln b_1}}, \zeta_1 t^{\frac{\ln(b_{m+1}b_1^{a_1}/\zeta_1)}{\ln b_1}}, \dots, \zeta_1 t^{\frac{\ln(b_{m+1}b_{m-1}^{a_1}/\zeta_1)}{\ln b_1}}, \dots,$$
$$\zeta_n t^{\frac{\ln(b_{m+n}/\zeta_n)}{\ln b_1}}, \zeta_n t^{\frac{\ln(b_{m+n}b_1^{a_n}/\zeta_n)}{\ln b_1}}, \dots, \zeta_n t^{\frac{\ln(b_{m+n}b_{m-1}^{a_n}/\zeta_n)}{\ln b_1}}\Bigg]\Bigg).$$

This curve passes by $R_0 = ([1, 0^{[m-1]}], [1; 0^{[nm]}])$ at $t = 0$ then by R'_0 at $t = b_1$ and finally by R_1 at $t = 1$. At these points, using (13), (17) and (25), we deduce that $(\varphi - \psi_m^0)$ is respectively negative, positive, and negative. The invariance by $\exp(i\theta)$ allows us to deduce that $(\varphi - \psi_m^0)$ reaches a maximum on the holomorphic curve given by the complexified version of the above described curve. This is in contradiction with the admissibility of φ.

• If there exist $j \in \{1, \dots, n\}$ such that $u_{m+j} > \zeta_j$.

In this case, we set for $k \in \{1, \dots, n\}$ and $i \in \{1, \dots, m-1\}$

$$\beta_{k,i} = \frac{\ln(b_{m+k}b_i^{a_k}/\zeta_k)}{\ln b_1}, \qquad \beta_{k,0} = \frac{\ln(b_{m+k}/\zeta_k)}{\ln b_1}.$$

Among the powers $\beta_{k,0} = \frac{\ln(b_{m+k}/\zeta_k)}{\ln b_1}$ and based on the assumption "there exist $j \in \{1, \dots, n\}$ such that $u_{m+j} > \zeta_j$", there are some negative. Consider the smallest among them and denote it γ which corresponds to some $\beta_{j_0,0} = \frac{\ln(b_{m+j_0}/\zeta_{j_0})}{\ln b_1}$. For this index j_0, we consider another auxiliary function, given by

$$\psi_{j_0}^0 = \ln \frac{x_0^{m-(a_1+\dots+a_n)}}{(x_0 + \dots + x_{m-1})^{m-(a_1+\dots+a_n)}}$$
$$\times \frac{(x_0^{a_{j_0}} x_{m+j_0})^{n+1}}{[x_m + (x_{m+1}x_0^{a_1} + \dots + x_{m+1}x_{m-1}^{a_1}) + \dots + (x_{m+n}x_0^{a_n} + \dots + x_{n+m}x_{m-1}^{a_n})]^{(n+1)}}.$$

We have

$$(\varphi - \psi_{j_0}^0)(R_0) > 0. \tag{17}$$

Denoting

$$R_\varepsilon = \Big([1, \varepsilon, \varepsilon^{(\ln b_2)/(\ln b_1)}, \ldots, \varepsilon^{(\ln b_{m-1})/(\ln b_1)}],$$
$$\Big[1; \zeta_1 \varepsilon^{\frac{\ln(b_{m+1}/\zeta_1)}{\ln b_1}}, \zeta_1 \varepsilon^{\frac{\ln(b_{m+1} b_1^{a_1}/\zeta_1)}{\ln b_1}}, \ldots, \zeta_1 \varepsilon^{\frac{\ln(b_{m+1} b_{m-1}^{a_1}/\zeta_1)}{\ln b_1}}$$
$$; \ldots; \zeta_n \varepsilon^{\frac{\ln(b_{m+n}/\zeta_n)}{\ln b_1}}, \zeta_n \varepsilon^{\frac{\ln(b_{m+n} b_1^{a_n}/\zeta_n)}{\ln b_1}}, \ldots, \zeta_n \varepsilon^{\frac{\ln(b_{m+n} b_{m-1}^{a_n}/\zeta_n)}{\ln b_1}}\Big]\Big),$$

we have

$$\psi_{j_0}^0(R_\varepsilon) = \ln\Big\{ \frac{1}{(1+\varepsilon^2+\varepsilon^{(2\ln b_2)/\ln(b_1)}+\cdots+\varepsilon^{(2\ln b_{m-1})/\ln(b_1)})^{m-(a_1+\cdots+a_n)}}$$
$$\times \frac{(\zeta_{j_0}\varepsilon^\gamma)^{2(n+1)}}{[1+\zeta_1^2\varepsilon^{2\beta_{1,0}}+\cdots+\zeta_1^2\varepsilon^{2\beta_{1,m-1}}+\cdots+\zeta_n^2\varepsilon^{2\beta_{n,0}}+\cdots+\zeta_n^2\varepsilon^{2\beta_{n,m-1}}]^{(n+1)}}\Big\}.$$

When ε approaches 0, we obtain:

$$\lim_{\varepsilon\to 0}\psi_{j_0}^0(R_\varepsilon) = \lim_{\varepsilon\to 0}\ln\Big\{ \frac{1}{(1+\varepsilon^2+\varepsilon^{(2\ln b_2)/\ln(b_1)}+\cdots+\varepsilon^{(2\ln b_{m-1})/\ln(b_1)})^{m-(a_1+\cdots+a_n)}}$$
$$\times \frac{(\zeta_{j_0}\varepsilon^\gamma)^{2(n+1)}}{[1+\zeta_1^2\varepsilon^{2\beta_{1,0}}+\cdots+\zeta_1^2\varepsilon^{2\beta_{1,m-1}}+\cdots+\zeta_n^2\varepsilon^{2\beta_{n,0}}+\cdots+\zeta_n^2\varepsilon^{2\beta_{n,m-1}}]^{(n+1)}}\Big\}$$
$$= \ln\lim_{t\to\infty}\frac{(\zeta_{j_0}t^{-\gamma})^{2(n+1)}}{[1+\zeta_1^2 t^{-2\beta_{1,0}}+\cdots+\zeta_1^2 t^{-2\beta_{1,m-1}}+\cdots+\zeta_n^2 t^{-2\beta_{n,0}}+\cdots+\zeta_n^2 t^{-2\beta_{n,m-1}}]^{(n+1)}}\}$$
$$= \ln\lim_{t\to\infty}\frac{(\zeta_{j_0}t^{-\gamma})^{2(n+1)}}{(\zeta_{j_0}t^{-\gamma})^{2(n+1)}}$$
$$= \ln 1 = 0$$

$(-\gamma)$ being the larger of the positive powers in the fraction above. Since $\varphi(R_\varepsilon) \le 0$ and taking into account (17), we deduce that there exists ε_0 such that

$$(\varphi-\psi_{j_0}^0)(R_{\varepsilon_0}) \le -\psi_{j_0}^0(R_{\varepsilon_0}) < (\varphi-\psi_{j_0}^0)(R_0). \tag{18}$$

On the other hand, the inequality (12), and the definitions of R_1, $\psi_{j_0}^0$, ψ_{j_0} and ψ yield:

$$(\varphi-\psi_{j_0}^0)(R_1) = (\varphi-\psi_{j_0})(R_1) \le (\varphi-\psi)(R_1) < 0. \tag{19}$$

The curve

$$[\varepsilon_0, 1] \ni t \to c(t) = \Big([1, t, t^{(\ln b_2)/(\ln b_1)}, \ldots, t^{(\ln b_{m-1})/(\ln b_1)}],$$
$$\Big[1; \zeta_1 t^{\frac{\ln(b_{m+1}/\zeta_1)}{\ln b_1}}, \zeta_1 t^{\frac{\ln(b_{m+1} b_1^{a_1}/\zeta_1)}{\ln b_1}}, \ldots, \zeta_1 t^{\frac{\ln(b_{m+1} b_{m-1}^{a_1}/\zeta_1)}{\ln b_1}}, \ldots,$$
$$\zeta_n t^{\frac{\ln(b_{m+n}/\zeta_n)}{\ln b_1}}, \zeta_n t^{\frac{\ln(b_{m+n} b_1^{a_n}/\zeta_n)}{\ln b_1}}, \ldots, \zeta_n t^{\frac{\ln(b_{m+n} b_{m-1}^{a_n}/\zeta_n)}{\ln b_1}}\Big]\Big)$$

passes by R_{ε_0} at $t=\varepsilon_0$, then by R_0 at $t=b_1$ and finally by R_1 at $t=1$. At these points, using (18), (17) and (19) we deduce that $(\varphi-\psi^0_{j_0})$ reaches a local maximum on the curve described above. This is in contradiction with the admissibility of φ.

Case B: $v_m=0$.

In this case, $i\in\{1,\dots,n\}$ is fixed and we use the coordinate system given by $\{v_0\neq 0, v_{m+i}\neq 0\}$.

We can write R_0, where φ reaches its maximum (equal to zero), in the form

$$R_0=([1,u_1\dots,u_{m-1}],[0;u_{m+1}(1,u_1^{a_1},\dots,u_{m-1}^{a_1});\dots;u_{m+i-1}(1,u_1^{a_{i-1}},\dots,u_{m-1}^{a_{i-1}}); (1,u_1^{a_i},\dots,u_{m-1}^{a_i});u_{m+i+1}(1,u_1^{a_{i+1}},\dots,u_{m-1}^{a_{i+1}});\dots;u_{m+n}(1,u_1^{a_n},\dots,u_{m-1}^{a_n})]),$$

using the G-invariance of φ, we can assume that the positive reals u_k verify $1\geq u_1\geq\cdots\geq u_{m-1}$. We shall prove an equivalent version of Lemma 2, that is

$$(\varphi-\psi)([1^{[m]}],[\mu_0;\mu_1^{[m]};\dots;\mu_{i-1}^{[m]},1^{[m]};\mu_{i+1}^{[m]}\dots;\mu_n^{[m]}])\geq 0, \tag{20}$$

where $\mu_k>0$, for $k\in\{0,1,\dots,i-1,i+1,\dots,n\}$.

Proceeding by contradiction, assume there exists a point

$$R_i=([1^{[m]}],[\mu_0;\mu_1^{[m]};\dots;\mu_{i-1}^{[m]},1^{[m]};\mu_{i+1}^{[m]};\dots;\mu_n^{[m]}])$$

with $\mu_k>0$, $\forall k\in\{0,\dots,i-1,i+1,\dots,n\}$ and

$$(\varphi-\psi)(R_i)<0. \tag{21}$$

Similar to the previous case, we distinguish two sub-cases:

•<u>If $u_{m+k}\leq\mu_k, \forall k\in\{1,\dots,i-1,i+1,\dots,n\}$.</u>

We introduce the auxiliary function

$$\psi_i^0=\ln\frac{x_0^{m-(a_1+\cdots+a_n)}}{(x_0+\cdots+x_{m-1})^{m-(a_1+\cdots+a_n)}} \times\frac{x_{m+i}^{n+1}}{[x_m+(x_{m+1}x_0^{a_1}+\cdots+x_{m+1}x_{m-1}^{a_1})+\cdots+(x_{m+n}x_0^{a_n}+\cdots+x_{n+m}x_{m-1}^{a_n})]^{(n+1)}}.$$

Since $\varphi\leq 0$, then we have

$$\begin{aligned}&(\varphi-\psi_i^0)([1,0^{[m-1]}],[0;0^{[m(i-1)]};1,0^{[m-1]};0^{[m(n-i)]}])\\&\quad=\varphi([1,0^{[m-1]}],[0;0^{[m(i-1)]};1,0^{[m-1]};0^{[m(n-i)]}])\leq 0.\end{aligned} \tag{22}$$

On the other hand, since $\varphi(R_0)=0$ and $\psi_i^0\leq 0$, we obtain

$$(\varphi-\psi_i^0)(R_0)\geq 0. \tag{23}$$

This inequality being strict as soon as

$$R_0\neq([1,0^{[m-1]}],[0;0^{[m(i-1)]};1,0^{[m-1]};0^{[m(n-i)]}]).$$

If $R_0=([1,0^{[m-1]}],[0;0^{[m(i-1)]};1,0^{[m-1]};0^{[m(n-i)]}])$, it suffices to consider a point close to R_0 on which we have $(\varphi-\psi_i^0)(R)>0$. Indeed, when $(\varphi-\psi_i^0)\leq 0$ in a neighborhood of R_0, then $(\varphi - \psi_i^0)$ admits a local maximum at R_0, which is in contradiction with the admissibility of φ at this point.

So, as in case A, there exists a point

$$R_0' = ([1, c_1 \ldots, c_{m-1}], [c_m; c_{m+1}(1, c_1^{a_1}, \ldots, c_{m-1}^{a_1}), \ldots, c_{m+i-1}(1, c_1^{a_{i-1}}; \ldots; c_{m-1}^{a_{i-1}}); (1, c_1^{a_i}, \ldots, c_{m-1}^{a_i}); c_{m+i+1}(1, c_1^{a_{i+1}}, \ldots, c_{m-1}^{a_{i+1}}); \ldots; c_{m+n}(1, c_1^{a_n}, \ldots, c_{m-1}^{a_n})]),$$

satisfying

$$(\varphi - \psi_i^0)(R_0') > 0. \tag{24}$$

By the continuity and the G-invariance of φ, we can assume $1 \geq c_1 \geq \cdots \geq c_{m-1}$, $\eta_0 > c_m > 0$. On the other hand, the inequality (12) and the definitions of R_i, ψ_i^0, ψ_i and ψ imply that

$$(\varphi - \psi_i^0)(R_i) = (\varphi - \psi_i)(R_i) \leq (\varphi - \psi)(R_i) < 0. \tag{25}$$

We now introduce another curve on Y, defined by

$$[0,1] \ni t \to c(t) = \Bigg([1, t, t^{(\ln c_2)/(\ln c_1)}, \ldots, t^{(\ln c_{m-1})/(\ln c_1)}], \Big[\mu_0 t^{\frac{\ln(c_m/\mu_0)}{\ln c_1}}; \mu_1 t^{\frac{\ln(c_{m+1}/\mu_1)}{\ln c_1}}, \mu_1 t^{\frac{\ln(c_{m+1}c_1^{a_1}/\mu_1)}{\ln c_1}}, \ldots, \mu_1 t^{\frac{\ln(c_{m+1}c_{m-1}^{a_1}/\mu_1)}{\ln c_1}}; \ldots; \mu_{i-1} t^{\frac{\ln(c_{m+i-1}/\mu_{i-1})}{\ln c_1}}, \mu_{i-1} t^{\frac{\ln(c_{m+i-1}c_1^{a_{i-1}}/\mu_{i-1})}{\ln c_1}}, \ldots, \mu_{i-1} t^{\frac{\ln(c_{m+i-1}c_{m-1}^{a_{i-1}}/\mu_{i-1})}{\ln c_1}}; 1, t^{(\ln c_1^{a_i})/(\ln c_1)}, \ldots, t^{(\ln c_{m-1}^{a_i})/(\ln c_1)}; \mu_{i+1} t^{\frac{\ln(c_{m+i+1}/\mu_{i+1})}{\ln c_1}}, \mu_{i+1} t^{\frac{\ln(c_{m+i+1}c_1^{a_{i+1}}/\mu_{i+1})}{\ln c_1}}, \ldots, \mu_{i+1} t^{\frac{\ln(c_{m+i+1}c_{m-1}^{a_{i+1}}/\mu_{i+1})}{\ln c_1}}; \ldots; \mu_n t^{\frac{\ln(c_{m+n}/\mu_n)}{\ln c_1}}, \mu_n t^{\frac{\ln(c_{m+n}c_1^{a_n}/\mu_n)}{\ln c_1}}, \ldots, \mu_n t^{\frac{\ln(c_{m+n}c_{m-1}^{a_n}/\mu_n)}{\ln c_1}}\Big]\Bigg),$$

the curve passes by $([1, 0^{[m-1]}], [0; 0^{[m(i-1)]}; 1, 0^{[m-1]}; 0^{[m(n-i)]}])$ at $t = 0$, then by R_0' at $t = c_1$ and finally by R_i at $t = 1$. Then by (22), (24) and (25), we deduce that $(\varphi - \psi_{i-1}^0)$ is respectively negative, positive and negative. The invariance by $\exp(i\theta)$, allows us to conclude that $(\varphi - \psi_i^0)$ reaches a maximum on the holomorphic curve given by the complexified version of the curve described above, which contradicts again admissibility of φ.

•<u>If there exist $j \neq i$ such that $u_{m+j} > \mu_j$.</u>

In this case, we set for $k \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m-1\}$,

$$\beta_{k,l} = \frac{\ln(c_{m+k}c_l^{a_k}/\mu_k)}{\ln c_1}, \qquad \beta_{k,0} = \frac{\ln(c_{m+k}/\mu_k)}{\ln c_1} \quad \text{and} \quad \beta_{0,0} = \frac{\ln(c_m/\mu_0)}{\ln c_1}.$$

Among the powers $\beta_{k,0} = \frac{\ln(u_{m+k}/\mu_k)}{\ln c_1}$ and according to the hypothesis "$\exists j \neq i$ such that $u_{m+j} > \mu_j$", there are some negative. Consider the smallest among them and denote it β which corresponds to some $\beta_{j_0,0} = \frac{\ln(c_{m+j_0}/\mu_{j_0})}{\ln c_1}$. For the index j_0 consider the auxiliary function:

$$\psi^0_{j_0} = \ln \frac{x_0^{m-(a_1+\cdots+a_n)}}{(x_0+\cdots+x_{m-1})^{m-(a_1+\cdots+a_n)}} \times \frac{(x_0^{a_{j_0}} x_{m+j_0})^{n+1}}{[x_m + (x_{m+1}x_0^{a_1} + \cdots + x_{m+1}x_{m-1}^{a_1}) + \cdots + (x_{m+n}x_0^{a_n} + \cdots + x_{n+m}x_{m-1}^{a_n})]^{(n+1)}}.$$

We have

$$(\varphi - \psi^0_{j_0})(R_0) > 0. \tag{26}$$

Denoting

$$R_\varepsilon = ([1, \varepsilon, \varepsilon^{(\ln c_2)/(\ln c_1)}, \ldots, \varepsilon^{(\ln c_{m-1})/(\ln c_1)}], [\mu_0\varepsilon^{\beta_{0,0}}; \mu_1\varepsilon^{\beta_{1,0}}, \ldots, \mu_1\varepsilon^{\beta_{1,m-1}}; \ldots; \mu_{i-1}\varepsilon^{\beta_{i-1,0}}, \ldots, \mu_{i-1}\varepsilon^{\beta_{i-1,m-1}}; 1, \varepsilon^{(\ln c_1^{a_i})/(\ln c_1)}, \ldots, \varepsilon^{(\ln c_{m-1}^{a_i})/(\ln c_1)}; \mu_{i+1}\varepsilon^{\beta_{i+1,0}}, \ldots, \mu_{i+1}\varepsilon^{\beta_{i+1,m-1}}; \mu_n\varepsilon^{\beta_{n,0}}, \ldots, \mu_n\varepsilon^{\beta_{n,m-1}}])$$

we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \psi^0_{j_0}(R_\varepsilon) \\
&= \lim_{\varepsilon\to 0} \ln\Bigg\{ \frac{1}{(1+\varepsilon^2+\varepsilon^{(2\ln c_2)/\ln(c_1)} + \cdots + \varepsilon^{(2\ln c_{m-1})/\ln(c_1)})^{m-(a_1+\cdots+a_n)}} \\
&\quad \times (\mu_{j_0}\varepsilon^{\beta})^{2(n+1)}[\mu_0^2\varepsilon^{2\beta_{0,0}} + \mu_1^2\varepsilon^{2\beta_{1,0}} + \cdots + \mu_1^2\varepsilon^{2\beta_{1,m-1}} + \cdots + \\
&\quad \mu_{i-1}^2\varepsilon^{2\beta_{i-1,0}} + \cdots + \mu_{i-1}^2\varepsilon^{2\beta_{i-1,m-1}} + 1 + \varepsilon^{2(\ln c_1^{a_i})/(\ln c_1)} + \cdots + \varepsilon^{2(\ln c_{m-1}^{a_i})/(\ln c_1)} \\
&\quad + \mu_{i+1}^2\varepsilon^{2\beta_{i+1,0}} + \cdots + \mu_{i+1}\varepsilon^{2\beta_{i+1,m-1}} + \cdots + \mu_n^2\varepsilon^{2\beta_{n,0}} + \cdots + \mu_n\varepsilon^{2\beta_{n,m-1}}]^{-(n+1)} \Bigg\} \\
&= \ln \lim_{t\to\infty} (\mu_{j_0}t^{-\beta})^{2(n+1)}[\mu_0^2 t^{-2\beta_{0,0}} + \mu_1^2 t^{-2\beta_{1,0}} + \cdots + \mu_1^2 t^{-2\beta_{1,m-1}} + \cdots \\
&\quad + \mu_{i-1}^2 t^{-2\beta_{i-1,0}} + \cdots + \mu_{i-1}^2 t^{-2\beta_{i-1,m-1}} + 1 + t^{-2(\ln c_1^{a_i})/(\ln c_1)} + \cdots + t^{-2(\ln c_{m-1}^{a_i})/(\ln c_1)} \\
&\quad + \mu_{i+1}^2 t^{-2\beta_{i+1,0}} + \cdots + \mu_{i+1} t^{-2\beta_{i+1,m-1}} + \cdots + \mu_n^2 t^{-2\beta_{n,0}} + \cdots + \mu_n t^{-2\beta_{n,m-1}}]^{-(n+1)} \\
&= \ln 1 = 0
\end{aligned}$$

$(-\beta)$ being the largest of the positive powers in the fraction above. Since $\varphi(R_\varepsilon) \leq 0$ and taking into account (26), we deduce that there exists ε_0 such that

$$(\varphi - \psi^0_{j_0})(R_{\varepsilon_0}) \leq -\psi^0_{j_0}(R_{\varepsilon_0}) < (\varphi - \psi^0_{j_0})(R_0). \tag{27}$$

On the other hand, the inequality (21), and the definitions of R_1, $\psi^0_{j_0}$, ψ_{j_0} and ψ yield:

$$(\varphi - \psi^0_{j_0})(R_1) = (\varphi - \psi_{j_0})(R_1) \leq (\varphi - \psi)(R_1) < 0. \tag{28}$$

The curve

$$\begin{aligned}
&[\varepsilon_0, 1] \ni t \to c(t) \\
&\quad = ([1, t, t^{(\ln c_2)/(\ln c_1)}, \ldots, t^{(\ln c_{m-1})/(\ln c_1)}], [\mu_0 t^{\beta_{0,0}}; \mu_1 t^{\beta_{1,0}}, \ldots, \mu_1 t^{\beta_{1,m-1}}; \ldots; \\
&\qquad \mu_{i-1}t^{\beta_{i-1,0}}, \ldots, \mu_{i-1}t^{\beta_{i-1,m-1}}; 1, t^{(\ln c_1^{a_i})/(\ln c_1)}, \ldots, t^{(\ln c_{m-1}^{a_i})/(\ln c_1)}; \\
&\qquad \mu_{i+1}t^{\beta_{i+1,0}}, \ldots, \mu_{i+1}t^{\beta_{i+1,m-1}}; \mu_n t^{\beta_{n,0}}, \ldots, \mu_n t^{\beta_{n,m-1}}])
\end{aligned}$$

passes by R_{ε_0} at $t = \varepsilon_0$, then by R_0 at $t = c_1$ and finally by R_i at $t = 1$. At these points, using (27), (26) and (28), we deduce that $(\varphi - \psi^0_{j_0})$ reaches a local maximum on the curve described above. This is in contradiction with the admissibility of φ. $\square$

2.2. *Proof of Corollary 1*

Let $\varphi \in \mathcal{C}^\infty(Y)$ be a g-admissible and G-invariant function with a null supremum on Y. According to Theorem 1, we have $\varphi \geq \psi$, therefore, for all $\alpha \geq 0$

$$\int_Y \exp(-\alpha\varphi)dv \leq \int_Y \exp(-\alpha\psi)dv.$$

To obtain the values of α for which the last integral converges we estimate the $\int_Y \exp(-\alpha\psi_k)dv$, for $k \in \{0, 1, \dots, n\}$.

Indeed

$$\int_Y \exp(-\alpha\psi)dv \leq \sum_{k=0}^{n} \int_Y \exp(-\alpha\psi_k)dv.$$

We mention that we can avoid the very hard computation of the element volume dv (as in [4]) by means of the following remark. If we write $g_{\lambda\overline{\mu}}$ in the form

$$g_{\lambda\overline{\mu}} = \partial_{\lambda\overline{\mu}} \log K.$$

The quantity $[K \det(g)]$ is intrinsic since we chose the metric g in $c_1(Y)$. Thus, we can deduce that there exist two constants C_1 and C_2 such that

$$\frac{C_1}{K} \leq \det(g) \leq \frac{C_2}{K}.$$

Using the above and setting

$$\begin{aligned} r &= x_1 + \cdots + x_m, \\ s &= 1 + (x_1^{a_1} + \cdots + x_m^{a_1}) + x_{m+1}(x_1^{a_2} + \cdots + x_m^{a_2}) \\ &\quad + \cdots + x_{m+n-1}(x_1^{a_n} + \cdots + x_m^{a_n}), \end{aligned}$$

we obtain that

$$dv \simeq \frac{C dx_1 \wedge \cdots \wedge dx_{m+n-1}}{r^{m-(a_1+\cdots+a_n)} s^{n+1}}.$$

Then $I_0 = \int_Y \exp(-\alpha\psi_0)dv$ which converges for $\alpha < \frac{1}{n+1}$ and for $k = \{1, \dots, n\}$, $I_k = \int_Y \exp(-\alpha\psi_k)dv$ converges for $\alpha < \frac{n}{n+1}$. In conclusion,

$$\int_Y \exp(-\alpha\psi)dv$$

converges for $\alpha < \frac{1}{n+1}$. See [3] and [6] for reference, and for the existence of K-E metrics refer to [7].

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