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The stable processes on symmetric matrices

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Abstract. This paper deals with a characterization of the stable processes on the space of symmetric matrices by means of its Laplace transform. This characterization is done using a mixture with the Wishart distribution and under some independence properties. This extends the results of reference Louati et al. (2015).

Keywords: Cumulant function; Stable process; Wishart distribution

Mathematics Subject Classification: 60G51; 60G52

1. INTRODUCTION

In the last decades, Paul Lévy during his studies on the sums of the random variables had introduced the class of the stable distributions. This class of distributions has drawn a considerable interest of researchers and several works have been realized on their different aspects. For several reasons, the stable distributions are particularly important to the daily practice of the statisticians in the analysis of data belonging to many areas of application. The first reason is met when there are solid theoretical reasons for expecting a non-Gaussian stable model, *e.g.*, hitting times for a Brownian motion yielding a Lévy distribution. The second reason is dealing with the generalized central limit theorem. Because of these probabilistic important properties, many studies have been carried out and some comprehensive books have been published. We may refer to [13]. We may also mention [11] which gave a

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complete description of the basic facts related to the stable distributions and their practical applications. The stable distributions appear in several areas, such as internet traffic and communication systems (see [13]), biology (see [12]), and ecology (see [1]). In connection with these studies, and recently, several works have been proposed to discuss and to describe the stability in terms of processes. In this direction, [7] gave a characterization of the real stable process $Y = (Y(t))_{t\geq 0}$ using a mixture with an exponential standard random variable T independent of Y. Louati et al. [10] extended this result by substituting the exponential distribution by a gamma distribution and found a new characterization of the stable processes on the real line. As extension of the gamma distributions, the Wishart distributions have received a lot of attention, since their importance in relevant applications using, mainly, the graphical Gaussian modeling. The essence of graphical models in multivariate analysis is to identify independencies and conditional independencies between various groups of variables. Moreover, the Wishart distribution plays a major role in the estimation of the covariance matrices in multivariate statistics. The structure of the Wishart distribution has been studied for a long time. Nevertheless, several results about the Wishart and its derived distributions (especially, the Riesz ones) were only obtained recently (see [2–4,6,8,9] and [14]).

Since the Wishart distribution represents the natural extension of the gamma one on the cone of symmetric matrices, then in this paper, we extend the works of [7] and [10] and we characterize the class of stable processes under some independence conditions related to these processes and using the mixture with a random Wishart matrix.

The article is organized as follows: In Section 2, we recall some definitions and give some preliminary results relevant to the Wishart distribution and the multivariate stable distributions. In Section 3, we establish, under some independent conditions and using the concept of the mixture, a characterization of the multivariate stable processes on the space of symmetric matrices by their Laplace transforms.

2. Preliminaries

To make clear the results of this paper, we first recall some notations and review some characteristic properties concerning the Wishart distribution on the space of real symmetric matrices and the multivariate stable processes.

2.1. The Wishart distribution

Let E be the Euclidean space of (r, r) real symmetric matrices with a dimension $n = \frac{r(r+1)}{2}$ and the scalar product $\langle x, y\rangle = \mathrm{tr}(xy)$. If μ is a positive random measure on E, we denote by

$$L_\mu(\theta) = \int_E \exp(\langle \theta, x\rangle)\mu(dx) < \infty$$

its Laplace transform and

$$\Theta(\mu) = int\{\theta \in E^*;\ L_\mu(\theta) < \infty\}.$$

Let μ be a probability measure such that $\Theta(\mu) \neq \emptyset$. We define the cumulant function of the measure μ by

$$k_\mu(\theta) = \ln\left(L_\mu(\theta)\right). \tag{2.1}$$

The class of the Wishart distributions on the space E depends on two parameters p and σ. The first one is the scale parameter $p > \frac{r-1}{2}$ for which we have a convolution semi-group.

The second one is the shape parameter σ which is an element of the cone Ω of definite positive elements of E. More precisely, the Wishart random matrix on E has the following continuous probability density function

$$W(p,\sigma)(dx) = \frac{e^{-\langle\sigma,x\rangle}\det(x)^{p-n/r}}{\Gamma_\Omega(p)\det(\sigma^{-p})}\mathbf{1}_\Omega(x)dx, \tag{2.2}$$

where

$$\Gamma_\Omega(s) = (2\pi)^{\frac{r(r-1)}{4}}\prod_{i=1}^{r}\Gamma\left(p-(i-1)/2\right).$$

Its Laplace transform is defined for θ in $\sigma - \Omega$, and given by

$$L_{W(p,\sigma)}(\theta) = \det\left(I_r - \sigma^{-1}\theta\right)^{-p}. \tag{2.3}$$

2.2. *Multivariate stable processes*

Let $\alpha \in (0, 2]$, a random variable X on an Euclidean space E is α-stable in the broad sense if for each $n \geq 2$, there exist $f_n \in E$ and n random variables $X_1, X_2, \dots, X_n$ i.i.d such that

$$X_1 + X_2 + \cdots + X_n \overset{d}{=} n^{1/\alpha}X + f_n.$$

(For more details about stable processes, the reader can see [11]).

It is worth reminding that a random variable X is $\frac{1}{p}$-stable, with $p > 1$, on the real line, if its Laplace transform is equal to

$$L_X(s) = e^{-(-s)^{1/p}}, \text{ for all } s \leq 0.$$

We say also that a process $(X(t))_{t\geq 0}$ is a drifted stable process, with index $\alpha \in (0, 1)$, if it has the following Laplace transform

$$L_{X_t}(s) = e^{t(1-(1-\beta s)^\alpha)}, \forall\beta > 0 \text{ and } \forall s < 1/\beta.$$

(For more information, we can see [5,7]).

3. Stable processes on symmetric matrices

In this section, we give some characteristic properties of a multivariate Lévy process $Y = (Y(t))_{t\geq 0}$ on Ω. Such properties are useful to characterize a stable one on the space of symmetric matrices. For this reason, in what follows, μ_t denotes the distribution of $Y(t)$.

Theorem 3.1. *Let $Y = (Y(t))_{t\geq 0}$ be a Lévy process on $\overline{\Omega}$ and let Z be a Wishart random matrix $W(n/r, I_r)$ independent from Y. If for any fixed $p \in \mathbf{N}\setminus\{0,1\}$, $\frac{Y(tr(Z))}{tr(Z)^p}$ and $Y(tr(Z))$ are independent, then,*
(i) $\mathbf{P}(Y(t) = 0) = 0$.
(ii) *For all $h \in \Omega$, we have*

$$\mathbf{E}\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right) = \frac{\sum_{\pi\in S_p} n^{m(\pi)}}{n\langle k'_{\mu_1}(0), h\rangle},$$

where S_p is the group of permutations on $\{1, \dots, p\}$, $\pi \in S_p$ and $m(\pi)$ is the number of cycles in π.

Proof. (i) Setting $p_t = \mathbf{P}(tr(Y(t)) = 0)$ and assume that $p_t > 0$. Using the infinite divisibility of $Y = (Y(t))_{t\geq 0}$, we get for all $s \in (-\infty, 0]$,

$$\begin{aligned} L_{tr(Y(t))}(s) &= \int_\Omega e^{str(y)}\mu_t(dy) = L_{Y(t)}(sI_r) = \left(L_{Y(1)}(sI_r)\right)^t \\ &= \left(L_{tr(Y(1))}(s)\right)^t = e^{tk_{tr(Y(1))}(s)}. \end{aligned}$$

If $s \longrightarrow -\infty$, we have

$$p_t = \lim_{s\to-\infty} e^{tk_{tr(Y(1))}(s)} = p_1^t. \tag{3.4}$$

Moreover, the independence of $\frac{Y(tr(Z))}{tr(Z)^p}$ and $Y(tr(Z))$ implies that

$$\begin{aligned} q = \mathbf{P}(tr(Y(tr(Z))) = 0) &= \mathbf{P}\left(\frac{tr(Y(tr(Z)))}{tr(Z)^p} = 0 \quad ; \quad tr(Y(tr(Z))) = 0\right) \\ &= \mathbf{P}\left(\frac{tr(Y(tr(Z)))}{tr(Z)^p} = 0\right)\mathbf{P}\left(tr(Y(tr(Z))) = 0\right) = q^2. \end{aligned}$$

It follows that

$$q^2 = q. \tag{3.5}$$

Since $Z \sim W(n/r, I_r)$, then, using (3.4) and (2.3), we deduce that

$$\begin{aligned} q &= \mathbf{P}(tr(Y(tr(Z))) = 0) = \int_\Omega \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)}\mathbf{P}(tr(Y(tr(z))) = 0)dz \\ &= \int_\Omega \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)} p_1^{tr(z)}dz = \int_\Omega \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)} e^{tr(z)\log(p_1)}dz \\ &= \int_\Omega \frac{e^{-\langle z,(1-\log(p_1))I_r\rangle}}{\Gamma_\Omega(n/r)}dz = (1-\log(p_1))^{-n} > 0. \end{aligned}$$

This, combined with (3.5), gives $q = 1$ which contradicts the fact that μ_t is not a Dirac measure of E. Hence $\mathbf{P}(tr(Y(t)) = 0) = 0$.

(ii) Since Z is Wishart distributed $W(n/r, I_r)$, then using the moment of Wishart distribution (see [6]), we obtain

$$\mathbf{E}(tr(Z)^p) = \int_\Omega tr(z)^p \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)}dz = \sum_{\pi\in S_p} n^{m(\pi)}. \tag{3.6}$$

Using (2.1), we get

$$\begin{aligned} \mathbf{E}(e^{\langle\theta, Y(tr(Z))\rangle}) &= \int_\Omega\int_\Omega e^{\langle\theta,y\rangle}\frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)}\mu_{tr(z)}(dy)dz \\ &= \int_\Omega \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)}\left(\int_\Omega e^{\langle\theta,y\rangle}\mu_{tr(z)}(dy)\right)dz \\ &= \int_\Omega \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)} L_{Y(tr(z))}(\theta)dz \\ &= \int_\Omega \frac{e^{-tr(z)}}{\Gamma_\Omega(n/r)} e^{tr(z)k_{\mu_1}(\theta)}dz = \int_\Omega \frac{e^{-\langle z,(1-k_{\mu_1}(\theta))I_r\rangle}}{\Gamma_\Omega(n/r)}dz. \end{aligned}$$

Then according to (2.2), we have

$$\mathbf{E}(e^{\langle\theta, Y(tr(Z))\rangle}) = \det((1-k_{\mu_1}(\theta))I_r)^{-n/r} = (1-k_{\mu_1}(\theta))^{-n}.$$

Differentiating this with respect to θ, we deduce that for all $h \in \Omega$,

$$\mathbf{E}(\langle Y(tr(Z)), h\rangle e^{\langle \theta, Y(tr(Z))\rangle}) = n(1 - k_{\mu_1}(\theta))^{-n-1}\langle k'_{\mu_1}(\theta), h\rangle.$$

Taking $\theta = 0$, we obtain

$$\mathbf{E}(\langle Y(tr(Z)), h\rangle) = n\langle k'_{\mu_1}(0), h\rangle.$$

Using (3.6) and the independence of $\frac{Y(tr(Z))}{tr(Z)^p}$ and $Y(tr(Z))$, we get for all $h \in \Omega$

$$\mathbf{E}\left(\langle Y(tr(Z)), h\rangle\right) \mathbf{E}\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right) = \mathbf{E}\left(tr(Z)^p\right).$$

This implies that

$$\begin{aligned}\mathbf{E}\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right) &= \frac{\mathbf{E}\left(\langle Y(tr(Z)), h\rangle\right)}{\mathbf{E}\left(\langle Y(tr(Z)), h\rangle\right)}\mathbf{E}\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right)\\ &= \frac{\mathbf{E}\left(tr(Z)^p\right)}{\mathbf{E}\left(\langle Y(tr(Z)), h\rangle\right)} = \frac{\sum_{\pi\in S_p} n^{m(\pi)}}{n\langle k'_{\mu_1}(0), h\rangle}.\end{aligned}$$

This achieves the result.

Next, we characterize, under an independence property, the stable processes on the space of symmetric matrices. More precisely, we have

Theorem 3.2. *Let $Y = (Y(t))_{t\geq 0}$ be a Lévy process on $\overline{\Omega}$ and let Z be a random variable with Wishart distribution $W(n/r, I_r)$ independent from Y. Then, for any fixed $p > 1$, $\frac{Y(tr(Z))}{tr(Z)^p}$ and $Y(tr(Z))$ are independent if, and only if Y is drifted stable on Ω and for all $h \in \Omega$ and $s \leq 0$*

$$L_{\langle Y(t),h\rangle}(s) = \exp\left(t\left(1 - (1 - \beta(h)s)^{1/p}\right)\right),$$

where

$$\frac{1}{\beta(h)} = \frac{n\Gamma(n)}{p\Gamma(n+p)}\mathbf{E}\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right). \tag{3.7}$$

Proof. To prove necessity: Since $Y = (Y(t))_{t\geq 0}$ is a Lévy process on Ω, then, for all $h \in \Omega$, $\langle Y(t), h\rangle$ is also a Lévy process on real line. Since Z is Wishart distributed, thus according (2.3), we can write for all $s \leq 0$

$$\mathbf{E}(e^{s\,tr(Z)}) = \int_\Omega e^{s\,tr(z)}\frac{e^{-tr(z)}}{\Gamma_\Omega(\frac{n}{r})}dz = \int_\Omega \frac{-\langle z, (1-s)I_r\rangle}{\Gamma_\Omega(\frac{n}{r})}dz = (1-s)^{-n}.$$

It follows that the variable $tr(Z)$ is gamma distributed with parameters n and 1. Using Theorem 3.2 of [10] and the fact that for all $h \in \Omega$, $\frac{\langle Y(tr(Z)),h\rangle}{tr(Z)^p}$ and $\langle Y(tr(Z)), h\rangle$ are independent, we deduce that

$$\mathbf{E}(e^{s\langle Y(t),h\rangle}) = \exp\left(t\left(1 - (1 - \beta(h)s)^{1/p}\right)\right), \tag{3.8}$$

where $\beta(h)$ is given by (3.7). This implies that $\langle Y(t), h\rangle$ is drifted stable for all $h \in \Omega$. It follows that $Y = (Y(t))_{t\geq 0}$ is a drifted stable process on Ω. As a consequence of the result (3.8), we have for all $s < 0$

$$\mathbf{E}(e^{s\langle Y(tr(Z)),h\rangle}) = (1 - \beta(h)s)^{-\frac{n}{p}} \tag{3.9}$$

and for all $a > 0$

$$\mathbf{E}\left(\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right)^a\right) = \frac{\Gamma(n/p)\Gamma(pa+n)}{\Gamma(n/p+a)\Gamma(n)\beta(h)^a}. \tag{3.10}$$

To prove sufficiency: Since for all $h \in \Omega$, and for all $s \leq 0$, we have

$$L_{\langle Y(t),h\rangle}(s) = \exp\left(t\left(1 - (1-\beta(h)s)^{1/p}\right)\right),$$

then, according to [10], we get (3.9) and (3.10). It follows that for all $h \in \Omega$ and $s \leq 0$

$$\mathbf{E}\left(\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right)^a e^{s\langle Y(tr(Z)),h\rangle}\right) = \mathbf{E}\left(\left(\frac{tr(Z)^p}{\langle Y(tr(Z)), h\rangle}\right)^a\right)\mathbf{E}\left(e^{s\langle Y(tr(Z)),h\rangle}\right).$$

This implies that the random variables $Y(tr(Z))$ and $\frac{Y(tr(Z))}{tr(Z)^p}$ are independent.

Note that for $p = 2$ and under the conditions of Theorem 3.2, the process Y follows an inverse Gaussian process on $\overline{\Omega}$. Then, the Laplace transform of $\langle Y(t), h\rangle$ is given by

$$L_{\langle Y(t),h\rangle}(s) = \exp\left(t\left(1 - \sqrt{1-\beta(h)s}\right)\right),$$

where

$$\beta(h) = \frac{2\Gamma(n+2)}{n\Gamma(n)\mathbf{E}\left(\frac{tr(Z)^2}{\langle Y(tr(Z)),h\rangle}\right)} = \frac{2\Gamma(n+2)n\langle k'_{\mu_1}(0), h\rangle}{n\Gamma(n)\sum_{\pi\in S_2} n^{m(\pi)}}.$$

4. Conclusion

In this paper, we have determined some characteristic properties of the stable processes on the cone of symmetric matrices using an independence property. More exactly, we have used the Wishart distribution as a mixing density with these stable processes and we have investigated this mixture to characterize an exponentially titled version of stable distribution called the drifted stable. These results extended the ones given by [10] and [7].

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