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The spectra and eigenvectors for the weighted mean matrix operator

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Abstract. In this paper, first we give conditions under which the weighted mean matrix operator is bounded on the weighted Hardy spaces, and we characterize the spectrum of the weighted mean matrix operator acting on some sequence spaces. Then we investigate eigenvectors of weighted mean matrix operator.

Keywords: Weighted Hardy spaces; Spectrum; Weighted mean matrix operator; Eigenvalue; Eigenvector

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1. INTRODUCTION

Let$\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and let $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series and the set of such series is denoted by $H^p(\beta)$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$.

For $1 \leq p < \infty$, $H^p(\beta) \cong l^p(\mu)$ where μ is the σ-finite measure defined on the positive integers by $\mu(K) = \Sigma_{n\in K}(\beta(n))^p$, $K \subseteq \mathbb{N}_0$, $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$.

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Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive numbers, and let $A_n = \sum_{i=0}^{n} a_i\beta(i)^p$. The weighted mean matrix operator on $H^p(\beta)$ is an infinite matrix $A = [a_{nk}]_{n,k}$ with

$$a_{nk} = \begin{cases} \dfrac{a_k\beta(n)^p}{A_n} & 0 \leq k \leq n \\ 0 & k > n. \end{cases}$$

Let X be a nontrivial complex Banach space of complex sequences. The spectrum $(\sigma(A))_X$ of a bounded operator A on X, is the set of all complex numbers λ such that the operator $A - \lambda I$ is not invertible on X. The resolvent $(\rho(A))_X$ of A is the complement of $(\sigma(A))_X$. A complex number λ is an eigenvalue of the operator A, whenever there exists a nonzero complex sequence $\{\hat{f}(n)\}_{n=0}^{\infty}$ in X such that $A\{\hat{f}(n)\} = \lambda\{\hat{f}(n)\}$. This nonzero complex sequence $\{\hat{f}(n)\}_{n=0}^{\infty}$ is called an eigenvector corresponding to λ. For some other sources on this topic see [1–9,11,10,21,13–15,19,20,16–18,12].

2. Boundedness of the weighted mean matrix operator

In this section we investigate the boundedness of the weighted mean operator matrix acting on some sequences spaces. Consider the fixed measure space $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), \mu)$ where $\mu(K) = \Sigma_{n\in K}(\beta(n))^p$. The collection of all complex sequences $f = \{\hat{f}(n)\}$ for which $\sum_{n=0}^{\infty} |\hat{f}(n)|\beta(n)^p < \infty$, will be denoted by $l^1(\mu)$. In this case put $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in l^1(\mu)$, and let

$$\|f\| = \sum_{n=0}^{\infty} |\hat{f}(n)|\beta(n)^p.$$

By the usual way we can define $l^r(\mu)$ for $r > 1$.

Lemma 2.1. *Suppose that $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$. Then $f(z) \in L^{\infty}(\mu)$ if and only if there exists $M > 0$ such that $|\hat{f}(n)| < M$ for all $n \geq 0$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in l^{\infty}(\mu)$. Observe that $|\hat{f}(n)| \leq \|f\|_{\infty}$ for all n (since by definition of $\mu(S)$, it follows that $\mu(S) = 0$ if and only if S is an empty set). So

$$\sup_{n\geq 0} |\hat{f}(n)| < \infty.$$

The converse is clear. Now the proof is complete. □

Thus $l^{\infty}(\mu)$ consists of all $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ such that $\|f\|_{\infty} = \sup_{n\geq 0} |\hat{f}(n)| < \infty$.

Corollary 2.2. *Let $1 \leq p < q \leq \infty$. Then the following statements holds*

(i) *If $\sum_{n=0}^{\infty} \beta(n) < \infty$, then $l^q(\mu) \subseteq l^p(\mu)$.*

(ii) *If $\beta(n) > 1$ for all $n \geq 0$, then $l^p(\mu) \subseteq l^q(\mu)$.*

Proof. Since $\sum_{n=0}^{\infty} \beta(n) < \infty$, there exists k such that $\beta(n) < 1$ for all $n \geq k$. Thus $\beta(n)^p < \beta(n)$ for all $n \geq k$, and so $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), \mu)$ is a finite measure space. So by Theorem 25.13 in [2], part (i) is true.

Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in L^p(\mu)$, then $\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$. So there exists k_1 such that $|\hat{f}(n)|^p \beta(n)^p \leq 1$ for all $n \geq k_1$. Since $\beta(n) > 1$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} |\hat{f}(n)|^q \beta(n)^p &= \sum_{n=0}^{\infty} |\hat{f}(n)|^q \beta(n)^q \beta(n)^{p-q} \\ &\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.\end{aligned}$$

Thus (ii) holds. This completes the proof. □

Theorem 2.3. *Let A be the weighted mean matrix operator. Assume that* $\limsup_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$. *Then A is bounded on $l^\infty(\mu)$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in l^\infty(\mu)$, thus

$$(Af)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} \hat{f}(k) \right) z^n.$$

Note that

$$\begin{aligned}|(\hat{Af})(n)| &= \left| \sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} \hat{f}(k) \right| \\ &\leq \sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} |\hat{f}(k)| \\ &\leq \sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} \|f\|_\infty\end{aligned}$$

for all $n \geq 0$. Since $\limsup_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$, there exists N such that $\beta(n) < \beta(k)$ for all $n \geq k \geq N$. Thus

$$\begin{aligned}\sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} &\leq \sum_{k=0}^{N-1} \frac{a_k \beta(n)^p}{A_n} + \sum_{k=N}^{n} \frac{a_k \beta(k)^p}{A_n} \\ &= \sum_{k=0}^{N-1} \frac{a_k \beta(k)^p}{A_k} \frac{A_k}{A_n} \frac{\beta(n)^p}{\beta(k)^p} \\ &\leq \sum_{k=0}^{N-1} \frac{\beta(n)^p}{\beta(k)^p} + 1 \\ &= \sum_{k=0}^{N-1} \frac{\beta(n)^p}{\beta(N)^p} \frac{\beta(N)^p}{\beta(k)^p} + 1 \\ &\leq \sum_{k=0}^{N-1} \frac{\beta(N)^p}{\beta(k)^p} + 1.\end{aligned}$$

So

$$|(\hat{Af})(n)| \le \left(\max \left\{ \sum_{k=0}^{m} \frac{a_k \beta(m)^p}{A_m} \right\}_{m=0}^{N-1} + \sum_{k=0}^{N-1} \frac{\beta(N)^p}{\beta(k)^p} + 1 \right) \|f\|_\infty,$$

for all $n \ge 0$. By Lemma 2.1, A is bounded on $l^\infty(\mu)$. Now the proof is complete. □

Lemma 2.4. *Suppose that* $B = [b_{nk}]$ $(n,k = 0,1,\dots)$ *is an infinite matrix of complex numbers such that* $b_{nk} \neq 0$ *for* $0 \le k \le n$*, and* $b_{nk} = 0$ *otherwise. If*

$$M_1 = \sup_{k\ge 0} \sum_{n \ge k} |b_{nk}| \left(\frac{\beta(n)}{\beta(k)} \right)^p < \infty,$$

then B *is a bounded operator on* $l^1(\mu)$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in l^1(\mu)$. Then

$$(Bf)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{nk} \hat{f}(k) \right) z^n.$$

We have

$$\begin{aligned}
\sum_{n\ge 0} |(\hat{Bf})(n)| \beta(n)^p &\le \sum_{n \ge 0} \sum_{k=0}^{n} |b_{nk}||\hat{f}(k)|\beta(n)^p \\
&= \sum_{n \ge 0} \sum_{k=0}^{n} |b_{nk}||\hat{f}(k)|\beta(k)^p \left(\frac{\beta(n)}{\beta(k)} \right)^p \\
&= \sum_{k \ge 0} |\hat{f}(k)|\beta(k)^p \left(\sum_{n=k}^{\infty} |b_{nk}| \left(\frac{\beta(n)}{\beta(k)} \right)^p \right) \\
&\le M_1 \|f\|.
\end{aligned}$$

Thus A is bounded on $l^1(\mu)$. Now the proof is complete. □

Theorem 2.5. *Let* $\liminf_{n\to\infty} \frac{a_n \beta(n)^p}{A_n} = \delta$*, where* $0 < \delta \le 1$*, and* $\limsup_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$*. Thus the weighted mean matrix operator is bounded on* $l^1(\mu)$.

Proof. By Lemma 2.4, it is sufficient to show that

$$\sup_{k \ge 0} \sum_{n \ge k} \frac{a_k \beta(n)^p}{A_n} \left(\frac{\beta(n)}{\beta(k)} \right)^p < \infty.$$

Since $\liminf_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = \delta$, there exists N_1 such that $\frac{a_n\beta(n)^p}{A_n} > \frac{\delta}{2}$ for all $n \ge N_1$. Thus

$$\left(1 - \frac{a_{n+1}\beta(n+1)^p}{A_{n+1}} \right) \le \left(1 - \frac{\delta}{2} \right) < 1,$$

for all $n \geq N_1$. For $k > N_1$ and $j \geq 1$, we have

$$\begin{aligned}\frac{a_k\beta(k)^p}{A_{k+j}} &= \frac{a_k\beta(k)^p}{A_k}\prod_{i=1}^{j}\left(1-\frac{a_{k+i}\beta(k+i)^p}{A_{k+i}}\right)\\ &\leq \frac{a_k\beta(k)^p}{A_k}\left(1-\frac{\delta}{2}\right)^j.\end{aligned}$$

So

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{a_k\beta(k)^p}{A_{k+j}} &\leq \frac{a_k\beta(k)^p}{A_k}\sum_{j=0}^{\infty}\left(1-\frac{\delta}{2}\right)^j\\ &\leq \sum_{j=0}^{\infty}\left(1-\frac{\delta}{2}\right)^j.\end{aligned}$$

Thus $\sum_{j=0}^{\infty}\frac{a_k\beta(k)^p}{A_{k+j}} < \infty$ for all $k+j \geq N_1$. Since $\limsup_{n\to\infty}\frac{\beta(n+1)}{\beta(n)} < 1$, there exists N_2 such that $\beta(n) < \beta(k)$ for all $n \geq k \geq N_2$. If $k \geq \max\{N_1, N_2\}$, then

$$\begin{aligned}\sum_{n\geq k}\frac{a_k\beta(k)^p}{A_n}\left(\frac{\beta(n)}{\beta(k)}\right)^{2p} &\leq \sum_{j=0}^{\infty}\frac{a_k\beta(k)^p}{A_{k+j}}\left(\frac{\beta(k+j)}{\beta(k)}\right)^{2p}\\ &\leq \sum_{j=0}^{\infty}\frac{a_k\beta(k)^p}{A_{k+j}}.\end{aligned}$$

Thus

$$\sup_{k\geq 0}\sum_{n\geq k}\frac{a_k\beta(n)^p}{A_n}\left(\frac{\beta(n)}{\beta(k)}\right)^p < \infty.$$

Thus A is bounded on $l^1(\mu)$. Now the proof is complete. □

3. THE SPECTRA FOR THE WEIGHTED MEAN MATRIX OPERATOR

In this section we investigate the spectra for the weighted mean matrix operator acting on some sequence spaces.

Lemma 3.1 ([4, *Lemma 1*]). *Let A be a weighted mean matrix operator, and define $T = A - \lambda I$ where $\lambda \in \mathbb{C}$ such that $a_{nn} \neq \lambda$ for each $n \geq 0$. Then $S(=[s_{nk}]) = T^{-1}$ exists and,*

$$s_{nk} = \begin{cases} \dfrac{-1}{\lambda^2}\dfrac{a_k\beta(n)^p}{A_n}\dfrac{1}{\prod_{j=k}^{n}(1-\frac{a_j\beta(j)^p}{\lambda A_j})} & 0 \leq k < n\\[2ex] \dfrac{1}{\frac{a_n\beta(n)^p}{A_n}-\lambda} & k = n\\ 0 & k > n.\end{cases}$$

Theorem 3.2. *Let* $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = \delta$ *where* $0 < \delta \leq 1$ *and let* $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$. *Then*

$$(\sigma(A))_{L^1(\mu)} \subseteq \left\{\lambda : \left|\lambda - \frac{1}{2-\delta}\right| \leq \frac{1-\delta}{2-\delta}\right\} \cup \left\{\frac{a_n\beta(n)^p}{A_n}\right\}_{n=0}^{\infty}.$$

Proof. Let $0 < \delta < 1$ and choose $0 < \epsilon < \delta - \frac{\delta}{2-\delta}$. Since $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$, there exists N such that

$$\delta - \epsilon < \frac{a_n\beta(n)^p}{A_n}, \qquad \frac{\beta(n+1)}{\beta(n)} < 1$$

for all $n \geq N$. Notice that we have

$$-\frac{1-\delta}{2-\delta} \leq \frac{a_n\beta(n)^p}{A_n} - \frac{1}{2-\delta} \leq \frac{1-\delta}{2-\delta},$$

for all $n \geq N$. Suppose λ is a complex number such that $\lambda \neq \frac{a_n\beta(n)^p}{A_n}$ for all $n \geq 0$, and $|\lambda - \frac{1}{2-\delta}| > \frac{1-\delta}{2-\delta}$. If $\delta < x \leq 1$, then $x \in D_\delta$, where

$$D_\delta = \left\{\lambda : \left|\lambda - \frac{1}{2-\delta}\right| \leq \frac{1-\delta}{2-\delta}\right\}.$$

Since $\lambda \notin D_\delta$, thus $|\lambda - x| \neq 0$. Define the function f by $f(x) = \frac{|\lambda|(1-x)}{|\lambda - x|}$ for all $\delta - \epsilon \leq x \leq 1$. By Lemma 4.9 in [4], f is continuous and $0 \leq f(x) < 1$. So there exists $0 < M_1 < 1$ such that $f(x) \leq M_1$ for all $\delta - \epsilon \leq x \leq 1$. If $k \geq N$, then $\frac{a_n\beta(n)^p}{A_n} \in D_\delta$ for all $n \geq k$. So

$$\begin{aligned}\left|\left(\frac{\beta(n+1)}{\beta(n)}\right)^p \frac{s_{(n+1)k}}{s_{nk}}\right| &\leq \frac{|\lambda|(1 - \frac{a_{n+1}\beta(n+1)^p}{A_{n+1}})}{|\lambda - \frac{a_{n+1}\beta(n+1)^p}{A_{n+1}}|} \\ &= f\left(\frac{a_{n+1}\beta(n+1)^p}{A_{n+1}}\right) \\ &\leq M_1.\end{aligned}$$

Now by the relation $|\lambda - \frac{a_n\beta(n)^p}{A_n}| > |\lambda - \frac{1}{2-\delta}| - \frac{1-\delta}{2-\delta} = M_2$ we get

$$\begin{aligned}\sum_{n=k+1}^{\infty} |s_{nk}| \frac{\beta(n)^p}{\beta(k)^p} &\leq \left(\frac{\beta(k+1)}{\beta(k)}\right)^p \frac{|s_{(k+1)k}|}{1-M} \\ &= \frac{1}{1-M} \frac{a_k\beta(k)^p}{A_k} \frac{1}{|\lambda - \frac{a_k\beta(k)^p}{A_k}|} \\ &\times \frac{1}{|\lambda - \frac{a_{k+1}\beta(k+1)^p}{A_{k+1}}|} \left(\frac{\beta(k+1)}{\beta(k)}\right)^{2p} \\ &\leq \frac{1}{(1-M_1)M_2^2}\end{aligned}$$

for all $k \geq N$. So by Lemma 2.4, $S \in B(l^1(\mu))$, i.e. $\lambda \in (\rho(A))_{l^1(\mu)}$.

Suppose that $\delta = 1$, and let λ be a complex number such that $\lambda = \lambda_1 + i\lambda_2$. Without loss of generality we can say that $0 < \lambda_1 < 1$ and $\lambda \neq \frac{a_n \beta(n)^p}{A_n}$ for all $n \geq 0$, and $|\lambda - 1| > 0$. Then for $\epsilon_1 < |\lambda - 1|$ and $1 - \epsilon_1 < x \leq 1$, we get $|x - 1| < |\lambda - 1|$, i.e. $|\lambda - x| \neq 0$. We know that $\frac{1}{2-x}$ and $\frac{x}{2-x}$ are two increasing functions, so there exists $\epsilon_2 > 0$ such that $0 < \lambda_1 < \frac{1}{2-x}$ and $\lambda_1 < \frac{x}{2-x}$ for all $1 - \epsilon_2 \leq x \leq 1$. Thus we obtain

$$\begin{aligned} \left|\lambda - \frac{1}{2-x}\right| &\geq \left|\lambda_1 - \frac{1}{2-x}\right| \\ &> \frac{1-x}{2-x}. \end{aligned}$$

Hence $\lambda \notin D_x$. If $\epsilon < \min\{\epsilon_1, \epsilon_2\}$, then f is continuous and $0 \leq f(x) < 1$ for all $1 - \epsilon \leq x \leq 1$. By a similar method used earlier we get $S \in B(l^1(\mu))$, i.e. $\lambda \in (\rho(A))_{l^1(\mu)}$. Thus the proof is complete. □

Lemma 3.3. *Let C be a space of convergent sequences and $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$. Thus the weighted mean matrix map $A : C \longrightarrow C$ is a bounded operator.*

Proof. Since $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$, so there exists a natural number N_1 such that $\beta(n+1) < \beta(n)$ for all $n \geq N_1$. If $x = \{x_n\} \in C$ and $\|x\|_\infty \leq 1$, then

$$\begin{aligned} |A(x)_n| &\leq \sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} |x_k| \\ &\leq \sum_{k=0}^{N_1-1} \frac{a_k \beta(n)^p}{A_n} |x_k| + \sum_{k=N_1}^{n} \frac{a_k \beta(k)^p}{A_n} \frac{\beta(n)^p}{\beta(k)^p} |x_k| \\ &\leq \sum_{k=0}^{N_1-1} \frac{a_k \beta(n)^p}{A_n} + 1 \\ &= \sum_{k=0}^{N_1-1} \frac{a_k \beta(k)^p}{A_k} \frac{A_k}{A_n} \frac{\beta(n)^p}{\beta(k)^p} + 1 \\ &\leq \sum_{k=0}^{N_1-1} \frac{\beta(n)^p}{\beta(k)^p} + 1 \\ &= \sum_{k=0}^{N_1-1} \frac{\beta(n)^p}{\beta(N_1)^p} \frac{\beta(N_1)^p}{\beta(k)^p} + 1 \\ &\leq \sum_{k=0}^{N_1-1} \frac{\beta(N_1)^p}{\beta(k)^p} + 1 \end{aligned}$$

for all $n \geq N_1$. Therefore

$$\|A\|_\infty \leq \left(\max\left\{ \sum_{k=0}^{m} \frac{a_k\beta(m)^p}{A_m} \right\}_{m=0}^{N_1-1} + \sum_{k=0}^{N_1-1} \frac{\beta(N_1)^p}{\beta(k)^p} + 1 \right).$$

This completes the proof. □

Theorem 3.4. *Let* $A : C \longrightarrow C$ *be a linear map, and* $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = \delta$ *where* $0 < \delta \leq 1$*. Assume that* $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$*. Then*

$$(\sigma(A))_C \subseteq \left\{ \lambda : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2} \right\}.$$

Proof. Suppose that $|\lambda - \frac{1}{2}| > \frac{1}{2}$. This inequality is equivalent to $\alpha > -1$, where $\frac{-1}{\lambda} = \alpha + i\beta$. Since $0 < \frac{a_n\beta(n)^p}{A_n} \leq 1$, we get

$$\begin{aligned} \left|1 - \frac{a_n\beta(n)^p}{\lambda A_n}\right| &> 1 + \alpha\frac{a_n\beta(n)^p}{A_n} \\ &> 1 - \frac{a_n\beta(n)^p}{A_n}, \end{aligned}$$

for all $n \geq 0$. Without loss of generality, suppose that $\beta(n)$ is a decreasing sequence, $\beta(n+1) \leq \beta(n)$ for all $n \geq 0$.

Consider $0 < \delta < 1$. Since

$$\frac{a_n\beta(n)^p}{A_{n-1}} = \frac{\frac{a_n\beta(n)^p}{A_n}}{1 - \frac{a_n\beta(n)^p}{A_n}},$$

is a monotone increasing sequence, there exists natural number N_1 such that

$$\frac{a_n\beta(n)^p}{A_{n-1}} < \left(\frac{\delta}{1-\delta}\right) + 1,$$

for all $n \geq N_1$. We have

$$\begin{aligned} \sum_{k=0}^{n} |s_{nk}| &\leq |s_{nn}| + \sum_{k=0}^{n-1} \frac{a_k\beta(k)^p}{A_n|\lambda|^2 \prod_{j=k}^{n}(1 + \alpha\frac{a_j\beta(j)^p}{A_j})} \\ &= |s_{nn}| + \frac{A_{n-1}}{A_n|\lambda|^2(1+\alpha)(1 + \alpha\frac{a_n\beta(n)^p}{A_n})} \\ &\leq \frac{1}{|\lambda|(1 + \alpha\frac{a_n\beta(n)^p}{A_n})} + \frac{A_{n-1}}{A_n|\lambda|^2(1+\alpha)(1 + \alpha\frac{a_n\beta(n)^p}{A_n})} \\ &\leq \left(1 + \frac{a_n\beta(n)^p}{A_{n-1}}\right)\left(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^2(1+\alpha)}\right) \\ &\leq \left(1 + \left(\frac{\delta}{1-\delta}\right) + 1\right)\left(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^2(1+\alpha)}\right). \end{aligned}$$

If $\delta = 1$, then there is a natural number N_2 such that

$$\frac{1}{2} < \frac{a_n\beta(n)^p}{A_n} \le 1.$$

When $0 \le \alpha$, we have $1 + \frac{\alpha}{2} < 1 + \alpha\frac{a_n\beta(n)^p}{A_n} \le 1 + \alpha$, and if $-1 < \alpha < 0$, then $1 + \alpha < 1 + \alpha\frac{a_n\beta(n)^p}{A_n} \le 1 + \frac{\alpha}{2}$, for all $n \ge N_2$. So

$$\begin{aligned}
\sum_{k=0}^{n} |s_{nk}| &\le |s_{nn}| + \sum_{k=0}^{n-1} \frac{a_k\beta(k)^p}{A_n|\lambda|^2 \prod_{j=k}^{n}(1 + \alpha\frac{a_j\beta(j)^p}{A_j})} \\
&= |s_{nn}| + \frac{A_{n-1}}{A_n|\lambda|^2(1+\alpha)(1 + \alpha\frac{a_n\beta(n)^p}{A_n})} \\
&\le \frac{1}{|\lambda|(1 + \alpha\frac{a_n\beta(n)^p}{A_n})} + \frac{A_{n-1}}{A_n|\lambda|^2(1+\alpha)(1 + \alpha\frac{a_n\beta(n)^p}{A_n})} \\
&\le M_3\left(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^2(1+\alpha)}\right),
\end{aligned}$$

where $M_3 = \text{Max}\{1 + \frac{\alpha}{2}, 1 + \alpha\}$.

Since

$$\begin{aligned}
\left|\frac{s_{n+1k}}{s_{nk}}\right| &= \left(\frac{\beta(n+1)}{\beta(n)}\right)^p \left(\frac{A_n}{A_{n+1}}\right)\left(\frac{1}{|1 - \frac{a_{n+1}\beta(n+1)^p}{\lambda A_{n+1}}|}\right) \\
&< \frac{A_n}{A_{n+1}} \frac{1}{1 - \frac{a_{n+1}\beta(n+1)^p}{A_{n+1}}} \\
&= 1,
\end{aligned}$$

for all $n, k \ge 0$. Therefore S has bounded columns. By Proof of Theorem 3 in [5] we get $\lambda \in (\rho(A))_c$. Now the proof is complete. $\square$

Theorem 3.5. *Let $A : C \longrightarrow C$ be a linear map, and $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = \delta$ where $0 < \delta \le 1$. If $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$, then*

$$(\sigma(A))_c \subseteq \left\{\lambda; \left|\lambda - \frac{1}{2-\delta}\right| \le \frac{1-\delta}{2-\delta}\right\} \cup \left\{\frac{a_n\beta(n)^p}{A_n}\right\}_{n=0}^{\infty}.$$

Proof. Let λ with $\frac{-1}{\lambda} = \alpha + i\beta$, be a fixed complex number which satisfies $|\lambda - \frac{1}{2-\delta}| > \frac{1-\delta}{2-\delta}$. Theorem 3.4 implies that $|\lambda - \frac{1}{2}| \le \frac{1}{2}$. If $\lambda \ne \frac{a_n\beta(n)^p}{A_n}$ for all $n \ge 0$, then $\alpha < -1$. Since

$$\left|1 - \frac{a_j\beta(j)^p}{\lambda A_j}\right| = \left|1 + \left(1 - \frac{1}{\lambda}\right)\frac{a_j\beta(j)^p}{A_{j-1}}\right| \frac{A_{j-1}}{A_j},$$

we see that

$$\begin{aligned}|s_{nk}| &= \frac{a_k\beta(n)^p}{A_n|\lambda|^2\prod_{j=k}^{n}|1-\frac{a_j\beta(j)^p}{\lambda A_j}|}\\ &= \frac{a_k\beta(n)^p}{A_{k-1}|\lambda|^2\prod_{j=k}^{n}|1+(1-\frac{1}{\lambda})\frac{a_j\beta(j)^p}{A_{j-1}}|}.\end{aligned} \tag{*}$$

Define

$$\begin{aligned}f(t) &= \left|1+\left(1-\frac{1}{\lambda}\right)t\right|^2\\ &= 1+2(1+\alpha)t+((1+\alpha)^2+\beta^2)t^2,\end{aligned}$$

for $t \geq 0$. It is clear that $f(t)$ at $t_0 = \frac{-(1+\alpha)}{(1+\alpha)^2+\beta^2}$ has a Minimum and $f(t_1) = 1$ where $t_1 = \frac{-2(1+\alpha)}{(1+\alpha)^2+\beta^2}$. If $0 < \delta < 1$, then we get $t_1 < \frac{\delta}{1-\delta}$ and

$$\begin{aligned}f\left(\frac{\delta}{1-\delta}\right) &= \left|1+\left(1-\frac{1}{\lambda}\right)t\right|^2\\ &= \left|\frac{1}{1-\delta}-\frac{\delta}{\lambda(1-\delta)}\right|^2\\ &> 1.\end{aligned}$$

Let $0 < \epsilon < \frac{\delta}{1-\delta} - t_1$, thus there exists natural number N_1 such that $\frac{\beta(n+1)}{\beta(n)} < 1$ and

$$\left(\frac{\delta}{1-\delta}\right) - \epsilon < \frac{a_n\beta(n)^p}{A_{n-1}} < \frac{\delta}{1-\delta} + 1,$$

for all $n \geq N_1$. Note that $f(t)$ is a continuous function so there is a positive number M_1 such that

$$\left|1+\left(1-\frac{1}{\lambda}\right)\frac{a_{n+1}\beta(n+1)^p}{A_n}\right| > M_1 > 1,$$

for all $n \geq N_1$. Using (*),

$$\begin{aligned}\left|\frac{s_{n+1k}}{s_{nk}}\right| &\leq \left(\frac{\beta(n+1)}{\beta(n)}\right)^p \frac{1}{|1+(1-\frac{1}{\lambda})\frac{a_{n+1}\beta(n+1)^p}{A_n}|}\\ &< 1,\end{aligned}$$

for all $n \geq N_1, k \geq 0$. Therefore S has bounded columns. From (*),

$$\sum_{k=0}^{n}|s_{nk}| \leq \frac{1+\frac{a_n\beta(n)^p}{A_{n-1}}}{|\lambda||1+(1-\frac{1}{\lambda})\frac{a_n\beta(n)^p}{A_{n-1}}|} + \sum_{k=0}^{n-1}\frac{a_k\beta(n)^p}{A_{k-1}|\lambda|^2\prod_{j=k}^{n}|1+(1-\frac{1}{\lambda})\frac{a_j\beta(j)^p}{A_{j-1}}|}$$

$$\leq \frac{1+\frac{\delta}{1-\delta}+1}{|\lambda|} + \frac{\beta(N_1)^p}{A_0}\sum_{k=0}^{N_1-1}\frac{a_k}{|\lambda|^2 M_2 M_1^{n-N_1+1}}$$

$$+ \sum_{k=N_1}^{n-1}\frac{a_k\beta(k)^p}{A_{k-1}|\lambda|^2\prod_{j=k}^{n}|1+(1-\frac{1}{\lambda})\frac{a_j\beta(j)^p}{A_{j-1}}|}$$

$$\leq \frac{1+\frac{\delta}{1-\delta}+1}{|\lambda|} + \frac{\beta(N_1)^p}{A_0}\sum_{k=0}^{N_1-1}\frac{a_k}{|\lambda|^2 M_2 M_1^{n-N_1+1}}$$

$$+ \frac{1}{|\lambda|^2}\left(\frac{\delta}{1-\delta}+1\right)\sum_{k=N_1}^{n-1}\frac{1}{M_1^{n-k+1}},$$

where $M_2 = \min\{\prod_{j=k}^{N_1}|1+(1-\frac{1}{\lambda})\frac{a_j\beta(j)^p}{A_{j-1}}|\}_{k=0}^{N_1}$. By the proof of Theorem 3 of [5], $\lambda \in (\rho(A))_C$.

If $\delta = 1$, then $\lim_{n\to\infty}\frac{a_n\beta(n)^p}{A_{n-1}} = \infty$. Since $\lim_{t\to\infty} f(t) = \infty$, there exists $M_2 > 0$, such that $f(M_2) > 1$. This implies that there is a natural number N_2 such that $\frac{a_n\beta(n)^p}{A_{n-1}} > M_2$, and $f(\frac{a_n\beta(n)^p}{A_{n-1}}) > 1$ for all $n \geq N_2$. Therefore

$$\left|1+\left(1-\frac{1}{\lambda}\right)\frac{a_{n+1}\beta(n+1)^p}{A_n}\right| > f(M_2) > 1,$$

and

$$\left|\frac{s_{n+1k}}{s_{nk}}\right| = \left(\frac{\beta(n+1)}{\beta(n)}\right)^p \frac{1}{|1+(1-\frac{1}{\lambda})\frac{a_{n+1}\beta(n+1)^p}{A_n}|}$$
$$< 1,$$

and hence S has bounded columns. With the relation

$$(1+2(1+\alpha)t+((1+\alpha)^2+\beta^2)t^2)^{\frac{1}{2}} \simeq ((1+\alpha)^2+\beta^2)^{\frac{1}{2}}t,$$

when $t \longrightarrow \infty$, we obtain

$$\frac{1+\frac{a_n\beta(n)^p}{A_{n-1}}}{|1+(1-\frac{1}{\lambda})\frac{a_n\beta(n)^p}{A_{n-1}}|} \longrightarrow \frac{1}{((1+\alpha)^2+\beta^2)^{\frac{1}{2}}},$$

as $n \longrightarrow \infty$. So there is a natural number N_3 such that

$$\frac{1+\frac{a_n\beta(n)^p}{A_{n-1}}}{|1+(1-\frac{1}{\lambda})\frac{a_n\beta(n)^p}{A_{n-1}}|} < \frac{1}{((1+\alpha)^2+\beta^2)^{\frac{1}{2}}} + 1,$$

for all $n \geq N_3$. Let $n \geq \mathrm{Max}\{N_2, N_3\}$. Then by a similar way, $\sum_{k=0}^{\infty} |s_{nk}| < \infty$. The Proof of Theorem 3 in [5] shows that $\lambda \in (\rho(A))_C$. Now the proof is complete. □

Corollary 3.6. *Let* $A : C \longrightarrow C$ *be a linear map, and* $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = \delta$ *where* $0 < \delta \leq 1$. *Assume that* $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$. *Then*

$$(\sigma(A))_{l^\infty(\mu)} \subseteq \left\{\lambda; \left|\lambda - \frac{1}{2-\delta}\right| \leq \frac{1-\delta}{2-\delta}\right\} \cup \left\{\frac{a_n\beta(n)^p}{A_n}\right\}_{n=0}^{\infty}.$$

Proof. By Theorem 2.3 $A \in B(l^\infty(\mu))$. So A is an element of $\Gamma(l^\infty(\mu))$, the Banach algebra of bounded, infinite matrices over $l^\infty(\mu)$. Let $\Gamma(c)$ be the Banach algebra of bounded, infinite matrices over c, thus it is a closed subalgebra of $\Gamma_{l^\infty(\mu)}$ which contains identity matrices. By Lemma 4.8 of [4], $(\sigma(A))_{l^\infty(\mu)} \subseteq (\sigma(A))_c$. Now the proof is complete. □

Proposition 3.7. *Let* $A : C \longrightarrow C$ *be a linear map, and* $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = \delta$, *where* $0 < \delta \leq 1$. *If* $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$, *then*

$$(\sigma(A))_{l^p(\mu)} \subseteq \left\{\lambda; \left|\lambda - \frac{1}{2-\delta}\right| \leq \frac{1-\delta}{2-\delta}\right\} \cup \left\{\frac{a_n\beta(n)^p}{A_n}\right\}_{n=0}^{\infty}.$$

Proof. By Theorems 2.3 and 2.5 we get $A \in B(l^1(\mu)) \cap B(l^\infty(\mu))$. If $\lambda \in (\rho(A))_{l^1(\mu)} \cap (\rho(A))_{l^\infty(\mu)}$, then the infinite matrix $S = (A - \lambda I)^{-1} \in B(l^1(\mu)) \cap B(l^\infty(\mu))$. Thus by the Riesz–Thorin Theorem, $S \in B(l^p(\mu))$. So $\lambda \in (\rho(A))_{l^p(\mu)}$, i.e. $(\sigma(A))_{l^p(\mu)} \subseteq (\sigma(A))_{l^1(\mu)} \cup (\sigma(A))_{l^\infty(\mu)}$. Thus the proof is complete. □

4. EIGENVECTORS AND DIAGONALIZATION OF THE WEIGHTED MEAN MATRIX OPERATOR

In this section we first characterize the eigenvectors and investigate the diagonalizability of the weighted mean matrix operator acting on weighted Hardy spaces.

Theorem 4.1. *Let* $A : C \longrightarrow C$ *be a linear map, and* $\lim_{n\to\infty} \frac{a_n\beta(n)^p}{A_n} = 1$. *If* $\lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)} < 1$, *then*

(i) *A is bounded on $H^p(\beta)$ and* $(\sigma(A))_{l^p(\mu)} = \{\frac{a_n\beta(n)^p}{A_n}\}_{n=0}^{\infty}$.

(ii) *For every* $j \geq 0$, $c_j = \frac{a_j\beta(j)^p}{A_j}$ *is an eigenvalue of A, the weighted mean matrix operator.*

(iii) *If* $\sum_{k=0}^{\infty}\beta(k)<\infty$, *then* $f(z)=\sum_{k=0}^{\infty}\beta(k)^p z^k\in L^p(\mu)$ *is the eigenvector for the eigenvalue* $c_0=1$.

Proof. By the Proposition 3.7, A is bounded on $H^p(\beta)$, and

$$(\sigma(A))_{L^p(\mu)}\subseteq\left\{\frac{a_n\beta(n)^p}{A_n}\right\}_{n=0}^{\infty}.$$

Theorem 3 of [3] implies that if $\lambda=\frac{a_n\beta(n)^p}{A_n}$, $n\geq 0$, then $\det(\lambda I-A)=0$. So (i) holds.

Suppose that $c_j=\lambda=\frac{a_j\beta(j)^p}{A_j}$, and $j\in\mathbb{N}$, we will show that there exists $f_\lambda(z)=\sum_{j=0}^{\infty}\widehat{f_\lambda}(j)z^j\in H^p(\beta)$, such that

$$A(f_\lambda)=\lambda(f_\lambda).$$

Define $\widehat{f_\lambda}(i)=0$ for $0\leq i<j$, and $\widehat{f_\lambda}(j)=a>0$. From $(\widehat{A(f_\lambda)})(j+1)=c_j\widehat{f_\lambda}(j+1)$, we have

$$\begin{aligned}(c_j-c_{j+1})\hat{f}_\lambda(j+1)&=\frac{a_j\beta(j+1)^p}{A_{j+1}}\widehat{f_\lambda}(j)\\&=\left(\frac{\beta(j+1)}{\beta(j)}\right)^p\frac{a_j\beta(j)^p}{A_{j+1}}\widehat{f_\lambda}(j)\\&=\left(\frac{\beta(j+1)}{\beta(j)}\right)^p c_j(1-c_{j+1})\widehat{f_\lambda}(j),\end{aligned}$$

thus

$$\widehat{f_\lambda}(j+1)=\left(\frac{\beta(j+1)}{\beta(j)}\right)^p\frac{(1-c_{j+1})}{(1-\frac{c_{j+1}}{c_j})}\widehat{f_\lambda}(j).$$

Let

$$\widehat{f_\lambda}(j+k)=\left(\frac{\beta(j+k)}{\beta(j)}\right)^p\left(\prod_{i=1}^{k}\frac{(1-c_{j+i})}{(1-\frac{c_{j+i}}{c_j})}\right)\widehat{f_\lambda}(j),$$

thus

$$(\widehat{A(f_\lambda)})(j+k+1)=c_j\widehat{f_\lambda}(j+k+1),$$

which implies that

$$\begin{aligned}&(c_j-c_{j+k+1})\widehat{f_\lambda}(j+k+1)\\&\quad=\sum_{i=1}^{k}\left(\frac{\beta(j+i)}{\beta(j)}\right)^p\frac{a_{j+i}\beta(j+k+1)^p}{A_{j+k+1}}\left(\prod_{l=1}^{i}\frac{(1-c_{j+l})}{(1-\frac{c_{j+l}}{c_j})}\right)\widehat{f_\lambda}(j)\end{aligned}$$

$$= \left(\frac{\beta(j+k+1)}{\beta(j)}\right)^p c_j^{k+1} \frac{\prod_{i=1}^{k+1}(1-c_{j+i})}{\prod_{i=1}^{k}(c_i - c_{j+i})} \widehat{f_\lambda}(j)$$

$$= \frac{\beta(j+k+1)^p}{\beta(j)^p} \left(\prod_{i=1}^{k+1} \frac{(1-c_{j+i})}{(1-\frac{c_{j+i}}{c_j})}\right) \widehat{f_\lambda}(j).$$

Thus $f_\lambda(z) = \sum_{j=0}^{\infty} \widehat{f_\lambda}(j) z^j$, is a formal power series such that $A(f_\lambda) = \lambda(f_\lambda)$.

For every $j \in \mathbb{N}$, put $\epsilon = 1 - \frac{2}{1+\frac{1}{c_j}} > 0$, so there exists a natural number N_1 such that

$$c_n > 1 - \epsilon = \frac{2c_j}{1+c_j}, \qquad \frac{\beta(n+1)^p}{\beta(n)^p} < 1$$

for all $n \geq N_1$. Let $m > 0$ such that $j + m > N_1$, thus

$$\frac{c_{j+m+1}}{c_j} - 1 > \frac{1-c_j}{1+c_j}, \qquad 1 - c_{j+m+1} < \frac{1-c_j}{1+c_j}.$$

Consider

$$\frac{|\widehat{f_\lambda}(j+m+1)|^p \beta(j+m+1)^p}{|\widehat{f_\lambda}(j+m)|^p \beta(j+m)^p} = \frac{\beta(j+m+1)^{2p}}{\beta(j+m)^{2p}} \left(\frac{1-c_{j+m+1}}{\frac{c_{j+m+1}}{c_j} - 1}\right)^p$$
$$< \left(\frac{1-c_{j+m+1}}{\frac{c_{j+m+1}}{c_j} - 1}\right)^p$$
$$< 1.$$

The formal power series $\sum_{k=1}^{\infty} \hat{f}_\lambda(k) z^k$ is in $H^p(\beta)$. Therefore (ii) holds.

Note that $\sum_{k=0}^{\infty} \beta(k) < \infty$, it is clear that $f_0(z) = \sum_{k=0}^{\infty} \beta(k)^p z^k \in L^p(\mu)$, thus

$$(\widehat{A(f_0)})(n) = \sum_{k=0}^{n} \frac{a_k \beta(n)^p}{A_n} \beta(k)^p$$
$$= \beta(n)^p,$$

so (iii) holds. Now the proof is complete. □

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