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The (exponential) multipartitional polynomials and polynomial sequences of multinomial type, Part II

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Abstract. We establish recursiveness properties for multipartitional polynomials and their connection with the derivatives of polynomials of multinomial type. Various comprehensive examples are illustrated.

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1. Introduction

In a precedent work [6], the authors established identities on multipartitional polynomials, generalizing [1]. They give a result concerning the preservation of sequences of multinomial type. These results appear as natural extension of Bell polynomials and of polynomials of binomial type (see for instance [2–4]). Finally, they give some connection between these two concepts.

Let us introduce, as in [6], some definitions and notations.

For $m, m_1, \ldots, m_r$ integers such that $\sum_{i=1}^{r} m_i = m$, we define

$$\binom{m}{m_1, \ldots, m_r} = \begin{cases} \frac{m!}{m_1! \cdots m_r!} & \text{if } m_i \geqslant 0, \quad i = 1, \ldots, r, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{R}$ the set of real numbers. We use, the following notations, for $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$, $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{R}^r$ we set

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$\mathbf{a}.b = a_1b_1 + \cdots + a_rb_r, \mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_r + b_r), \lambda\mathbf{a} = (\lambda\, a_1, \ldots, \lambda a_r)$,
$(\mathbf{a} \geqslant \mathbf{b}) \iff (a_1 \geqslant b_1, \ldots, a_r \geqslant b_r), (\mathbf{a} > \mathbf{b}) \iff (a_1 > b_1, \ldots, a_r > b_r)$,

$\mathbf{1}_{(\mathbf{a}\geqslant\mathbf{b})} = \begin{cases} 1 & \text{if } \mathbf{a} \geqslant \mathbf{b}, \\ 0 & \text{otherwise,} \end{cases}$ for $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ we set

$\mathbf{t}^{\mathbf{n}} = t_1^{n_1} \cdots t_r^{n_r}, \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}, \mathbf{n}! = n_1! \cdots n_r!, |\mathbf{n}| = n_1 + \cdots + n_r, \mathbf{1} = (1, \ldots, 1)$,

$\binom{\mathbf{n}}{\mathbf{i}}_\pi = \binom{n_1}{i_1} \cdots \binom{n_r}{i_r}, \binom{\mathbf{n}}{\mathbf{i}, \mathbf{j}, \ldots, \mathbf{k}}_\pi = \binom{n_1}{i_1, j_1, \ldots, k_1} \cdots \binom{n_r}{i_r, j_r, \ldots, k_r}$,

$B_{\mathbf{n},k}(x_{\mathbf{i}}) = B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r})$, $\mathbf{n} \geqslant \mathbf{0}, k \geqslant 0$ and $=0$ otherwise,
$A_{\mathbf{n}}(x_{\mathbf{i}}) = A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r})$, $\mathbf{n} \geqslant \mathbf{0}$ and $=0$ otherwise,
$f_{\mathbf{n}}(x) = f_{n_1,\ldots,n_r}(x)$ if $\mathbf{n} \geqslant \mathbf{0}$, $f_{\mathbf{0}}(x) = 1$ and $=0$ otherwise, for $(m, n) \in \mathbb{N}^2$ we set
$B_{m,n;k}(x_{i,j}) = B_{m,n;k}(x_{0,1}, x_{1,0}, \ldots, x_{m,n})$,
$A_{m,n}(x_{i,j}) = A_{m,n}(x_{0,1}, x_{1,0}, \ldots, x_{m,n})$ and we set also $D_{z=0} = \frac{d}{dz}\big|_{z=0}$.

The complete (exponential) multipartitional polynomials $A_{\mathbf{n}}$ in the variables ($x_{\mathbf{i}}$, $\mathbf{i} \neq \mathbf{0}$), are defined by the sum

$$A_{\mathbf{n}}(x_{\mathbf{i}}) := \sum \frac{\mathbf{n}!}{\prod_{\mathbf{i}=\mathbf{0},\mathbf{i}\neq\mathbf{0}}^{\mathbf{n}} k_{\mathbf{i}}!} \prod_{\mathbf{i}=\mathbf{0},\mathbf{i}\neq\mathbf{0}}^{\mathbf{n}} \left(\frac{x_{\mathbf{i}}}{\mathbf{i}!}\right)^{k_{\mathbf{i}}}, \tag{1}$$

where the summation is extended over all partitions of the multipartite number $\mathbf{n} = (n_1, \ldots, n_r)$, that is, over all nonnegative integers $(k_{\mathbf{i}}, \mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n})$ solution of the equations

$$\sum\nolimits_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} i_j k_{\mathbf{i}} = n_j, \ j = 1, \ldots, r, \quad \text{with convention } k_{\mathbf{0}} = 0. \tag{2}$$

The partial (exponential) multipartitional polynomials $B_{\mathbf{n},k}$ in the variables $(x_{\mathbf{i}}, \mathbf{i} \neq \mathbf{0})$, of degree k, are defined by the sum

$$B_{\mathbf{n},k}(x_{\mathbf{i}}) := \sum \frac{\mathbf{n}!}{\prod_{\mathbf{i}=\mathbf{0},\mathbf{i}\neq\mathbf{0}}^{\mathbf{n}} (k_{\mathbf{i}}!)} \prod\nolimits_{\mathbf{i}=\mathbf{0},\mathbf{i}\neq\mathbf{0}}^{\mathbf{n}} \left(\frac{x_{\mathbf{i}}}{\mathbf{i}!}\right)^{k_{\mathbf{i}}}, \tag{3}$$

where the summation is extended over all partitions of the multipartite number $\mathbf{n} = (n_1, \ldots, n_r)$ into k parts, that is, over all nonnegative integers $(k_{\mathbf{i}}, \mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{n})$ solution of the equations

$$\sum\nolimits_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} k_{\mathbf{i}} = k, \quad \sum\nolimits_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} i_j k_{\mathbf{i}} = n_j, \quad j = 1, \ldots, r, \quad \text{with convention } k_{\mathbf{0}} = 0. \tag{4}$$

Also, we can verify that the exponential generating function for $A_{n_1,\ldots,n_r}$ is given by

$$\sum\nolimits_{\mathbf{n}\geqslant\mathbf{0}} A_{\mathbf{n}}(x_{\mathbf{j}}) \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \exp\left(\sum\nolimits_{|\mathbf{i}|\geqslant 1} x_{\mathbf{i}} \frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right) \tag{5}$$

and the exponential generating function for $B_{\mathbf{n};\ k}$ is given by

$$\sum\nolimits_{|\mathbf{n}|\geqslant k} B_{\mathbf{n};k}(x_{\mathbf{j}}) \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \frac{1}{k!}\left(\sum\nolimits_{|\mathbf{i}|\geqslant 1} x_{\mathbf{i}} \frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k. \tag{6}$$

The polynomials of multinomial type $(f_{\mathbf{n}}(x))$ are defined by $f_{\mathbf{0}}(x):=1$ and

$$\left(\sum\nolimits_{\mathbf{i}\geqslant\mathbf{0}} f_{\mathbf{i}}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{x} = \sum\nolimits_{\mathbf{n}\geqslant 0} f_{\mathbf{n}}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}. \tag{7}$$

In the following $\mathbf{S}$ design a vector of $\mathbb{N}^r \setminus \{\mathbf{0}\}$.

We will use the two following results, given in [6]:

Theorem 1. *Let $r \geqslant 1$ be an integer and $\mathbf{a} \in \mathbb{R}^r$. If $(f_{\mathbf{n}}(x))$ is a multinomial type sequence of polynomials, then the polynomials $(h_{\mathbf{n}}(x))$ and $(p_{\mathbf{n}}(x))$ given by*

$$h_{\mathbf{n}}(x) := \frac{x}{\mathbf{a}.\mathbf{n}+x} f_{\mathbf{n}}(\mathbf{a}.\mathbf{n}+x) \tag{8}$$

$$p_{\mathbf{n}}(x) := f_{|\mathbf{n}|}(x) \tag{9}$$

are of multinomial type.

Theorem 2. *Let $\alpha \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^r$ and $(f_{\mathbf{n}}(x))$ be a sequence of polynomials of multinomial type. We have*

$$A_{\mathbf{n}}\left(\alpha\frac{f_{\mathbf{i}}(\mathbf{a}.\mathbf{i})}{\mathbf{a}.\mathbf{i}}\right) = \alpha\frac{f_{\mathbf{n}}(\mathbf{a}.\mathbf{n}+\alpha)}{\mathbf{a}.\mathbf{n}+\alpha}. \tag{10}$$

In this paper, we start with two light extensions concerning multipartitional polynomials given in [6]. The main results of the paper concern the recursiveness properties for multipartitional polynomials and their connections with the derivatives of polynomials of multinomial type. Each result is followed by various applications.

2. Identities related to multipartitional polynomials

In [6], Theorems 3 and 4 are established when $\mathbf{S} \in \{0,1\}^r$, $\mathbf{S} \neq \mathbf{0}$. Here, we extend the results to any vector $\mathbf{S} \in \mathbb{N}^r$ such that $|\mathbf{S}| \geqslant 1$. These extensions give an other application illustrated by vectorial multinomial coefficient.

Theorem 3. *Let (x_n) be a sequence of real numbers. Then*

$$\binom{|\mathbf{n}|+k}{|\mathbf{n}|} B_{\mathbf{n}+k\mathbf{S},k}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi} x_{|\mathbf{j}|}\right) = \frac{1}{k!}\binom{\mathbf{n}+k\mathbf{S}}{\mathbf{n},\mathbf{S},\ldots,\mathbf{S}}_{\pi} B_{|\mathbf{n}|+k,k}(jx_{j+|\mathbf{S}|-1}). \tag{11}$$

Proof. Identity (11) follows from the following expansion:

$$\frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k=\frac{1}{k!}\left(\sum_{j\geqslant 1}x_j\sum_{|\mathbf{i}|=j}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k$$
$$=\frac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^k}\left(\sum_{j\geqslant 1}x_j\sum_{|\mathbf{i}|=j}\frac{\mathbf{t}^{\mathbf{i}-\mathbf{S}}}{(\mathbf{i}-\mathbf{S})!}\right)^k$$
$$=\frac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^k}\left(\sum_{j\geqslant 1}x_{j+|\mathbf{S}|-1}\sum_{|\mathbf{i}|=j-1}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k$$
$$=\frac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^k}\left(\sum_{j\geqslant 1}jx_{j+|\mathbf{S}|-1}\frac{(|\mathbf{t}|)^{j-1}}{j!}\right)^k$$
$$=\frac{\mathbf{t}^{k\mathbf{S}}}{(\mathbf{S}!)^k}\sum_{m\geqslant k}B_{m,k}(jx_{j+|\mathbf{S}|-1})\frac{(|\mathbf{t}|)^{m-k}}{m!}$$
$$=\frac{\mathbf{t}^{k\mathbf{S}}}{(\mathbf{S}!)^k}\sum_{m\geqslant k}\frac{(m-k)!}{m!}B_{m,k}(jx_{j+|\mathbf{S}|-1})\sum_{|\mathbf{m}|=m-k}\frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!}$$
$$=\frac{\mathbf{t}^{k\mathbf{S}}}{(\mathbf{S}!)^k}\sum_{\mathbf{m}\geqslant\mathbf{0}}\frac{(|\mathbf{m}|)!}{(|\mathbf{m}|+k)!}B_{|\mathbf{m}|+k,k}(jx_{j+|\mathbf{S}|-1})\frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!}$$
$$=\frac{1}{k!(\mathbf{S}!)^k}\sum_{\mathbf{m}\geqslant\mathbf{0}}\frac{\mathbf{n}!}{(\mathbf{n}-k\mathbf{S})!}\frac{B_{|\mathbf{n}|-k|\mathbf{S}|+k,k}(jx_{j+|\mathbf{S}|-1})}{\binom{|\mathbf{n}|-(|\mathbf{S}|-1)k}{k}}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}$$
$$=\frac{1}{k!}\sum_{\mathbf{n}\geqslant k\mathbf{S}}\binom{\mathbf{n}}{\mathbf{n}-\mathbf{k}\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\frac{B_{|\mathbf{n}|-(|\mathbf{S}|-1)k,k}(jx_{j+|\mathbf{S}|-1})}{\binom{|\mathbf{n}|-(|\mathbf{S}|-1)k}{k}}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}.\qquad\square$$

Corollary 4. *Let (x_n) be a sequence of real numbers. Then, for $r = 1$ and $\mathbf{S} = u$, we get*

$$\binom{n+k}{n}B_{n+ku,k}\left(\binom{j}{u}x_j\right)=\frac{1}{k!}\binom{n+ku}{n,u,\ldots,u}B_{n+k,k}(jx_{j+u-1}).$$

Corollary 5. *Let (x_n) be a sequence of real numbers. Then, for $r = 2$ and $\mathbf{S} = (u,0)$, we get*

$$\binom{m+n+k}{m+n}B_{m+ku,n;k}\left(\binom{i}{u}x_{i+j}\right)=\frac{1}{k!}\binom{m+ku}{m,u,\ldots,u}B_{m+n+k,k}(jx_{j+u-1}),$$

for $r = 2$ and $\mathbf{S} = (0,v)$, we get

$$\binom{m+n+k}{m+n}B_{m,n+kv;k}\left(\binom{j}{v}x_{i+j}\right)=\frac{1}{k!}\binom{n+kv}{n,v,\ldots,v}B_{m+n+k,k}(jx_{j+v-1}),$$

for $r = 2$ and $\mathbf{S} = (u,v)$, we get

$$\binom{m+n+k}{m+n}B_{m+ku,n+kv,k}\left(\binom{i}{u}\binom{j}{v}x_{i+j}\right)$$
$$=\frac{1}{k!}\binom{m+ku}{m,u,\ldots,u}\binom{n+kv}{n,v,\ldots,v}B_{m+n+k,k}(jx_{j+u+v-1}).$$

Theorem 6. *Let $\alpha\in\mathbb{R}$, $\mathbf{a}\in\mathbb{R}^r$ and $(f_{\mathbf{n}}(x))$ be a multinomial type sequence of polynomials. We have*

$$B_{\mathbf{n},k}\left(\alpha\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{f_{\mathbf{i}-\mathbf{S}}(\mathbf{a}.(\mathbf{i}-\mathbf{S})+\alpha)}{\mathbf{a}.(\mathbf{i}-\mathbf{S})+\alpha}\right)=\frac{\alpha}{(k-1)!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\frac{f_{\mathbf{n}-k\mathbf{S}}(\mathbf{a}.(\mathbf{n}-k\mathbf{S})+\alpha k)}{\mathbf{a}.(\mathbf{n}-k\mathbf{S})+\alpha k}. \qquad (12)$$

Proof. We have

$$\begin{aligned}\sum_{|\mathbf{n}|\geqslant k}B_{\mathbf{n},k}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}f_{\mathbf{i}-\mathbf{S}}(\alpha)\right)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}&=\frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}f_{\mathbf{i}-\mathbf{S}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{k}\\&=\frac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\left(\sum_{\mathbf{i}\geqslant\mathbf{0}}f_{\mathbf{i}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{k}\\&=\frac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\sum_{\mathbf{n}\geqslant\mathbf{0}}f_{\mathbf{n}}(\alpha k)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\\&=\frac{1}{k!(\mathbf{S}!)^{k}}\sum_{\mathbf{n}\geqslant k\mathbf{S}}\frac{\mathbf{n}!}{(\mathbf{n}-k\mathbf{S})!}f_{\mathbf{n}-k\mathbf{S}}(\alpha k)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},\end{aligned}$$

i.e.

$$B_{\mathbf{n},k}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}f_{\mathbf{i}-\mathbf{S}}(\alpha)\right)=\frac{1}{k!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}f_{\mathbf{n}-k\mathbf{S}}(\alpha k). \qquad (13)$$

It suffices to replace $(f_{\mathbf{n}}(x))$ by $(h_{\mathbf{n}}(x))$ as in (8). □

Corollary 7. *Let $\alpha, a \in \mathbb{R}$ and $(f_n(x))$ be a binomial type sequence of polynomials. Then, for $r = 1$ and $\mathbf{S} = u$, we have*

$$B_{n,k}\left(\alpha\binom{i}{u}\frac{f_{i-u}(a(i-u)+\alpha)}{a(i-u)+\alpha}\right)=\frac{\alpha}{(k-1)!}\binom{n}{n-ku,u,\ldots,u}\frac{f_{n-ku}(a(n-ku)+\alpha k)}{a(n-ku)+\alpha k}.$$

Corollary 8. *Let $\alpha \in \mathbb{R}, \mathbf{a} = (a,b) \in \mathbb{R}^2$ and $(f_{m,n}(x))$ be a trinomial type sequence of polynomials. Then, for $r = 2$ and $\mathbf{S} = (u,0)$, we have*

$$B_{m,n;k}\left(\alpha\binom{i}{u}\frac{f_{i-u,j}(a(i-u)+bj+\alpha)}{a(i-u)+bj+\alpha}\right)=\frac{\alpha}{(k-1)!}\binom{m}{m-ku,u,\ldots,u}\frac{f_{m-ku,n}(a(m-ku)+bn+\alpha k)}{a(m-ku)+bn+\alpha k},$$

for $r = 2$ and $\mathbf{S} = (0,v)$, we have

$$B_{m,n;k}\left(\alpha\binom{j}{v}\frac{f_{i,j-v}(ai+b(j-v)+\alpha)}{ai+b(j-v)+\alpha}\right)=\frac{\alpha}{(k-1)!}\binom{n}{n-kv,v,\ldots,v}\frac{f_{m,n-kv}(am+b(n-kv)+\alpha k)}{am+b(n-kv)+\alpha k},$$

for $r = 2$ and $\mathbf{S} = (u,v)$, we have

$$\begin{aligned}&B_{m,n;k}\left(\alpha\binom{i}{u}\binom{j}{v}\frac{f_{i-u,j-v}(a(i-u)+b(j-v)+\alpha)}{a(i-u)+b(j-v)+\alpha}\right)\\&=\frac{\alpha}{(k-1)!}\frac{m!n!}{(m-ku)!(n-kv)!(u!v!)^{k}}\frac{f_{m-ku,n-kv}(a(m-ku)+b(n-kv)+\alpha k)}{a(m-ku)+b(n-kv)+\alpha k}.\end{aligned}$$

3. Recursiveness in multipartitional polynomials

This section is devoted to the twice composition of multipartitional polynomials.

Theorem 9. *Let $(x_{\mathbf{n}})$ be a sequence of real numbers with $x_{\mathbf{S}} = 1, \mathbf{d} \in \mathbb{N}^r$ and $d \in \mathbb{N}^{\star}$. We have*

$$B_{\mathbf{n},k}\left(\mathbf{1}_{(\mathbf{i}\geqslant\mathbf{S})}(t-1)!d\binom{\mathbf{i}+(t-1)\mathbf{S}}{\mathbf{i},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{i}+(t-1)\mathbf{S},t}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{\mathbf{j}}\right)\right)$$
$$=\mathbf{1}_{(\mathbf{n}\geqslant k\mathbf{S})}\frac{(T-1)!d}{(k-1)!}\binom{\mathbf{n}+(T-k)\mathbf{S}}{\mathbf{n},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{n}+(T-k)\mathbf{S},T}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{\mathbf{j}}\right), \tag{14}$$

where $t = \mathbf{d}\cdot(\mathbf{i}-\mathbf{S}) + dT = \mathbf{d}\cdot(\mathbf{n}-k\mathbf{S}) + kd$.

Proof. Let $(f_{\mathbf{n}}(x))$ be a multinomial type sequence of polynomials with $f_{\mathbf{n}}(1){:=}x_{\mathbf{n}+\mathbf{S}}$. For $\mathbf{a} = \mathbf{0}$ and $\alpha = 1$ in (12), we get

$$\frac{f_{\mathbf{n}}(k)}{k}=(k-1)!\binom{\mathbf{n}+k\mathbf{S}}{\mathbf{n},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{n}+k\mathbf{S},k}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\right). \tag{15}$$

For $\mathbf{a} = \mathbf{d}$ and $\alpha = d$, Formula (12) becomes

$$B_{\mathbf{n},k}\left(d\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{f_{\mathbf{i}-\mathbf{S}}(t)}{t}\right)=\frac{d}{(k-1)!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\frac{f_{\mathbf{n}-k\mathbf{S}}(T)}{T}. \tag{16}$$

Therefore, form (15), we obtain

$$\frac{f_{\mathbf{i}-\mathbf{S}}(t)}{t}=(t-1)!\binom{\mathbf{i}+(t-1)\mathbf{S}}{\mathbf{i}-\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{i}+(t-1)\mathbf{S},t}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\right)\text{ and}$$
$$\frac{f_{\mathbf{n}-k\mathbf{S}}(T)}{T}=(T-1)!\binom{\mathbf{n}+(T-k)\mathbf{S}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{n}+(T-1)\mathbf{S},T}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\right).$$

Now, replacing in (16) $f_{\mathbf{i}-\mathbf{S}}(t)$ and $f_{\mathbf{n}-k\mathbf{S}}(T)$ by their above expressions, we get

$$B_{\mathbf{n},k}\left((t-1)!d\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\binom{\mathbf{i}+(t-1)\mathbf{S}}{\mathbf{i}-\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{i}+(t-1)\mathbf{S},t}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\right)\right)$$
$$=\frac{(T-1)!d}{(k-1)!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\binom{\mathbf{n}+(T-k)\mathbf{S}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{n}+(T-k)\mathbf{S},T}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\right).$$

Then, to obtain (14), it suffices to observe that

$$\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\binom{\mathbf{i}+(t-1)\mathbf{S}}{\mathbf{i}-\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}=\mathbf{1}_{(\mathbf{i}\geqslant\mathbf{S})}\binom{\mathbf{i}+(t-1)\mathbf{S}}{\mathbf{i},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1},$$
$$\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\binom{\mathbf{n}+(T-k)\mathbf{S}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}=\mathbf{1}_{(\mathbf{n}\geqslant k\mathbf{S})}\binom{\mathbf{n}+(T-k)\mathbf{S}}{\mathbf{n},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}. \qquad \square$$

Corollary 10. *Let $(f_n(x))$ be a binomial type sequence of polynomials and $d \in \mathbb{N}^{\star}$. Then, for $r = 1$, $\mathbf{S} = u$ and $\mathbf{d} = p \in \mathbb{N}$, and $B_{n,k}(x_i)$ be the partial Bell polynomials, we get*

$$B_{n,k}\left(\mathbf{1}_{(i\geqslant u)}(t_1-1)!d\binom{i+(t_1-1)u}{i,u,\ldots,u}^{-1}B_{i+(t_1-1)u,t_1}\left(\binom{j}{u}x_j\right)\right)$$
$$=\mathbf{1}_{(n\geqslant ku)}\frac{(T_1-1)!d}{(k-1)!}\binom{n+(T_1-k)u}{n,u,\ldots,u}^{-1}B_{n+(T_1-k)u,T_1}\left(\binom{j}{u}x_j\right),$$

where $t_1 = p(i-u)+d, T_1 = p(n-ku)+kd$.

Corollary 11. *Let $p,q,d \geqslant 1$ be nonnegative integers and $(f_{m,n}(x))$ be a trinomial type sequence of polynomials. Then, for $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{S} = (u,0)$, we get*

$$B_{m,n,k}\left(\mathbf{1}_{(i\geqslant u)}(t_{10}-1)!d\binom{i+(t_{10}-1)u}{i,u,\ldots,u}^{-1}B_{i+(t_{10}-1)u,j;t_{10}}\left(\binom{i'}{u}x_{i',j'}\right)\right)$$
$$=\mathbf{1}_{(m\geqslant ku)}\frac{(T_{10}-1)!d}{(k-1)!}\binom{m+(T_{10}-k)u}{m,u,\ldots,u}^{-1}B_{m+(T_{10}-k)u,n;T_{10}}\left(\binom{i'}{u}x_{i',j'}\right),$$

where $t_{10} = p(i-u)+qj+d$ and $T_{10} = p(m-ku)+qn+kd$; for $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{S} = (0,v)$, we get

$$B_{m,n,k}\left(\mathbf{1}_{(j\geqslant v)}(t_{01}-1)!d\binom{j+(t_{01}-1)v}{j,v,\ldots,v}^{-1}B_{i,j+(t_{01}-1)v;t_{01}}\left(\binom{j'}{v}x_{i',j'}\right)\right)$$
$$=\mathbf{1}_{(n\geqslant kv)}\frac{(T_{01}-1)!d}{(k-1)!}\binom{n+(T_{01}-k)v}{n,v,\ldots,v}^{-1}B_{m,n+(T_{01}-k)v;T_{01}}\left(\binom{j'}{v}x_{i',j'}\right),$$

where $t_{01} = pi+q(j-v)+d$ and $T_{01} = pm+q(n-kv)+kd$; for $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{S} = (u,v)$, we get

$$B_{m,n,k}\left(\mathbf{1}_{(i\geqslant u)}\mathbf{1}_{(j\geqslant v)}(t_{11}-1)!d\frac{B_{i+(t_{11}-1)u,j+(t_{11}-1)v;t_{11}}\left(\binom{i'}{u}\binom{j'}{v}x_{i',j'}\right)}{\binom{i+(t_{11}-1)u}{i,u,\ldots,u}\binom{j+(t_{11}-1)v}{j,v,\ldots,v}}\right)$$
$$=\mathbf{1}_{(m\geqslant ku)}\mathbf{1}_{(n\geqslant kv)}\frac{(T_{11}-1)!d}{(k-1)!}\frac{B_{m+(T_{11}-k)u,n+(T_{11}-k)v;T_{11}}\left(\binom{i'}{u}\binom{j'}{v}x_{i',j'}\right)}{\binom{m+(T_{11}-k)u}{m,u,\ldots,u}\binom{n+(T_{11}-k)v}{n,v,\ldots,v}},$$

where $t_{11} = p(i-u)+q(j-v)+d$ and $T_{11} = p(m-ku)+q(n-kv)+dk$.

For $u = 1$ in Corollary 10 we obtain Proposition 4 of [3] and for $u = v = 1$ in Corollary 11 we obtain Theorems 7 and 10 of [5].

Using some particular cases related to Bell polynomials. Indeed, for the choice $x_{\mathbf{n}}:=x_{|\mathbf{n}|}$ in Theorem 9 and using relation (11), we obtain:

Corollary 12. *Let (x_n) be a sequence of real numbers with $x_{\mathbf{S}} = 1, \mathbf{d} \in \mathbb{N}^r$ and $d \in \mathbb{N}^{\star}$. We have*

$$B_{\mathbf{n},k}\left(\frac{d}{t}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{B_{|\mathbf{i}|-|\mathbf{S}|+t,t}(x_j)}{\binom{|\mathbf{i}|-|\mathbf{S}|+t}{t}}\right)=\frac{d}{(k-1)!T}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\frac{B_{|\mathbf{n}|-k|\mathbf{S}|+T,T}(x_j)}{\binom{|\mathbf{n}|-k|\mathbf{S}|+T}{T}},\tag{17}$$

where $t = \mathbf{d}\cdot(\mathbf{i}-\mathbf{S}) + d, T = \mathbf{d}\cdot(\mathbf{n}-k\mathbf{S}) + kd.$

Example 13. Let a,x be real numbers and $(f_n(x))$ be a binomial type sequence of polynomials. Then, for $x_n = \frac{nx}{an+x}f_{n-1}(an+x)$, we obtain from the identity (see Proposition given in [3])

$$B_{n,k}\left(\frac{nx}{an+x}f_{n-1}(an+x)\right)=\binom{n}{k}\frac{kx}{a(n-k)+kx}f_{n-k}(a(n-k)+kx)$$

the following identity

$$B_{\mathbf{n},k}\left(dx\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{f_{n-k}(a(|\mathbf{i}|-|\mathbf{S}|)+tx)}{a(|\mathbf{i}|-|\mathbf{S}|)+tx}\right)=\frac{dx}{(k-1)!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}\frac{f_{n-k}(a(|\mathbf{n}|-k|\mathbf{S}|)+Tx)}{a(|\mathbf{n}|-k|\mathbf{S}|)+Tx},$$

where $t = \mathbf{d}\cdot(\mathbf{i}-\mathbf{S}) + d,\ T = \mathbf{d}\cdot(\mathbf{n}-k\mathbf{S}) + kd.$

Theorem 14. *Let $(x_{\mathbf{n}})$ be a sequence of real numbers with $x_{\mathbf{S}} = 1$, d be an integer and $\mathbf{d} \in (\mathbb{N}^{\star})^r$. Then, for $\mathbf{d}.\mathbf{n} + d \geqslant 1$, we have*

$$\begin{aligned}&A_{\mathbf{n}}\left((\mathbf{d}.\mathbf{i}-1)!d\binom{\mathbf{i}+(\mathbf{d}.\mathbf{i})\mathbf{S}}{\mathbf{i},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{i}+(\mathbf{d}.\mathbf{i})\mathbf{S},\mathbf{d}.\mathbf{i}}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{\mathbf{j}}\right)\right)\\&\quad=(\mathbf{d}.\mathbf{n}+d-1)!d\binom{\mathbf{n}+(\mathbf{d}.\mathbf{n}+d)\mathbf{S}}{\mathbf{n},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{n}+(\mathbf{d}.\mathbf{n}+d)\mathbf{S},\mathbf{d}.\mathbf{n}+d}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{\mathbf{j}}\right).\end{aligned}\tag{18}$$

Proof. Set $\mathbf{a} = \mathbf{d}$ and $x = d$, Identity (10) becomes

$$A_{\mathbf{n}}\left(d\frac{f_{\mathbf{i}}(\mathbf{d}.\mathbf{i})}{\mathbf{d}.\mathbf{i}}\right)=d\frac{f_{\mathbf{n}}(\mathbf{d}.\mathbf{n}+d)}{\mathbf{d}.\mathbf{n}+d}.\tag{19}$$

From (15) we obtain

$$\begin{aligned}\frac{f_{\mathbf{i}}(\mathbf{d}.\mathbf{i})}{\mathbf{d}.\mathbf{i}}&=(\mathbf{d}.\mathbf{i}-1)!\binom{\mathbf{i}+(\mathbf{d}.\mathbf{i})\mathbf{S}}{\mathbf{i},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{i}+(\mathbf{d}.\mathbf{i})\mathbf{S},\mathbf{d}.\mathbf{i}}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{\mathbf{j}}\right)\\\frac{f_{\mathbf{n}}(\mathbf{d}.\mathbf{n}+d)}{\mathbf{d}.\mathbf{n}+d}&=(\mathbf{d}.\mathbf{n}+d-1)!\binom{\mathbf{n}+(\mathbf{d}.\mathbf{n}+d)\mathbf{S}}{\mathbf{n},\mathbf{S},\ldots,\mathbf{S}}_{\pi}^{-1}B_{\mathbf{n}+(\mathbf{d}.\mathbf{n}+d)\mathbf{S},\mathbf{d}.\mathbf{n}+d}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{\mathbf{j}}\right).\end{aligned}$$

Now, to obtain (18), replace in (19) $f_{\mathbf{i}}(\mathbf{d}.i)$ and $f_{\mathbf{n}}(\mathbf{d}.n + d)$ by their above expressions. □

Corollary 15. *Let (x_n) be a sequence of real numbers with $x_{\mathbf{S}} = 1$. Then, for $r = 1$, $\mathbf{S} = u$ and $\mathbf{d} = p \geqslant 1$, with p, d are integers and $pn + d \geqslant 1$, and $A_n(x_i)$ be the complete Bell polynomials and we get*

$$A_n\left((pi-1)!d\binom{(pu+1)i}{i,u,\ldots,u}^{-1}B_{(pu+1)i,pi}\left(\binom{j}{u}x_j\right)\right)$$
$$=(pn+d-1)!d\binom{(pu+d+1)n}{n,u,\ldots,u}^{-1}B_{(pu+d+1)n,pn+d}\left(\binom{j}{u}x_j\right).$$

Corollary 16. *Let $(x_{m,n})$ be a sequence of real numbers with $x_{\mathbf{S}} = 1$ and $p \geqslant 1$, $q \geqslant 1$, d be integers such that $pm + qn + d \geqslant 1$. Then, for $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{S} = (u,0)$, we get*

$$A_{m,n}\left((pi+qj-1)!d\binom{i+(pi+qj)u}{i,u,\ldots,u}^{-1}B_{i+(pi+qj)u,j;pi+qj}\left(\binom{i'}{u}x_{i',j'}\right)\right)$$
$$=(pm+qn+d-1)!d\binom{m+(pm+qn)u}{m,u,\ldots,u}^{-1}B_{m+(pm+qn+d)u,n,pm+qn+d}\left(\binom{i'}{u}x_{i',j'}\right),$$

for $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{S} = (0,v)$, we get

$$A_{m,n}\left((pi+qj-1)!d\binom{j+(pi+qj)v}{j,v,\ldots,v}^{-1}B_{i,j+(pi+qj)v;pi+qj}\left(\binom{j'}{v}x_{i',j'}\right)\right)$$
$$=(pm+qn+d-1)!d\binom{n+(pm+qn)v}{n,v,\ldots,v}^{-1}B_{m,n+(pm+qn+d)v,pm+qn+d}\left(\binom{j'}{v}x_{i',j'}\right),$$

for $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{S} = (u,v)$, we get

$$A_{m,n}\left((pi+qj-1)!d\frac{B_{i+(pi+qj)u,j+(pi+qj)v;pi+qj}\left(\binom{i'}{u}\binom{j'}{v}x_{i',j'}\right)}{\binom{i+(pi+qj)u}{i,u,\ldots,u}\binom{j+(pi+qj)v}{j,v,\ldots,v}}\right)$$
$$=(pm+qn+d-1)!d\frac{B_{m+(pm+qn+d)u,n+(pm+qn+d)v,pm+qn+d}\left(\binom{i'}{u}\binom{j'}{v}x_{i',j'}\right)}{\binom{m+(pm+qn)u}{m,u,\ldots,u}\binom{n+(pm+qn)v}{n,v,\ldots,v}}.$$

For $u = 1$ in Corollary 15 we obtain Proposition 4 of [3] and for $u = v = 1$ in Corollary 16 we obtain Theorem 13 and 16 of [7].

For the choice $x_{\mathbf{n}} := x_{|\mathbf{n}|}$ in Theorem 14 and using relation (11), we obtain:

Corollary 17. *Let (x_n) be a sequence of real numbers with $x_{\mathbf{S}} = 1$, d be an integer and $\mathbf{d} \in \mathbb{N}^r$. Then, for $\mathbf{d}.n + d \geqslant 1$, we have*

$$A_{\mathbf{n}}\left(\frac{d}{\mathbf{d}.\mathbf{i}}\frac{B_{(\mathbf{d}+\mathbf{1}).\mathbf{i},\mathbf{d}.\mathbf{i}}(x_j)}{\binom{(\mathbf{d}+\mathbf{1}).\mathbf{i}}{\mathbf{d}.\mathbf{i}}}\right)=\frac{d}{\mathbf{d}.\mathbf{n}+d}\frac{B_{(\mathbf{d}+\mathbf{1}).\mathbf{n}+d,\mathbf{d}.\mathbf{n}+d}(x_j)}{\binom{(\mathbf{d}+\mathbf{1}).\mathbf{n}+d}{\mathbf{d}.\mathbf{n}+d}}.$$

Example 18. Let a, x be real numbers and $(f_n(x))$ be a binomial type sequence of polynomials. For the (x_n) given in Example 13, we obtain

$$A_{\mathbf{n}}\left(dx\frac{f_{|\mathbf{i}|}(a|\mathbf{i}|+(\mathbf{d}.\mathbf{i})x)}{a|\mathbf{i}|+(\mathbf{d}.\mathbf{i})x}\right)=dx\frac{f_{|\mathbf{n}|}(a|\mathbf{n}|+(\mathbf{d}.\mathbf{n}+d)x)}{a|\mathbf{n}|+(\mathbf{d}.\mathbf{n}+d)x}.$$

4. Multipartitional polynomials and derivatives of polynomials of multinomial type

Some identities related to the derivatives of polynomial sequences of multinomial type are given. We use the convention $D_{z=0}^{-1}g(z)=0$.

Theorem 19. *Let $\alpha\in\mathbb{R}$ be a real number and $(f_{\mathbf{n}}(x))$ is of multinomial type sequence of polynomials. We have*

$$B_{\mathbf{n},k}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi}D_{z=0}(e^{\alpha z}f_{\mathbf{i}-\mathbf{S}}(x+z))\right)=\frac{1}{k!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}D_{z=0}^{k}(e^{\alpha z}f_{\mathbf{n}-k\mathbf{S}}(kx+z)). \tag{20}$$

Proof. Let $x_{\mathbf{n}}=\binom{\mathbf{n}}{\mathbf{S}}_{\pi}D_{z=0}(e^{\alpha z}f_{\mathbf{n}-\mathbf{S}}(x+z))$. Then

$$\begin{aligned}
\tfrac{1}{k!}\left(\textstyle\sum_{|\mathbf{i}|\geqslant 1}x_{\mathbf{i}}\tfrac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{k} &= \tfrac{1}{k!}\left(\textstyle\sum_{\mathbf{i}\geqslant\mathbf{S}}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}D_{z=0}(e^{\alpha z}f_{\mathbf{i}-\mathbf{S}}(x+z))\tfrac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{k}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\left(\textstyle\sum_{\mathbf{i}\geqslant\mathbf{0}}D_{z=0}(e^{\alpha z}f_{\mathbf{i}}(x+z))\tfrac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{k}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\left(D_{z=0}e^{\alpha z}\textstyle\sum_{\mathbf{i}\geqslant\mathbf{0}}f_{\mathbf{i}}(x+z)\tfrac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{k}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\left(D_{z=0}(e^{\alpha z}F(\mathbf{t})^{x+z})\right)^{k}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\left(F(\mathbf{t})^{x}D_{z=0}e^{(\alpha+\ln F(\mathbf{t}))z}\right)^{k}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}F(\mathbf{t})^{kx}(\alpha+\ln F(\mathbf{t}))^{k}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}F(\mathbf{t})^{kx}D_{z=0}^{k}e^{(\alpha+\ln F(\mathbf{t}))z}\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}D_{z=0}^{k}(e^{\alpha z}F(\mathbf{t})^{kx+z})\\
&= \tfrac{\mathbf{t}^{k\mathbf{S}}}{k!(\mathbf{S}!)^{k}}\textstyle\sum_{\mathbf{i}\geqslant\mathbf{0}}D_{z=0}^{k}(e^{\alpha z}f_{\mathbf{i}}(kx+z))\tfrac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\\
&= \tfrac{1}{k!}\textstyle\sum_{\mathbf{n}\geqslant k\mathbf{S}}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}D_{z=0}^{k}(e^{\alpha z}f_{\mathbf{n}-k\mathbf{S}}(kx+z))\tfrac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},
\end{aligned}$$

so, we obtain the desired identity by identification. □

Theorem 20. *Let* $\mathbf{d} \in \mathbb{N}^r$, $d \in \mathbb{N}^{\star}$, $\mathbf{a} \in \mathbb{R}^r$, α, x *be real numbers and* $(f_{\mathbf{n}}(x))$ *be a multinomial type sequence of polynomials. Then we have*

$$B_{\mathbf{n},k}\left(d\binom{\mathbf{i}}{\mathbf{S}}_{\pi}D^{\mathbf{d}.(\mathbf{i}-\mathbf{S})+d-1}(xD+1)\left(\frac{f_{\mathbf{i}-\mathbf{S}}(\bar{t}+z)}{\bar{t}+z}e^{\alpha z}\right)\Big|_{z=0}\right)$$
$$=\frac{d}{(k-1)!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}D^{\mathbf{d}.(\mathbf{n}-k\mathbf{S})+kd-1}(xD+1)\left(\frac{f_{\mathbf{n}-k\mathbf{S}}(\overline{T}+z)}{(\overline{T}+z)}e^{\alpha z}\right)\Big|_{z=0}, \tag{21}$$

where $\bar{t}=\mathbf{a}.(\mathbf{i}-\mathbf{S})+dx$ *and* $\overline{T}=\mathbf{a}.(\mathbf{n}-k\mathbf{S})+kdx$.

Proof. Set in (14) $x_{\mathbf{n}} := D_{z=0}(e^{\alpha z}f_{\mathbf{n}-\mathbf{S}}(x+z))$ and use Theorem 19, we obtain

$$B_{\mathbf{n},k}\left(\frac{d}{t}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}D^{t}_{z=0}(e^{\alpha z}f_{\mathbf{i}-\mathbf{S}}(tx+z))\right)=\frac{d}{(k-1)!T}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}$$
$$D^{T}_{z=0}(e^{\alpha z}f_{\mathbf{n}-k\mathbf{S}}(Tx+z)),$$

with $t=\mathbf{d}\cdot(\mathbf{i}-\mathbf{S})+d$, $T=\mathbf{d}\cdot(\mathbf{n}-k\mathbf{S})+kd$.

To obtain (21), use in this identity the multinomial type sequence $(h_{\mathbf{n}}(x))$ instead $(f_{\mathbf{n}}(x))$, where $h_{\mathbf{n}}(x)$ is defined, as in (8), by $h_{\mathbf{n}}(x) := \frac{x}{(\mathbf{a}-x\mathbf{d}).\mathbf{n}+x}f_{\mathbf{n}}((\mathbf{a}-x\mathbf{d}).\mathbf{n}+x)$. □

Corollary 21. *Let* α, x, a *be real numbers,* $p \geqslant 0$, $d \geqslant 1$ *be integers and* $(f_n(x))$ *be a binomial type sequence of polynomials. For* $r=1$, $\mathbf{d}=p$, $\mathbf{a}=a$ *and* $\mathbf{S}=u$, *we have*

$$B_{n,k}\left(d\binom{i}{u}D^{p(i-u)+d-1}(xD+1)\left(\frac{f_{i-u}(\bar{t}_1+z)}{\bar{t}_1+z}e^{\alpha z}\right)\Big|_{z=0}\right)$$
$$=\frac{d}{(k-1)!}\binom{n}{ku}D^{p(n-ku)+kd-1}(xD+1)\left(\frac{f_{n-ku}(\overline{T}_1+z)}{\overline{T}_1+z}e^{\alpha z}\right)\Big|_{z=0},$$

where $\bar{t}_1=a(i-u)+dx$ *and* $\overline{T}_1=a(n-ku)+kdx$.

Corollary 22. *Let* α,x,a, b *be real numbers,* $p \geqslant 0$, $q \geqslant 0$, $d \geqslant 1$ *be integers and* $(f_{m,n}(x))$ *be a trinomial type sequence of polynomials. Then, for* $r=2$, $\mathbf{d}=(p,q)$, $\mathbf{a}=(a,b)$ *and* $\mathbf{S}=(u,0)$, *we get*

$$B_{m,n,k}\left(d\binom{i}{u}D^{p(i-u)+qj+d-1}(xD+1)\left(\frac{f_{i-u,j}(\bar{t}_{10}+z)}{\bar{t}_{10}+z}e^{\alpha z}\right)\Big|_{z=0}\right)$$
$$=\frac{d}{(k-1)!}\binom{m}{ku}D^{p(m-ku)+qn+kd-1}(xD+1)\left(\frac{f_{m-ku,n}(\overline{T}_{10}+z)}{\overline{T}_{10}+z}e^{\alpha z}\right)\Big|_{z=0},$$

where $\bar{t}_{10}=a(i-u)+bj+dx$ *and* $\overline{T}_{10}=a(m-ku)+bn+kdx$; *for* $r=2$, $\mathbf{d}=(p,q)$, $\mathbf{a}=(a,b)$ *and* $\mathbf{S}=(0,v)$, *we get*

$$B_{m,n,k}\left(d\binom{j}{v}D^{pi+q(j-v)+d-1}(xD+1)\left(\frac{f_{i,j-v}(\bar{t}_{01}+z)}{\bar{t}_{01}+z}e^{\alpha z}\right)\Big|_{z=0}\right)$$
$$=\frac{d}{(k-1)!}\binom{n}{kv}D^{pm+q(n-kv)+kd-1}(xD+1)\left(\frac{f_{m,n-kv}(\overline{T}_{01}+z)}{\overline{T}_{01}+z}e^{\alpha z}\right)\Big|_{z=0},$$

where $\bar{t}_{01}=ai+b(j-v)+dx$ *and* $\overline{T}_{01}=am+b(n-kv)+kdx$; *for* $r=2$, $\mathbf{d}=(p,q)$, $\mathbf{a}=(a,b)$ *and* $\mathbf{S}=(u,v)$, *we get*

$$B_{m,n,k}\left(d\binom{i}{u}\binom{j}{v}D^{p(i-u)+q(j-v)+d-1}(xD+1)\left(\frac{f_{i-u,j-v}(\bar{t}_{11}+z)}{\bar{t}_{11}+z}e^{\alpha z}\right)\Big|_{z=0}\right)$$
$$=\frac{d}{(k-1)!}\binom{m}{ku}\binom{n}{kv}D^{p(m-ku)+q(n-kv)+kd-1}(xD+1)\left(\frac{f_{m-ku,n-kv}(\overline{T}_{11}+z)}{(\overline{T}_{11}+z)}e^{\alpha z}\right)\Big|_{z=0},$$

where $\bar{t}_{11}=a(i-u)+b(j-vt)+dx$ *and* $\overline{T}_{11}=a(m-ku)+b(n-kv)+kdx$.

For $u=1$ in Corollary 21 we obtain Corollary 19 of [5] and for $u=v=1$ in Corollary 22 we obtain Theorems 16 and 17 of [5].

Now, when we use a multinomial type sequence as in (9), Theorem 20 becomes:

Corollary 23. *Let* $\mathbf{d}\in\mathbb{N}^r$, $d\in\mathbb{N}^{\star}$, $\in\mathbb{R}^r$, α, x *be real numbers and* $(f_{\mathbf{n}}(x))$ *be a multinomial type sequence of polynomials. Then we have*

$$B_{\mathbf{n},k}\left(d\binom{\mathbf{i}}{\mathbf{S}}_{\pi}D^{\mathbf{d}.(\mathbf{i}-\mathbf{S})+d-1}(xD+1)\left(\frac{f_{|\mathbf{i}-\mathbf{S}|}(\bar{t}+z)}{\bar{t}+z}e^{\alpha z}\right)|_{z=0}\right)$$
$$=\frac{d}{(k-1)!}\binom{\mathbf{n}}{\mathbf{n}-k\mathbf{S},\mathbf{S},\ldots,\mathbf{S}}_{\pi}D^{\mathbf{d}.(\mathbf{n}-k\mathbf{S})+kd-1}(xD+1)\left(\frac{f_{|\mathbf{n}-k\mathbf{S}|}(\overline{T}+z)}{(\overline{T}+z)}e^{\alpha z}\right)\Big|_{z=0}, \quad (22)$$

where $\bar{t}=\mathbf{a}.(\mathbf{i}-\mathbf{S})+dx$ *and* $\overline{T}=\mathbf{a}.(\mathbf{n}-k\mathbf{S})+kdx$.

Theorem 24. *Let* $\mathbf{d}\in(\mathbb{N}^{\star})^r$, *d be an integer,* $\mathbf{a}\in\mathbb{R}^r,\alpha,x$ *be real numbers and* $(f_{\mathbf{n}}(x))$ *be a multinomial type sequence of polynomials. Then, for* $\mathbf{d}.n+d\geqslant 1$, *we have*

$$A_{\mathbf{n}}\left(dD^{\mathbf{d}.\mathbf{i}-1}(xD+1)\left(\frac{f_{\mathbf{i}}(\mathbf{a}.\mathbf{i}+z)}{\mathbf{a}.\mathbf{i}+z}e^{\alpha z}\right)\Big|_{z=0}\right)$$
$$=dD^{\mathbf{d}.\mathbf{n}+d-1}(xD+1)\left(\frac{f_{\mathbf{n}}(\mathbf{a}.\mathbf{n}+dx+z)}{\mathbf{a}.\mathbf{n}+dx+z}e^{\alpha z}\right)\Big|_{z=0}. \quad (23)$$

Proof. Set in (18) $x_{\mathbf{n}}:=D_{z=0}(e^{\alpha z}f_{\mathbf{n}-S}(x+z))$ and use Lemma 19, we obtain

$$A_{\mathbf{n}}\left(\frac{d}{\mathbf{d}.\mathbf{i}}D^{\mathbf{d}.\mathbf{i}}_{z=0}(e^{\alpha z}f_{\mathbf{i}}((\mathbf{d}.\mathbf{i})x+z))\right)=\frac{d}{\mathbf{d}.\mathbf{n}+d}D^{\mathbf{d}.\mathbf{n}+d}_{z=0}(e^{\alpha z}f_{\mathbf{n}}((\mathbf{d}.\mathbf{n}+d)x+z)).$$

To obtain (23), use in this identity the multinomial type sequence $(h_{\mathbf{n}}(x))$ instead of $(f_{\mathbf{n}}(x))$, where $h_{\mathbf{n}}(x)$ is defined, as in (8), by $h_{\mathbf{n}}(x):=\frac{x}{(\mathbf{a}-x\mathbf{d}).\mathbf{n}+x}f_{\mathbf{n}}((\mathbf{a}-x\mathbf{d}).\mathbf{n}+x)$. $\square$

Corollary 25. *Let α, x, a be real numbers, $p \geqslant 1$, d be integers such that $pn + d \geqslant 1$ and $(f_n(x))$ be a binomial type sequence of polynomials. For $r = 1$, $\mathbf{d} = p$ and $\mathbf{a} = a$, we get*

$$A_n\left(dD^{pi-1}(xD+1)\left(\frac{f_i(ai+z)}{ai+z}e^{\alpha z}\right)\bigg|_{z=0}\right) = dD^{pn+d-1}(xD+1)\left(\frac{f_n(an+dx+z)}{an+dx+z}e^{\alpha z}\right)\bigg|_{z=0}.$$

Corollary 26. *Let α, x, a, b be real numbers, $p \geqslant 1$, $q \geqslant 1$, d be integers such that $pm + qn + d \geqslant 1$ and $(f_{m,n}(x))$ be a trinomial type sequence of polynomials. For $r = 2$, $\mathbf{d} = (p,q)$ and $\mathbf{a} = (a,b)$, we get*

$$A_{m,n}\left(dD^{pi+qj-1}(xD+1)\left(\frac{f_{i,j}(ai+bj+z)}{ai+bj+z}e^{\alpha z}\right)\bigg|_{z=0}\right)$$
$$= dD^{pm+qn+d-1}(xD+1)\left(\frac{f_{m,n}(am+bn+dx+z)}{am+bn+dx+z}e^{\alpha z}\right)\bigg|_{z=0}.$$

Corollary 25 is Corollary 20 of [5] and Corollary 26 is Theorem 18 of [5].

Now, when we use a multinomial type sequence as in (9), Theorem 24 becomes:

Corollary 27. *Let $\mathbf{d} \in (\mathbb{N}^{\star})^r$, d be an integer, $\mathbf{a} \in \mathbb{R}^r$, α, x be real numbers and $(f_{\mathbf{n}}(x))$ be a multinomial type sequence of polynomials. Then, for $\mathbf{d}.n + d \geqslant \mathbf{1}$, we have*

$$A_{\mathbf{n}}\left(dD^{\mathbf{d}.\mathbf{i}-1}(xD+1)\left(\frac{f_{|\mathbf{i}|}(\mathbf{a}.\mathbf{i}+z)}{\mathbf{a}.\mathbf{i}+z}e^{\alpha z}\right)\bigg|_{z=0}\right)$$
$$= dD^{\mathbf{d}.\mathbf{n}+d-1}(xD+1)\left(\frac{f_{|\mathbf{n}|}(\mathbf{a}.\mathbf{n}+dx+z)}{\mathbf{a}.\mathbf{n}+dx+z}e^{\alpha z}\right)\bigg|_{z=0}.$$

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