Arab Journal of Mathematical Sciences

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Arab J Math Sci 20(2) (2014), 233–245

The (exponential) multipartitional polynomials and polynomial sequences of multinomial type, Part I

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Available online 24 September 2013

Abstract. We establish some formulas relating multipartitional polynomials to multinomial polynomials. They appear, respectively, as a natural extension of Bell polynomials and of polynomials of binomial type. Our results are illustrated by some comprehensive examples.

Keywords: Multipartitional polynomials; Polynomial sequences of multinomial type; Bell polynomials

1. Introduction

Recently, Mihoubi [4,5] studies the connection between Bell polynomials and binomial type sequences and deduces identities for complete and partial Bell polynomials.

As an extension of our previous results on bipartitional polynomials, see [1,6], we establish some connections between multipartitional polynomials and polynomials of multinomial type. They appear, respectively, as a natural extension of Bell polynomials and the polynomials of binomial type.

Let us introduce some definitions and notations.

We define the complete (exponential) multipartitional polynomial $A_{n_1,\ldots,n_r}$ in the variables $x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}$ as

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[1] The research is supported by IFORCE Laboratory of USTHB University.
[2] The research is supported by IFORCE Laboratory of USTHB University and CMEP Tassili bilateral Algerian-French program 09MDU765.
Peer review under responsibility of King Saud University.

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http://dx.doi.org/10.1016/j.ajmsc.2012.12.001

$$A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},x_{0,\ldots,0,2},\ldots,x_{n_1,\ldots,n_r}):=\sum\frac{n_1!\cdots n_r!}{k_{0,\ldots,0,1}!k_{0,\ldots,0,2}!\cdots k_{n_1,\ldots,n_r}!}\times\left(\frac{x_{0,\ldots,0,1}}{0!\cdots 0!1!}\right)^{k_{0,\ldots,0,1}}\cdots\left(\frac{x_{n_1,\ldots,n_r}}{n_1!\cdots n_r!}\right)^{k_{n_1,\ldots,n_r}}, \quad (1)$$

where the summation is extended over all partitions of the multipartite number $(n_1,\ldots,n_r)$, that is, over all nonnegative integers $(k_{0,\ldots,0,1},\ldots,k_{n_1,\ldots,n_r})$ solution of the equations

$$\sum_{i_r=0}^{n_r}\cdots\sum_{i_1=0}^{n_1} i_j k_{i_1,\ldots,i_r}=n_j,\quad j=1,\ldots,r,\ \text{with the convention } k_{0,\ldots,0}=0. \quad (2)$$

Also, for a given k integer, we define the partial (exponential) multipartitional polynomial of degree k : $B_{n_1,\ldots,n_r,k}$ in the variables $x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}$ as the sum

$$B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},x_{0,\ldots,0,2},\ldots,x_{n_1,\ldots,n_r}):=\sum\frac{n_1!\cdots n_r!}{k_{0,\ldots,0,1}!k_{0,\ldots,0,2}!\ldots k_{n_1,\ldots,n_r}!}\times\left(\frac{x_{0,\ldots,0,1}}{0!\cdots 0!1!}\right)^{k_{0,\ldots,0,1}}\cdots\left(\frac{x_{n_1,\ldots,n_r}}{n_1!\cdots n_r!}\right)^{k_{n_1,\ldots,n_r}}, \quad (3)$$

where the summation is extended over all partitions of the multipartite number $(n_1,\ldots,n_r)$ into k parts, that is, over all nonnegative integers $(k_{0,\ldots,0,1},\ldots,k_{n_1,\ldots,n_r})$ solution of the equations

$$\begin{aligned}&\sum_{i_r=0}^{n_r}\cdots\sum_{i_1=0}^{n_1} i_j k_{i_1,\ldots,i_r}=n_j,\quad j=1,\ldots,r,\\&\sum_{i_r=0}^{n_r}\cdots\sum_{i_1=0}^{n_1} k_{i_1,\ldots,i_r}=k.\end{aligned} \quad (4)$$

These polynomials generalize the partial and complete Bell polynomials, see [2,3,7,8], and for other recent results, see [4,5]. Some properties can be deduced from the above definitions, thus: for all real numbers α, β, γ we have

$$A_{n_1,\ldots,n_r}(\alpha_r x_{0,\ldots,0,1},\ldots,\alpha_1^{n_1}\cdots\alpha_r^{n_r}x_{n_1,\ldots,n_r})=\alpha_1^{n_1}\cdots\alpha_r^{n_r}A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}), \quad (5)$$

$$B_{n_1,\ldots,n_r,k}(\beta\alpha_r x_{0,\ldots,0,1},\ldots,\beta\alpha_1^{n_1}\cdots\alpha_r^{n_r}x_{n_1,\ldots,n_r})=\beta^k\alpha_1^{n_1}\cdots\alpha_r^{n_r}B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}). \quad (6)$$

The exponential generating functions for $A_{n_1,\ldots,n_r}$ and $B_{n_1,\ldots,n_r,k}$ are given by

$$\sum_{n_1,\ldots,n_r\geqslant 0}A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r})\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}=\exp\left(\sum_{i_1+\cdots+i_r\geqslant 1}x_{i_1,\ldots,i_r}\frac{t_1^{i_1}}{i_1!}\cdots\frac{t_r^{i_r}}{i_r!}\right), \quad (7)$$

$$\sum_{n_1+\cdots+n_r\geqslant k}B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r})\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}=\frac{1}{k!}\left(\sum_{i_1+\cdots+i_r\geqslant 1}x_{i_1,\ldots,i_r}\frac{t_1^{i_1}}{i_1!}\cdots\frac{t_r^{i_r}}{i_r!}\right)^k. \quad (8)$$

The polynomials of multinomial type $(f_{n_1,\ldots,n_r}(x))$ have the following property: $f_{0,\ldots,0}(x):=1$ and

$$\left(\sum_{i_1,\ldots,i_r\geqslant 0} f_{i_1,\ldots,i_r}(1)\frac{t_1^{i_1}}{i_1!}\cdots\frac{t_r^{i_r}}{i_r!}\right)^x = \sum_{n_1+\cdots+n_r\geqslant 1} f_{n_1,\ldots,n_r}(x)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}. \tag{9}$$

We use the following notations

$$\mathbf{t}^{\mathbf{n}} = t_1^{n_1}\cdots t_r^{n_r} \quad \text{and} \quad \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!},$$

$$\mathbf{a} = (a_1,\ldots,a_r) \quad \text{and} \quad \mathbf{1} = (1,\ldots,1),$$

$$|\mathbf{n}| = n_1+\cdots+n_r, \mathbf{n}! = n!\cdots n_r!,$$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1+\cdots+a_rb_r,$$

$$\mathbf{a}+\mathbf{b} = (a_1+b_1,\ldots,a_r+b_r),$$

$$\lambda\mathbf{a} = (\lambda a_1,\ldots,\lambda a_r),$$

$$(\mathbf{a}\geqslant\mathbf{b}) \Longleftrightarrow (a_1\geqslant b_1,\ldots,a_r\geqslant b_r),$$

$$(\mathbf{a}>\mathbf{b}) \Longleftrightarrow (a_1>b_1,\ldots,a_r>b_r),$$

$$D_{\mathbf{t}} = \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_r},$$

$$D_{z=0} = \left.\frac{d}{dz}\right|_{z=0},$$

$$\binom{x}{k} = \begin{cases} \frac{1}{k!}x(x-1)\cdots(x-k+1) & \text{if } k\geqslant 1,\\ 1 & \text{if } k=0, \qquad x\in\mathbb{R},\\ 0 & \text{otherwise}\end{cases}$$

$$\binom{\mathbf{a}}{\mathbf{i}}_{\pi} = \binom{a_1}{i_1}\cdots\binom{a_r}{i_r,} a_1,\ldots,a_r \text{ are real numbers},$$

$$B_{\mathbf{n},k}(x_{\mathbf{i}}) = B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}), \quad \mathbf{n}\geqslant\mathbf{0},\ k\geqslant 0 \text{ and } 0 \text{ otherwise},$$

$$A_{\mathbf{n}}(x_{\mathbf{i}}) = A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}), \quad \mathbf{n}\geqslant\mathbf{0} \text{ and } 0 \text{ otherwise},$$

$$B_{m,n;k}(x_{i,j}) = B_{m,n;k}(x_{0,1},x_{1,0},\ldots,x_{m,n}),$$

$$A_{m,n}(x_{i,j}) = A_{m,n}(x_{0,1},x_{1,0},\ldots,x_{m,n}),$$

$$B_{m,n,p;k}(x_{i,j,l}) = B_{m,n,p;k}(x_{0,0,1},x_{0,1,0},\ldots,x_{m,n,p}),$$

$$A_{m,n,p}(x_{i,j,l}) = A_{m,n,p}(x_{0,0,1},x_{0,1,0},\ldots,x_{m,n,p}),$$

$$f_{\mathbf{n}}(x) = f_{n_1,\ldots,n_r}(x) \quad \text{if } \mathbf{n}\geqslant\mathbf{0},\ f_{\mathbf{0}}(x)=1 \text{ and } 0 \text{ otherwise},$$

$e_s = (0, \ldots, 0, 1, 0, \ldots, 0)$ s-th position. Let $\mathbf{S} = (s_1, \ldots, s_r)$ be a vector with $s_i \in \{0, 1\}$ such that $|\,\mathbf{S}\,| \geqslant 1$.

Let $\mathbb{N}$ be the set of nonnegative integers, $\mathbb{N}^* = \mathbb{N} - \{0\}$ and $\mathbb{R}$ be the set of real numbers.

2. Identities on multipartitional polynomials

Theorem 1. *Let $(x_{\mathbf{n}})$ be a sequence of real numbers. Then for $j = 1, \ldots, r$, we have*

$$\sum_{\mathbf{i}} \binom{\mathbf{n}}{\mathbf{i}}_{\pi} x_{\mathbf{i}} B_{\mathbf{n}-\mathbf{i},k-1}(x_{\mathbf{j}}) = k B_{\mathbf{n},k}(x_{\mathbf{j}}), \tag{10}$$

$$\sum_{\mathbf{i}} \binom{\mathbf{n}}{\mathbf{i}}_{\pi} i_j x_{\mathbf{i}} B_{\mathbf{n}-\mathbf{i},k-1}(x_{\mathbf{j}}) = n_j B_{\mathbf{n},k}(x_{\mathbf{j}}), \tag{11}$$

$$\sum_{\mathbf{i}} \binom{\mathbf{n}}{\mathbf{i}}_{\pi} i_j x_{\mathbf{i}} A_{\mathbf{n}-\mathbf{i}}(x_{\mathbf{j}}) = n_j A_{\mathbf{n}}(x_{\mathbf{j}}). \tag{12}$$

Proof. From (8) we have

$$\sum_{|\mathbf{n}| \geqslant k} \frac{d}{dx_{\mathbf{j}}} B_{\mathbf{n},k}(x_{\mathbf{i}}) \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \frac{1}{k!} \frac{d}{dx_{\mathbf{j}}} \left(\sum_{|\mathbf{i}| \geqslant 1} x_{\mathbf{i}} \frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!} \right)^k = \left(\sum_{|\mathbf{m}| \geqslant k-1} B_{\mathbf{m},k-1}(x_{\mathbf{i}}) \frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!} \right) \frac{\mathbf{t}^{\mathbf{j}}}{\mathbf{j}!} = \sum_{|\mathbf{n}| \geqslant |\mathbf{j}|+k-1} \binom{\mathbf{n}}{\mathbf{j}}_{\pi} B_{\mathbf{n}-\mathbf{j},k-1}(x_{\mathbf{i}}) \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}.$$

Then

$$\frac{d}{dx_{\mathbf{j}}} B_{\mathbf{n},k}(x_{\mathbf{i}}) = \binom{\mathbf{n}}{\mathbf{j}}_{\pi} B_{\mathbf{n}-\mathbf{j},k-1}(x_{\mathbf{i}}). \tag{13}$$

Now, for $\alpha_1 = \cdots = \alpha_r = 1$ in (6), use (13) and take the derivatives with respect to β of the two sides of (6), this gives Identity (10). We obtain similarly Identity (11). Identity (12) can be deduced by summing the two sides of Identity (11) over all values of k. □

Theorem 2. *Let $(x_{\mathbf{n}})$ be a sequence of real numbers. For given $\mathbf{d} = (d_1, \ldots, d_r) \in (\mathbb{N}^*)^r$ and $y_j := \sum_{\mathbf{d}.\mathbf{i}=j} \frac{j!}{\mathbf{i}!} x_{\mathbf{i}}$ we have*

$$\sum_{\mathbf{d}.\mathbf{n}=n} \frac{n!}{\mathbf{n}!} B_{\mathbf{n},k}(x_{\mathbf{j}}) = B_{n,k}(y_j), \tag{14}$$

$$\sum_{\mathbf{d}.\mathbf{n}=n} \frac{n!}{\mathbf{n}!} A_{\mathbf{n}}(x_{\mathbf{j}}) = A_n(y_j), \tag{15}$$

$$B_{n\mathbf{e}_s;k}(x_{\mathbf{j}}) = B_{n,k}(x_{j\mathbf{e}_s}), \quad s = 1, \ldots, r, \tag{16}$$

$$A_{n\mathbf{e}_s}(x_{\mathbf{j}}) = A_n(x_{j\mathbf{e}_s}), \quad s = 1, \ldots, r. \tag{17}$$

Proof. From (8), setting $t_1 = t^{d_1}, \ldots, t_r = t^{d_r}$, we obtain

$$\frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1} x_{\mathbf{i}}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k = \sum_{|\mathbf{n}|\geqslant k} B_{\mathbf{n},k}(x_{\mathbf{j}})\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{n\geqslant k}\frac{t^n}{n!}\sum_{\mathbf{d}.\mathbf{n}=n}\frac{n!}{\mathbf{n}!}B_{\mathbf{n},k}(x_{\mathbf{j}}),$$

$$\frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1} x_{\mathbf{i}}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k = \frac{1}{k!}\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}\sum_{\mathbf{d}.\mathbf{i}=j}\frac{j!}{\mathbf{i}!}x_{\mathbf{i}}\right)^k = \sum_{n\geqslant k}\frac{t^n}{n!}B_{n,k}\left(\sum_{\mathbf{d}.\mathbf{i}=j}\frac{j!}{\mathbf{i}!}x_{\mathbf{i}}\right),$$

which give Identity (14). For $t_i = 0$, $i = 1, \ldots, r$, $i \neq s$, $t_s = t$ in (8), we get

$$\frac{1}{k!}\left(\sum_{i\geqslant 1} x_{i\mathbf{e}_s}\frac{t^i}{i!}\right)^k = \sum_{n\geqslant k} B_{n\mathbf{e}_s,k}(x_{\mathbf{i}})\frac{t^n}{n!} = \sum_{n\geqslant k} B_{n,k}(x_{j\mathbf{e}_s})\frac{t^n}{n!},$$

which gives, by identification, Identity (16). The sum over all possible values of k in the two sides of each of Identities (14) and (16), gives Identities (15) and (17) respectively. $\square$

Theorem 3. *Let (x_n) be a sequence of real numbers. Then*

$$B_{\mathbf{n},k}(x_{|\mathbf{j}|}) = B_{|\mathbf{n}|,k}(x_j), \tag{18}$$

$$A_{\mathbf{n}}(x_{|\mathbf{j}|}) = A_{|\mathbf{n}|}(x_j), \tag{19}$$

$$\binom{|\mathbf{n}|+k}{|\mathbf{n}|}B_{\mathbf{n}+k\mathbf{S},k}\left(\binom{\mathbf{j}}{\mathbf{S}}_{\pi}x_{|\mathbf{j}|}\right) = (k!)^{|\mathbf{S}|-1}\binom{\mathbf{n}+k\mathbf{S}}{\mathbf{n}}_{\pi}B_{|\mathbf{n}|+k,k}\left(jx_{j+|\mathbf{S}|-1}\right). \tag{20}$$

Proof. By (8), we have

$$\frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1} x_{|\mathbf{i}|}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k = \frac{1}{k!}\left(\sum_{j\geqslant 1}\frac{x_j}{j!}(|\mathbf{t}|)^j\right)^k = \sum_{m\geqslant k} B_{m,k}(x_j)\frac{(|\mathbf{t}|)^m}{m!} = \sum_{|\mathbf{n}|\geqslant k} B_{|\mathbf{n}|,k}(x_j)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}.$$

Then, we obtain Identity (18). Identity (19) follows when we sum over all possible values of k the two sides of Identity (18). Identity (20) follows from the following expansion:

$$\begin{aligned}
\frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}x_{\mathbf{i}}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k &= \frac{1}{k!}\left(\sum_{j\geqslant 1}x_j\sum_{|\mathbf{i}|=j}\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k = \frac{\mathbf{t}^{k\mathbf{S}}}{k!}\left(\sum_{j\geqslant 1}\frac{x_j}{j!}D_{\mathbf{t}}\sum_{|\mathbf{i}|=j}j!\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k \\
&= \frac{\mathbf{t}^{k\mathbf{S}}}{k!}\left(\sum_{j\geqslant|\mathbf{S}|}\frac{x_j}{(j-|\mathbf{S}|)!}(|\mathbf{t}|)^{j-|\mathbf{S}|}\right)^k = \frac{\mathbf{t}^{k\mathbf{S}}}{k!}\left(\sum_{j\geqslant 1}jx_{j+|\mathbf{S}|-1}\frac{(|\mathbf{t}|)^{j-1}}{j!}\right)^k \\
&= \mathbf{t}^{k\mathbf{S}}\sum_{m\geqslant k}B_{m,k}(jx_{j+|\mathbf{S}|-1})\frac{(|\mathbf{t}|)^{m-k}}{m!} = \mathbf{t}^{k\mathbf{S}}\sum_{m\geqslant k}\frac{(m-k)!}{m!}B_{m,k}(jx_{j+|\mathbf{S}|-1})\sum_{|\mathbf{m}|=m-k}\frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!} \\
&= \mathbf{t}^{k\mathbf{S}}\sum_{\mathbf{m}\geqslant\mathbf{0}}\frac{(|\mathbf{m}|)!}{(|\mathbf{m}|+k)!}B_{|\mathbf{m}|+k,k}(jx_{j+|\mathbf{S}|-1})\frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!} \\
&= (k!)^{|\mathbf{S}|-1}\sum_{\mathbf{n}\geqslant k\mathbf{S}}\binom{\mathbf{n}}{k\mathbf{S}}_{\pi}\frac{B_{|\mathbf{n}|-(|\mathbf{S}|-1)k,k}(jx_{j+|\mathbf{S}|-1})}{\binom{|\mathbf{n}|-(|\mathbf{S}|-1)k}{k}}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},
\end{aligned}$$

using the fact that $\prod_i(ks_i)! = (k!)^{\sum_i s_i}$. $\square$

Corollary 4. *For $r = 2$ and $\mathbf{S} = (1,0),(0,1)$ or $(1,1)$ we get respectively*

$$B_{m+k,n,k}(ix_{i+j}) = \frac{\binom{m+k}{k}}{\binom{m+n+k}{k}} B_{m+n+k,k}(jx_j),$$

$$B_{m,n+k,k}(jx_{i+j}) = \frac{\binom{n+k}{k}}{\binom{m+n+k}{k}} B_{m+n+k,k}(jx_j),$$

$$B_{m+k,n+k,k}(ijx_{i+j}) = k! \frac{\binom{m+k}{k}\binom{n+k}{k}}{\binom{m+n+k}{k}} B_{m+n+k,k}(jx_{j+1}).$$

The two first identities of Corollary 4 are those of Theorem 3 in [6].

Corollary 5. *Let (x_n) be a sequence of real numbers. Then, for $r = 3$ and $\mathbf{S} = (1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ or $(1,1,1)$, we get respectively*

$$B_{m+k,n,p,k}(ix_{i+j+l}) = \frac{\binom{m+k}{k}}{\binom{m+n+p+k}{k}} B_{m+n+p+k,k}(jx_j),$$

$$B_{m,n+k,p,k}(jx_{i+j+l}) = \frac{\binom{n+k}{k}}{\binom{m+n+p+k}{k}} B_{m+n+p+k,k}(jx_j),$$

$$B_{m,n,p+k,k}(lx_{i+j+l}) = \frac{\binom{p+k}{k}}{\binom{m+n+p+k}{k}} B_{m+n+p+k,k}(jx_j),$$

$$B_{m+k,n+k,p,k}(ijx_{i+j+l}) = k! \frac{\binom{m+k}{k}\binom{n+k}{k}}{\binom{m+n+p+k}{k}} B_{m+n+p+k,k}(jx_{j+1}),$$

$$B_{m+k,n,p+k,k}(ilx_{i+j+l}) = k!\frac{\binom{m+k}{k}\binom{p+k}{k}}{\binom{m+n+p+k}{k}}B_{m+n+p+k,k}(jx_{j+1}),$$

$$B_{m,n+k,p+k,k}(jlx_{i+j+l}) = k!\frac{\binom{n+k}{k}\binom{p+k}{k}}{\binom{m+n+p+k}{k}}B_{m+n+p+k,k}(jx_{j+1}),$$

$$B_{m+k,n+k,p+k,k}(ijlx_{i+j+l}) = (k!)^2\frac{\binom{m+k}{k}\binom{n+k}{k}\binom{p+k}{k}}{\binom{m+n+p+k}{k}}B_{m+n+p+k,k}(jx_{j+2}).$$

3. Polynomials of multinomial type

Theorem 6. *Let $r \geqslant 1$ be an integer and $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$. If $(f_{\mathbf{n}}(x))$ is of multinomial type, then the sequence $(h_{\mathbf{n}}(x))$ given by*

$$h_{\mathbf{n}}(x) := \frac{x}{\mathbf{a}\cdot\mathbf{n}+x}f_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{n}+x) \tag{21}$$

is of multinomial type.

Proof. By induction on r. The theorem is true for $r = 1$ [4, Proposition 1]. Assume the theorem true for $k = 1, \ldots, r$. Let $(f_{(\mathbf{n},n)}(x))$ be a sequence of multinomial type with $(\mathbf{n}, n) = (n_1, \ldots, n_r, n)$, then by (9), we get

$$\sum_{\mathbf{n}\geqslant\mathbf{0}}\left(\sum_{n\geqslant 0} f_{\mathbf{n},n}(x)\frac{t_{r+1}^n}{n!}\right)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{\mathbf{n}\geqslant\mathbf{0},n\geqslant 0} f_{\mathbf{n},n}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\frac{t_{r+1}^n}{n!} = \left(\sum_{\mathbf{i}\geqslant\mathbf{0},i\geqslant 0} f_{\mathbf{i},i}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\frac{t_{r+1}^i}{i!}\right)^x$$
$$= \left(\sum_{\mathbf{i}\geqslant\mathbf{0}}\left(\sum_{i\geqslant 0} f_{\mathbf{i},i}(1)\frac{t_{r+1}^i}{i!}\right)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^x.$$

If we set

$$u_{\mathbf{n}}(x) = \sum_{n\geqslant 0} f_{\mathbf{n},n}(x)\frac{t_{r+1}^n}{n!},$$

and thus

$$\sum_{\mathbf{n}\geqslant\mathbf{0}} u_{\mathbf{n}}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \left(\sum_{\mathbf{i}\geqslant\mathbf{0}} u_{\mathbf{i}}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^x,$$

that is the sequence $(u_{\mathbf{n}}(x))$ is of multinomial type.

Under the induction hypothesis, the sequence $(U_{\mathbf{n}}(x))$ given by

$$U_{\mathbf{n}}(x) = x\frac{u_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{n}+x)}{\mathbf{a}\cdot\mathbf{n}+x} = \frac{x}{\mathbf{a}\cdot\mathbf{n}+x}\sum_{n\geqslant 0} f_{\mathbf{n},n}(\mathbf{a}\cdot\mathbf{n}+x)\frac{t_{r+1}^{n}}{n!}$$

is of multinomial type. We then have

$$\left(\sum_{\mathbf{n}\geqslant\mathbf{0}} U_{\mathbf{n}}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\right) = \left(\sum_{\mathbf{i}\geqslant\mathbf{0}} U_{\mathbf{i}}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{x},$$

or equivalently

$$\sum_{n\geqslant 0} v_n(x)\frac{t_{r+1}^{n}}{n!} = \left(\sum_{i\geqslant 0} v_i(1)\frac{t_{r+1}^{i}}{i!}\right)^{x} \quad \text{with } v_n(x) = \sum_{\mathbf{n}\geqslant\mathbf{0}} x\frac{f_{\mathbf{n},n}(\mathbf{a}\cdot\mathbf{n}+x)}{\mathbf{a}\cdot\mathbf{n}+x}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}.$$

This implies that the sequence $(v_n(x))$ is of binomial type and the sequence $(V_n(x))$ given by

$$V_n(x) = \frac{x}{a_{r+1}n+x}v_n(a_{r+1}n+x) = \sum_{\mathbf{n}\geqslant\mathbf{0}}\frac{x}{\mathbf{a}\cdot\mathbf{n}+a_{r+1}n+x}f_{\mathbf{n},n}(\mathbf{a}\cdot\mathbf{n}+a_{r+1}n+x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}$$

is of binomial type too. We have $\left(\sum_{i=0}^{\infty} V_i(1)\frac{t^i}{i!}\right)^x = \sum_{n=0}^{\infty} V_n(x)\frac{t^n}{n!}$, or equivalently

$$\left(\sum_{\mathbf{i}\geqslant\mathbf{0},i\geqslant 0} x\frac{f_{\mathbf{i},i}(\mathbf{a}\cdot\mathbf{i}+a_{r+1}i+x)}{\mathbf{a}\cdot\mathbf{i}+a_{r+1}i+x}\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\frac{t_{r+1}^{i}}{i!}\right)^{x} = \sum_{\mathbf{n}\geqslant\mathbf{0},n_{r+1}\geqslant 0} x\frac{f_{\mathbf{n},n}(\mathbf{a}\cdot\mathbf{n}+a_{r+1}n+xt)}{\mathbf{a}\cdot\mathbf{n}+a_{r+1}n+x}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\frac{t_{r+1}^{n}}{n!}.$$

This proves that the sequence $(h_{\mathbf{n},n}(x))$ given by

$$h_{\mathbf{n},n}(x) := \frac{x}{\mathbf{a}\cdot\mathbf{n}+a_{r+1}n+x}f_{\mathbf{n},n}(\mathbf{a}\cdot\mathbf{n}+a_{r+1}n+x)$$

is of multinomial type. $\square$

From (9), one can infer that if the sequence $(f_{\mathbf{n}}(x))$ is of multinomial type, then

$$f_{\mathbf{n}}(x+y) = \sum_{\mathbf{i}}\binom{\mathbf{n}}{\mathbf{i}}_{\pi} f_{\mathbf{i}}(x)f_{\mathbf{n}-\mathbf{i}}(y). \tag{22}$$

Theorem 7. *Let* $(f_n(x))$, $(g_n^{(1)}(x)), \ldots, (g_n^{(r)}(x))$ *be sequences of binomial type. Then the polynomials* $(p_{\mathbf{n}}(x))$ *and* $(q_{\mathbf{n}}(x))$ *defined by*

$$p_{\mathbf{n}}(x) = f_{|\mathbf{n}|}(x) \quad \text{and} \quad q_{\mathbf{n}}(x) = g_{n_1}^{(1)}(x)\cdots g_{n_r}^{(r)}(x) \tag{23}$$

are of multinomial type.

Proof. We have $p_{\mathbf{0}}(x) = q_{\mathbf{0}}(x) = 1$,

$$\sum_{\mathbf{n}\geqslant\mathbf{0}} p_{\mathbf{n}}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{k\geqslant 0} f_k(x)\sum_{|\mathbf{n}|=k}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{k\geqslant 0} f_k(x)\frac{(|\mathbf{t}|)^k}{k!} = \left(\sum_{k\geqslant 0} f_k(1)\frac{(|\mathbf{t}|)^k}{k!}\right)^{x},$$

and

$$\sum_{\mathbf{n}\geqslant\mathbf{0}} q_{\mathbf{n}}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \prod_{k=1}^{r}\left(\sum_{n_k\geqslant 0} g_{n_k}^{(k)}(x)\frac{t_k^{n_k}}{n_k!}\right) = \left(\prod_{k=1}^{r}\left(\sum_{n_k\geqslant 0} g_{n_k}^{(k)}(1)\frac{t_k^{n_k}}{n_k!}\right)\right)^{x}. \qquad \square$$

Remark 8. Let $(f_n(x))$, $(g_n^{(1)}(x)), \ldots, (g_n^{(r)}(x))$ be sequences of binomial type. Then, using Theorem 6, the sequences

$$P_{\mathbf{n}}(x) = \frac{x}{\mathbf{a}\cdot\mathbf{n}+x} f_{|\mathbf{n}|}(\mathbf{a}\cdot\mathbf{n}+x) \quad \text{and} \quad Q_{\mathbf{n}}(x) = \frac{x}{\mathbf{a}\cdot\mathbf{n}+x}\prod_{i=1}^{r} g_{n_i}^{(i)}(\mathbf{a}\cdot\mathbf{n}+x)$$

are of multinomial type.

4. Multipartitional polynomials and multinomial type polynomials

Roman [8, p. 82], proved that any sequence of binomial type $(f_n(x))$, $f_0(x) = 1$, is related to the partial Bell polynomials in the form

$$f_n(x) = \sum_{k=1}^{n} B_{n,k}(D_{\alpha=0}\, f_j(\alpha))x^k, \quad n \geqslant 1. \tag{24}$$

Similarly, we will establish that a sequence of multinomial type $(f_{\mathbf{n}}(x))$, and the partial multipartitional polynomials are related in the form given by the following theorem:

Theorem 9. *Let* $(f_{\mathbf{n}}(x))$ *be a multinomial type sequence. Then, for* $|\mathbf{n}| \geqslant 1$, *we have*

$$f_{\mathbf{n}}(x) = \sum_{k=1}^{|\mathbf{n}|} B_{\mathbf{n},k}(D_{\alpha=0}\, f_{\mathbf{i}}(\alpha))x^k = A_{\mathbf{n}}(xD_{\alpha=0}\, f_{\mathbf{i}}(\alpha)). \tag{25}$$

Proof. We have

$$\begin{aligned}
\sum_{\mathbf{n}\geqslant\mathbf{0}}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\sum_{k=0}^{|\mathbf{n}|} B_{\mathbf{n},k}(D_{\alpha=0}\, f_{\mathbf{i}}(\alpha))x^k &= \sum_{k=0}^{\infty} x^k \sum_{|\mathbf{n}|\geqslant k} B_{\mathbf{n},k}(D_{\alpha=0}\, f_{\mathbf{i}}(\alpha))\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\sum_{\mathbf{i}\geqslant\mathbf{0}} D_{\alpha=0}\, f_{\mathbf{i}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k \\
&= \exp\left(x\left(\sum_{\mathbf{i}\geqslant\mathbf{0}} D_{\alpha=0}\, f_{\mathbf{i}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)\right) = \exp\left(xD_{\alpha=0}\sum_{\mathbf{i}\geqslant\mathbf{0}} f_{\mathbf{i}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right) \\
&= \exp\left(xD_{\alpha=0}\left(\sum_{\mathbf{i}\geqslant\mathbf{0}} f_{\mathbf{i}}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{\alpha}\right) = \exp\left(x\ln\left(\sum_{\mathbf{i}\geqslant\mathbf{0}} f_{\mathbf{i}}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)\right) \\
&= \left(\sum_{\mathbf{i}\geqslant\mathbf{0}} f_{\mathbf{i}}(1)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^{x} = \sum_{\mathbf{n}\geqslant\mathbf{0}} f_{\mathbf{n}}(x)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},
\end{aligned}$$

so, the desired identity follows by identification. $\square$

Theorem 10. *Let* $\alpha \in \mathbb{R}$, $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$ *and* $(f_{\mathbf{n}}(x))$ *be a multinomial type sequence. Then for* $\mathbf{n} \geqslant k\mathbf{S}$, *we have*

$$B_{\mathbf{n},k}\left(\alpha\binom{\mathbf{i}}{\mathbf{S}}_{\pi}\frac{f_{\mathbf{i}-\mathbf{S}}(\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha)}{\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha}\right) = \alpha k(k!)^{|\mathbf{S}|-1}\binom{\mathbf{n}}{k\mathbf{S}}_{\pi}\frac{f_{\mathbf{n}-k\mathbf{S}}(\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k)}{\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k}. \quad (26)$$

Proof. We have

$$\begin{aligned}\sum_{|\mathbf{n}|\geqslant k} B_{\mathbf{n},k}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi} f_{\mathbf{i}-\mathbf{S}}(\alpha)\right)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} &= \frac{1}{k!}\left(\sum_{|\mathbf{i}|\geqslant 1}\binom{\mathbf{i}}{\mathbf{S}}_{\pi} f_{\mathbf{i}-\mathbf{S}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k \\ &= \frac{\mathbf{t}^{k\mathbf{S}}}{k!}\left(\sum_{\mathbf{i}\geqslant\mathbf{0}} f_{\mathbf{i}}(\alpha)\frac{\mathbf{t}^{\mathbf{i}}}{\mathbf{i}!}\right)^k \\ &= \frac{\mathbf{t}^{k\mathbf{S}}}{k!}\sum_{\mathbf{n}\geqslant\mathbf{0}} f_{\mathbf{n}}(\alpha k)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \\ &= (k!)^{|\mathbf{S}|-1}\sum_{\mathbf{n}\geqslant k\mathbf{S}}\binom{\mathbf{n}}{k\mathbf{S}}_{\pi} f_{\mathbf{n}-k\mathbf{S}}(\alpha k)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},\end{aligned}$$

i.e.

$$B_{\mathbf{n},k}\left(\binom{\mathbf{i}}{\mathbf{S}}_{\pi} f_{\mathbf{i}-\mathbf{S}}(\alpha)\right) = (k!)^{|\mathbf{S}|-1}\binom{\mathbf{n}}{k\mathbf{S}}_{\pi} f_{\mathbf{n}-k\mathbf{S}}(\alpha k). \quad (27)$$

It suffices to replace $(f_{\mathbf{n}}(x))$ by $(h_{\mathbf{n}}(x))$ given by (21). □

Corollary 11. *Let* $(f_n(x))$ *be a binomial type sequence. Then, for* $r = 1$, $\mathbf{S} = (1)$, $\mathbf{a} = a$, *the polynomials* $B_{n,k}(x_i)$ *represent the partial Bell polynomials and*

$$B_{n,k}\left(\alpha i\frac{f_{i-1}(a((i-1))+\alpha)}{a(i-1)+\alpha}\right) = \alpha k\binom{n}{k}\frac{f_{n-k}(a(n-k)+\alpha k)}{a(n-k)+\alpha k}.$$

Corollary 12. *Let* $(f_{m,n}(x))$ *be a trinomial type sequence. Then, for* $r = 2$, $\mathbf{S} = (1, 0)$, $\mathbf{a} = (a, b)$, *we get*

$$B_{m,n,k}\left(\alpha i\frac{f_{i-1,j}(a(i-1)+bj+\alpha)}{a(i-1)+bj+\alpha}\right) = \alpha k\binom{m}{k}\frac{f_{m-k,n}(a(m-k)+bn+\alpha k)}{a(m-k)+bn+\alpha k},$$

for $r = 2$, $\mathbf{S} = (0, 1)$, $\mathbf{a} = (a, b)$, *we get*

$$B_{m,n,k}\left(\alpha j\frac{f_{i,j-1}(ai+b(j-1)+\alpha)}{ai+b(j-1)+\alpha}\right) = \alpha k\binom{m}{k}\frac{f_{m,n-k}(am+b(n-k)+\alpha k)}{am+b(n-k)+\alpha k},$$

and for $r = 2$, $\mathbf{S} = (1, 1)$, $\mathbf{a} = (a, b)$, *we get*

$$\begin{aligned}&B_{m,n,k}\left(\alpha ij\frac{f_{i-1,j-1}(a(i-1)+b(j-1)+\alpha)}{a(i-1)+b(j-1)+\alpha}\right) \\ &= \alpha k\binom{m}{k}\binom{n}{k}\frac{f_{m-k,n-k}(a(m-k)+b(n-k)+\alpha k)}{a(m-k)+b(n-k)+\alpha k}.\end{aligned}$$

The identity of Corollary 11 is Proposition 1 in [4] and the identities of Corollary 12 are those of Theorems 10 and 11 in [6].

Corollary 13. *Let* $(f_{m,n,p}(x))$ *be a multinomial type sequence with* $f_{0,0,0}(x) = 1$. *Then, for* $r = 3$, $\mathbf{S} = (1,0,0)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha i \frac{f_{i-1,j,l}(a(i-1)+bj+cl+\alpha)}{a(i-1)+bj+cl+\alpha}\right) = \alpha k \binom{m}{k} \frac{f_{m-k,n,p}(a(m-k)+bn+cp+\alpha k)}{a(m-k)+bn+cp+\alpha k},$$

for $r = 3$, $\mathbf{S} = (0,1,0)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha j \frac{f_{i,j-1,l}(ai+b(j-1)+cl+\alpha)}{ai+b(j-1)+cl+\alpha}\right) = \alpha k \binom{n}{k} \frac{f_{m,n-k,p}(am+b(n-k)+cp+\alpha k)}{am+b(n-k)+cp+\alpha k},$$

for $r = 3$, $\mathbf{S} = (0,0,1)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha l \frac{f_{i,j,l-1}(ai+bj+c(l-1)+\alpha)}{ai+bj+c(l-1)+\alpha}\right) = \alpha k \binom{p}{k} \frac{f_{m,n,p-k}(am+bn+c(p-k)+\alpha k)}{am+bn+c(p-k)+\alpha k},$$

for $r = 3$, $\mathbf{S} = (1,1,0)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha ij \frac{f_{i-1,j-1,l}(a(i-1)+b(j-1)+cl+\alpha)}{a(i-1)+b(j-1)+cl+\alpha}\right)$$
$$= \alpha k! \binom{m}{k}\binom{n}{k} \frac{f_{m-k,n-k,p}(a(m-k)+b(n-k)+cp+\alpha k)}{a(m-k)+b(n-k)+cp+\alpha k},$$

for $r = 3$, $\mathbf{S} = (1,0,1)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha il \frac{f_{i-1,j,l-1}(a(i-1)+bj+c(l-1)+\alpha)}{a(i-1)+bj+c(l-1)+\alpha}\right)$$
$$= \alpha k! \binom{m}{k}\binom{p}{k} \frac{f_{m-k,n,p-k}(a(m-k)+bn+c(p-k)+\alpha k)}{a(m-k)+bn+c(p-k)+\alpha k},$$

for $r = 3$, $\mathbf{S} = (0,1,1)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha jl \frac{f_{i,j-1,l-1}(ai+b(j-1)+c(l-1)+\alpha)}{ai+b(j-1)+c(l-1)+\alpha}\right)$$
$$= \alpha k! \binom{n}{k}\binom{p}{k} \frac{f_{m,n-k,p-k}(am+b(n-k)+c(p-k)+\alpha k)}{am+b(n-k)+c(p-k)+\alpha k},$$

and for $r = 3$, $\mathbf{S} = (1,1,1)$, $\mathbf{a} = (a,b,c)$, *we get*

$$B_{m,n,p,k}\left(\alpha ijl \frac{f_{i-1,j-1,l-1}(a(i-1)+b(j-1)+c(l-1)+\alpha)}{a(i-1)+b(j-1)+c(l-1)+\alpha}\right)$$
$$= \alpha (k!)^2 \binom{m}{k}\binom{n}{k}\binom{p}{k} \frac{f_{m-k,n-k,p-k}(a(m-k)+b(n-k)+c(p-k)+\alpha k)}{a(m-k)+b(n-k)+c(p-k)+\alpha k}.$$

Example 14. When $f_{\mathbf{n}}(\alpha) = \alpha^{|\mathbf{n}|}$ in (26), we get

$$B_{\mathbf{n},k}\left(\alpha\binom{\mathbf{i}}{\mathbf{S}}_\pi(\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha)^{|\mathbf{i}-\mathbf{S}|-1}\right) = \alpha k(k!)^{|\mathbf{S}|-1}\binom{\mathbf{n}}{k\mathbf{S}}_\pi(\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k)^{|\mathbf{n}|-1}$$

and when $f_{\mathbf{n}}(\alpha) = \mathbf{n}!\binom{\alpha\mathbf{1}}{\mathbf{n}}_\pi$, we get

$$B_{\mathbf{n},k}\left(\alpha\frac{(\mathbf{i}-\mathbf{S})!}{\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha}\binom{\mathbf{i}}{\mathbf{S}}_\pi\binom{(\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha)\mathbf{1}}{\mathbf{i}-\mathbf{S}}_\pi\right)$$
$$= \alpha k(k!)^{|\mathbf{S}|-1}\frac{(\mathbf{n}-k\mathbf{S})!}{\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k}\binom{\mathbf{n}}{k\mathbf{S}}_\pi\binom{(\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k)\mathbf{1}}{\mathbf{n}-k\mathbf{S}}_\pi.$$

Using Theorem 7, Theorem 10 becomes

Corollary 15. *Let $\alpha \in \mathbb{R}$, $\mathbf{a} = (a_1,\dots,a_r) \in \mathbb{R}^r$ and $(f_n(x))$ be a sequence of binomial type. Then, for $\mathbf{n} \geqslant k\mathbf{S}$, we have*

$$B_{\mathbf{n},k}\left(\alpha\binom{\mathbf{i}}{\mathbf{S}}_\pi\frac{f_{|\mathbf{i}-\mathbf{S}|}(\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha)}{\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha}\right) = \alpha k(k!)^{|\mathbf{S}|-1}\binom{\mathbf{n}}{k\mathbf{S}}_\pi\frac{f_{|\mathbf{n}-k\mathbf{S}|}(\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k)}{\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k}. \tag{28}$$

Example 16. When $f_n(\alpha) = n!\binom{\alpha}{n}$ in (28), one has

$$B_{\mathbf{n},k}\left(\alpha\frac{(|\mathbf{i}|)!}{\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha}\binom{\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha}{|\mathbf{i}-\mathbf{S}|}\right) = \frac{\alpha k}{k!}\frac{(|\mathbf{n}|)!}{\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k}\binom{\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k}{|\mathbf{n}-k\mathbf{S}|}$$

and when $f_n(\alpha) = B_n(\alpha)$, (the single variable Bell polynomials),

$$B_{\mathbf{n},k}\left(\alpha\binom{\mathbf{i}}{\mathbf{S}}_\pi\frac{B_{|\mathbf{i}-\mathbf{S}|}(\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha)}{\mathbf{a}\cdot(\mathbf{i}-\mathbf{S})+\alpha}\right) = \alpha k(k!)^{|\mathbf{S}|-1}\binom{\mathbf{n}}{k\mathbf{S}}_\pi\frac{B_{|\mathbf{n}-k\mathbf{S}|}(\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k)}{\mathbf{a}\cdot(\mathbf{n}-k\mathbf{S})+\alpha k}.$$

Theorem 17. *Let $\alpha \in \mathbb{R}$, $\mathbf{a} = (a_1,\dots,a_r) \in \mathbb{R}^r$ and $(f_{\mathbf{n}}(x))$ be a multinomial type sequence. We have*

$$A_{\mathbf{n}}\left(\alpha\frac{f_{\mathbf{i}}(\mathbf{a}\cdot\mathbf{i})}{\mathbf{a}\cdot\mathbf{i}}\right) = \alpha\frac{f_{\mathbf{n}}(\mathbf{a}\cdot\mathbf{n}+\alpha)}{\mathbf{a}\cdot\mathbf{n}+\alpha}. \tag{29}$$

Proof. From (25), we infer

$$A_{\mathbf{n}}(\alpha D_{x=0}\, f_{\mathbf{i}}(x)) = f_{\mathbf{n}}(\alpha),$$

which gives the desired identity by replacing $f_{\mathbf{n}}(\alpha)$ by $h_{\mathbf{n}}(\alpha)$ given by (21). □

Corollary 18. *Let $(f_n(x))$ be a binomial type sequence. For $r = 1$, $\mathbf{a} = a$, the polynomials $A_n(x_i)$ represent the complete Bell polynomials, and we have*

$$A_m\left(\alpha\frac{f_i(ai)}{ai}\right) = \alpha\frac{f_m(am+\alpha)}{am+\alpha}.$$

Corollary 19. *Let $(f_{m,n}(x))$ be a trinomial type sequence. Then, for $r = 2$, $\mathbf{a} = (a, b)$,*

$$A_{m,n}\left(\alpha\frac{f_{i,j}(ai+bj)}{ai+bj}\right) = \alpha\frac{f_{m,n}(am+bn+\alpha)}{am+bn+\alpha},$$

and for $r = 3$, $\mathbf{a} = (a, b, c)$,

$$A_{m,n,p}\left(\alpha\frac{f_{i,j}(ai+bj+cl)}{ai+bj+cl}\right) = \alpha\frac{f_{m,n,p}(am+bn+cp+\alpha)}{am+bn+cp+\alpha}.$$

The identity of Corollary 18 is Proposition 3 in [5] and the identities of Corollary 19 are those of Theorem 14 in [6].

Example 20. For $f_{\mathbf{n}}(\alpha) = \alpha^{|\mathbf{n}|}$ in (29), we get

$$A_{\mathbf{n}}(\alpha(\mathbf{a}\cdot\mathbf{i})^{|\mathbf{i}|-1}) = \alpha(\mathbf{a}\cdot\mathbf{n}+\alpha)^{|\mathbf{n}|-1},$$

and for $f_{\mathbf{n}}(\alpha) = \mathbf{n}!\binom{x\mathbf{1}}{\mathbf{n}}$, we have

$$A_{\mathbf{n}}\left(\alpha\frac{\mathbf{i}!}{\mathbf{a}\cdot\mathbf{i}}\binom{(\mathbf{a}\cdot\mathbf{i})\mathbf{1}}{\mathbf{i}}_{\pi}^{\pi}\right) = \alpha\frac{\mathbf{n}!}{\mathbf{a}\cdot\mathbf{n}+\alpha}\binom{(\mathbf{a}\cdot\mathbf{n}+\alpha)\mathbf{1}}{\mathbf{n}}_{\pi}.$$

Using Theorem 7, Theorem 17 becomes

Corollary 21. *Let $\alpha \in \mathbb{R}$, $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$ and $(f_n(x))$ be a sequence of binomial type. We have*

$$A_{\mathbf{n}}\left(\alpha\frac{f_{|\mathbf{i}|}(\mathbf{a}\cdot\mathbf{i})}{\mathbf{a}\cdot\mathbf{i}}\right) = \alpha\frac{f_{|\mathbf{n}|}(\mathbf{a}\cdot\mathbf{n}+\alpha)}{\mathbf{a}\cdot\mathbf{n}+\alpha}. \tag{30}$$

Example 22. For $f_n(\alpha) = n!\binom{\alpha}{n}$, we get from (30):

$$A_{\mathbf{n}}\left(\alpha\frac{(|\mathbf{i}|)!}{\mathbf{a}\cdot\mathbf{i}}\binom{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{i}|}\right) = \alpha\frac{(|\mathbf{n}|)!}{\mathbf{a}\cdot\mathbf{n}+\alpha}\binom{\mathbf{a}\cdot\mathbf{n}+\alpha}{|\mathbf{n}|},$$

and for $f_n(\alpha) = B_n(\alpha)$,

$$A_{\mathbf{n}}\left(\alpha\frac{B_{|\mathbf{i}|}(\mathbf{a}\cdot\mathbf{i})}{\mathbf{a}\cdot\mathbf{i}}\right) = \alpha\frac{B_{|\mathbf{n}|}(\mathbf{a}\cdot\mathbf{n}+\alpha)}{\mathbf{a}\cdot\mathbf{n}+\alpha}.$$

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