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The differential pencils with turning point on the half line

A. Neamaty *, Y. Khalili

Department of Mathematics, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran

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Abstract. We investigate the inverse spectral problem of recovering pencils of second-order differential operators on the half-line with turning point. Using the asymptotic distribution of the Weyl function, we give a formulation of the inverse problem and prove the uniqueness theorem for the solution of the inverse problem.

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1. Introduction

We consider the differential equation

$$y''(x) + (\rho^2 R(x) + i\rho q_1(x) + q_0(x))y(x) = 0, \quad x \geqslant 0, \tag{1}$$

on the half-line with nonlinear dependence on the spectral parameter ρ. Let $a \geqslant 1$, and

$$R(x) = \begin{cases} -1, & 0 \leqslant x < a, \\ x - 1, & x \geqslant a, \end{cases} \tag{2}$$

i.e., the sign of the weight-function changes in an interior point $x = a$, which is called the turning point. The functions $q_j(x)$, $j = 0,1$, are complex-valued, $q_1(x)$ is absolutely continuous and $(1 + x)q_j^{(l)} \in L(0, \infty)$ for $0 \leqslant l \leqslant j \leqslant 1$.

Differential equations with spectral parameter and turning point arise in various problems of mathematics (see, for example, Tamarkin [7]). The classical Sturm–Liouville operators with turning points in the finite interval have been studied fairly completely in Freiling and Schneider [2]. Indefinite differential pencils produce

* Corresponding author.

E-mail addresses: namaty@umz.ac.ir (A. Neamaty), y.khalili@stu.umz.ac.ir (Y. Khalili).

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significant qualitative modification in the investigation of the inverse problem. Some aspects of the inverse problem theory for differential pencils without turning points were studied in Khruslov and Shepelsky [5] and Yurko [8]. In Freiling and Yurko [3,4], the inverse problem was investigated for differential equations with m turning points. Also the inverse problem was investigated for differential pencils with turning point and nonlinear dependence on the spectral parameter in Yurko [11,12]. Here we investigate the uniqueness solution of the inverse problem with turning point when the weight-function changes in the linear form after turning point. As the main spectral characteristic for the boundary value problem, we introduce the so-called Weyl function.

In this paper, we will study the uniqueness theorem for Eq. (1) with spectral boundary condition. In Section 2, we determine the asymptotic forms of the solutions of Eq. (1) and derive characteristic function. In Section 3, we obtain the Weyl function and establish a formulation of the inverse problem. In Section 4, we prove the uniqueness theorem.

2. Primary results

We consider the boundary value problem L for Eq. (1) on the half-line with the boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{3}$$

where the coefficients β_1 and β_0 are complex numbers and $\beta_1 \neq \pm 1$. Denote $\Pi\pm :=\{\rho: \pm Im\rho > 0\}$, $\Pi_0:=\{\rho:Im\rho = 0\}$. By the well-known method (see, Mennicken and Moller [6]; Tamarkin [7] and Freiling and Yurko [4]), we obtain a solution $e(x,\rho)$ of the Eq. (1) (which is called the Jost-type solution) with the following properties:

Theorem 2.1. *Eq. (1) has a unique solution $y = e(x,\rho)$, $\rho \in \Pi\pm$, $x \geqslant a$, with the following properties:*

1. *For each fixed $x \geqslant a$, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$ (i.e., they are piecewise holomorphic).*
2. *The functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuous for $x \geqslant a, \rho \in \overline{\Pi}_+$ and $\rho \in \overline{\Pi}_-$ (we differ the sides of the cut Π_0). In other words, for real ρ, there exist the finite limits*

$$e_{\pm}^{(\nu)}(x,\rho) = \lim_{z\to\rho,\ z\in\Pi_\pm} e^{(\nu)}(x,z).$$

Moreover, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuously differentiable with respect to $\rho \in \overline{\Pi}_+ \setminus \{0\}$ and $\rho \in \overline{\Pi}_- \setminus \{0\}$.

3. *For $x \to \infty, \rho \in \overline{\Pi}_\pm \setminus \{0\},\ \nu = 0,1$,*

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^\nu R(x)^{\nu-\frac{1}{2}} exp(\pm(i\rho x - Q(x)))(1 + o(1)), \tag{4}$$

where $Q(x) = \frac{1}{2}\int_0^x q_1(t)dt$.

4. *For $|\rho| \to \infty, \rho \in \overline{\Pi}_\pm,\ \nu = 0,1$, uniformly in $x \geqslant a$,*

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^\nu R(x)^{\nu-\frac{1}{2}} exp(\pm(i\rho x - Q(x)))[1], \tag{5}$$

where [1]:=1 + $O(\rho^{-1})$.We extend $e(x,\rho)$ to the segment [0,a] as a solution of Eq. (1) *which is smooth for $x \geqslant 0$, i.e.,*

$$e^{(v)}(a-0,\rho) = e^{(v)}(a+0,\rho),\ v = 0,1. \tag{6}$$

Then the properties 1 − 2 remain true for $x \geqslant 0$.Let the functions $\varphi(x,\rho)$ and $S(x,\rho)$ be the solutions of Eq. (1) under the initial conditions $\varphi(0,\rho) = 1$, $U(\varphi) = 0$, $S(0,\rho) = 0$, $S'(0,\rho) = 1$. For each fixed $x \geqslant 0$, the functions $\varphi^{(v)}(x,\rho)$ and $S^{(v)}(x,\rho)$, $v = 0,1$, are entire in ρ.

Lemma 2.2. *For $m = 0,1, |\rho| \to \infty$, uniformly in x, the following asymptotic formulae are valid:*

$$\begin{cases} \varphi^{(m)}(x,\rho) = \frac{\rho^m}{2}((-1)^m(1+\beta_1)exp(-\rho x + iQ(x))[1] + (1-\beta_1)exp(\rho x - iQ(x))[1]),\ x \in [0,a], \\ \varphi^{(m)}(x,\rho) = \Big(\frac{1+\beta_1}{-4}\Big(1+i(a-1)^{\frac{1}{2}}\Big)exp\Big(-\rho a\Big(1+i(a-1)^{\frac{1}{2}}\Big)+iQ(a)\Big(1-i(a-1)^{\frac{-1}{2}}\Big)\Big)[1] \\ \quad +\frac{1-\beta_1}{4}\Big(1-i(a-1)^{\frac{1}{2}}\Big)exp\Big(\rho a\Big(1-i(a-1)^{\frac{1}{2}}\Big)-iQ(a)\Big(1+i(a-1)^{\frac{-1}{2}}\Big)\Big)[1]\Big) \\ \quad \times\Big(i\rho(x-1)^{\frac{1}{2}}\Big)^m exp\Big(i\rho(x-1)^{\frac{1}{2}}x-(x-1)^{\frac{-1}{2}}Q(x)\Big)+\Big(\frac{1+\beta_1}{-4}\Big(1+i(a-1)^{\frac{1}{2}}\Big) \\ \quad \times exp\Big(-\rho a\Big(1-i(a-1)^{\frac{1}{2}}\Big)+iQ(a)\Big(1+i(a-1)^{\frac{-1}{2}}\Big)\Big)[1]+\frac{1-\beta_1}{4}\Big(1-i(a-1)^{\frac{1}{2}}\Big) \\ \quad \times exp\Big(\rho a\Big(1+i(a-1)^{\frac{1}{2}}\Big)-iQ(a)\Big(1-i(a-1)^{\frac{-1}{2}}\Big)\Big)[1]\Big)\Big(-i\rho(x-1)^{\frac{1}{2}}\Big)^m \\ \quad \times exp\Big(-i\rho(x-1)^{\frac{1}{2}}x+(x-1)^{\frac{-1}{2}}Q(x)\Big), \qquad x \geqslant a. \end{cases}$$

Proof. Denote $\Pi^1_\pm := \{\rho : \pm Re\rho > 0\}$. It is known (see, Mennicken and Moller [6] and Tamarkin [7]) that for $x \geqslant a,\ m = 0,1,\ \rho \in \overline{\Pi}^1_\pm,\ |\rho| \to \infty$, there exists a fundamental system of solutions $\{Y_k(x,\rho)\}_{k=1,2}$ of Eq. (1) of the form

$$Y_k^{(m)}(x,\rho) = \Big((-1)^{k-1}i\rho(x-1)^{\frac{1}{2}}\Big)^m exp\Big((-1)^{k-1}\Big(i\rho x(x-1)^{\frac{1}{2}} - (x-1)^{\frac{-1}{2}}Q(x)\Big)\Big)[1]. \tag{7}$$

Similarly for $x \in [0,a],\ m = 0,1,\ \rho \in \overline{\Pi}^1_\pm,\ |\rho| \to \infty$, there exists a fundamental system of solutions $\{y_k(x,\rho)\}_{k=1,2}$ of Eq. (1) of the form

$$y_k^{(m)}(x,\rho) = ((-1)^k\rho)^m exp((-1)^k(\rho x - iQ(x)))[1]. \tag{8}$$

Using the Birkhoff-type fundamental system of solutions, one has

$$\varphi^{(m)}(x,\rho) = A_1(\rho)y_1^{(m)}(x,\rho) + A_2(\rho)y_2^{(m)}(x,\rho), \quad x \in [0,a], \tag{9}$$

$$\varphi^{(m)}(x,\rho) = B_1(\rho)Y_1^{(m)}(x,\rho) + B_2(\rho)Y_2^{(m)}(x,\rho), \quad x \geqslant a. \tag{10}$$

Taking (8) and the initial conditions $\varphi(0,\rho) = 1$, $\varphi'(0,\rho) = -(\beta_1\rho + \beta_0)$ into account, we calculate

$$A_1(\rho) = \frac{1+\beta_1}{2}[1], \quad A_2(\rho) = \frac{1-\beta_1}{2}[1]. \tag{11}$$

Substituting (8) and (11) in (9), we obtain the asymptotic formulae $\varphi^{(m)}(x,\rho),\ m=0,1,$ as $|\rho|\to\infty$, uniformly in $x\in[0,a]$.

Now using (7), (10) and the smooth condition $\varphi^{(m)}(a-0,\rho)=\varphi^{(m)}(a+0,\rho),\ m=0,1,$ we have

$$B_1(\rho)=\frac{1}{2}exp\Big(-i\rho a(a-1)^{\frac{1}{2}}+Q(a)(a-1)^{\frac{-1}{2}}\Big)\Big(i\rho(a-1)^{\frac{1}{2}}\varphi(a,\rho)+\varphi'(a,\rho)\Big)[1],$$
$$B_2(\rho)=\frac{1}{2}exp\Big(i\rho a(a-1)^{\frac{1}{2}}-Q(a)(a-1)^{\frac{-1}{2}}\Big)\Big(-i\rho(a-1)^{\frac{1}{2}}\varphi(a,\rho)+\varphi'(a,\rho)\Big)[1].$$

Substituting the coefficients $B_1(\rho)$, $B_2(\rho)$ and (7) in (10), we obtain $\varphi^{(m)}(x,\rho)$, $m=0,1,$ as $x\geqslant a$, $|\rho|\to\infty$. Lemma 2.2 is proved. □

Corollary 2.3. *It follows from Lemma* 2.2 *that*

$$|\varphi^{(m)}(x,\rho)|\leqslant C|\rho|^m exp(|Re\rho|x),\qquad x\in[0,a],\tag{12}$$
$$|\varphi^{(m)}(x,\rho)|\leqslant C|\rho|^m exp(|Re\rho|a)exp(|Im\rho|(a-x)),\qquad x\geqslant a.\tag{13}$$

Denote

$$\Delta(\rho):=U(e(x,\rho)).\tag{14}$$

The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ, there exist the finite limits

$$\Delta\pm(\rho)_{=}\lim_{z\to\rho,\ z\in\Pi_\pm}\Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho\in\overline{\Pi}_\pm\setminus\{0\}$.

Theorem 2.4. *For $|\rho|\to\infty,\ \rho\in\overline{\Pi}_\pm$, the following asymptotical formula holds:*

$$\Delta(\rho)=\frac{\rho}{2}(a-1)^{\frac{-1}{2}}exp(\pm(i\rho a-Q(a)))((-1\pm i(a-1))(1-\beta_1)exp(\rho a-iQ(a))[1]$$
$$-(-1\mp i(a-1))(1+\beta_1)exp(-\rho a+iQ(a))[1]).$$

Proof. Taking the Birkhoff-type fundamental system of solutions $\{y_k(x,\rho)\}_{k=1,2}$ of Eq. (1) on the interval $[0,a]$, one has

$$e^{(m)}(x,\rho)=H_1(\rho)y_1^{(m)}(x,\rho)+H_2(\rho)y_2^{(m)}(x,\rho),\qquad x\in[0,a].\tag{15}$$

Using the Cramers rule, we calculate

$$H_1(\rho)=\frac{1}{2}(a-1)^{\frac{-1}{2}}(1\mp i(a-1))exp(\pm(i\rho a-Q(a)))exp(\rho a-iQ(a))[1],$$
$$H_2(\rho)=\frac{1}{2}(a-1)^{\frac{-1}{2}}(1\pm i(a-1))exp(\pm(i\rho a-Q(a)))exp(-\rho a+iQ(a))[1].$$

Now, taking (8), (15) and coefficients $H_j(\rho)$, $j = 1,2$, we have for $|\rho| \to \infty$, $\rho \in \overline{\Pi}_\pm$, $x \in [0, a]$,

$$
\begin{aligned}
e^{(m)}(x,\rho) = \frac{\rho^m}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-1)^m(1 \mp i(a-1))exp(\rho a - iQ(a)) \\
\times exp(-\rho x + iQ(x))[1] + (1 \pm i(a-1)) \\
\times exp(-\rho a + iQ(a))exp(\rho x - iQ(x))[1]).
\end{aligned}
$$

Together with (3) and (14), this yields the characteristic function $\Delta(\rho)$. □

Definition 2.5. The values of the parameter ρ, for which Eq. (1) has nontrivial solutions satisfying the conditions $U(y) = 0$, $y(\infty) = 0$ (i.e., $\lim_{x\to\infty} y(x) = 0$) are called eigenvalues of L, and the corresponding solutions are called eigenfunctions.

Denote

$$
\begin{aligned}
&\Lambda'_\pm = \{\rho \in \Pi_\pm; \Delta(\rho) = 0\}, \quad && \Lambda' = \Lambda'_+ \cup \Lambda'_-,\\
&\Lambda''_\pm = \{\rho \in R; \Delta_\pm(\rho) = 0\}, \quad && \Lambda'' = \Lambda''_+ \cup \Lambda''_-,\\
&\Lambda_\pm = \Lambda'_\pm \cup \Lambda''_\pm, && \Lambda = \Lambda_+ \cup \Lambda_-.
\end{aligned}
$$

Theorem 2.6.

(1) *For sufficiently large k, the function $\Delta(\rho)$ has simple zeros of the form*

$$\rho_k = \frac{1}{a}(k\pi i + iQ(a) + \kappa_1 \pm \kappa_2) + O(k^{-1}), \tag{16}$$

where

$$\kappa_1 = \frac{1}{2} ln \frac{\beta_1 + 1}{\beta_1 - 1}, \qquad \kappa_2 = \frac{1}{2} ln \frac{i(a-1)+1}{i(a-1)-1}.$$

(2) *The set Λ' coincides with the set of nonzero eigenvalues of L. For $\rho_k \in \Lambda'$, the functions $e(x,\rho_k)$, $\varphi(x,\rho_k)$ are eigenfunctions and*

$$e(x,\rho_k) = \gamma_k \varphi(x,\rho_k), \quad \gamma_k \neq 0. \tag{17}$$

(3) *For real $\rho \neq 0$, L has no eigenvalues.*

Proof. By virtue of Theorem 2.4, we have

$$
\begin{aligned}
&\Delta(\rho) = f(\rho) + g(\rho),\\
&f(\rho) = \frac{\rho}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-1 \pm i(a-1))(1-\beta_1)exp(\rho a - iQ(a))\\
&\quad -(-1 \mp i(a-1))(1+\beta_1)exp(-\rho a + iQ(a))),\\
&g(\rho) = \frac{\rho}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-1 \pm i(a-1))(1-\beta_1)exp(\rho a - iQ(a))O(\rho^{-1})\\
&\quad -(-1 \mp i(a-1))(1+\beta_1)exp(-\rho a + iQ(a))O(\rho^{-1})).
\end{aligned}
$$

Whereas for sufficiently large ρ, $|f(\rho)| > |g(\rho)|$, applying Rouche's theorem (see, Conway [1]), the number of zeros of $\Delta(\rho)$ coincides with the number of zeros of $f(\rho)$. Now, solving the equation $f(\rho) = 0$, we obtain for sufficiently large k,

$$\rho_k = \frac{1}{a}\left(k\pi i + iQ(a) + \frac{1}{2} ln\frac{\beta_1 + 1}{\beta_1 - 1} \pm \frac{1}{2}\ln\frac{(i(a-1)+1)}{(i(a-1)-1)}\right).$$

Since $g(\rho) = f(\rho)O(\rho^{-1})$, we obtain zeros of the form (16). For $\rho_k \in \Lambda'$, by virtue of (5) and (6) we result that $\lim_{x\to\infty} e(x,\rho_k) = 0$. Thus ρ_k is an eigenvalue. Since $\langle\varphi(x,\rho), e(x,\rho)\rangle = \triangle(\rho)$, we arrive at (17). Overhand, let ρ_k (complex value) be an eigenvalue and $y(x,\rho_k)$ be a corresponding eigenfunction. Since $U(y(x,\rho_k)) = 0$, $\lim_{x\to\infty} y(x,\rho_k) = 0$, one gets $y(x,\rho_k) = c_k\varphi(x,\rho_k)$, and $y(x,\rho_k) = c_k e(x,\rho_k)$ for $c_k \neq 0$. Therefore this yields (17) and $\Delta(\rho) = U(e(x,\rho_k)) = 0$. To prove part 3, let $\rho_0 \neq 0$ be real, then the function $e(x,\rho)$ does not vanish at infinity. Thus for real $\rho \neq 0$, BVP(L) has no eigenvalues. □

3. Weyl solution and function

We put

$$\phi(x,\rho) = \frac{e(x,\rho)}{\Delta(\rho)}, \tag{18}$$

that the function $\phi(x,\rho)$ be a solution of Eq. (1) under the conditions $U(\phi) = 1$, $\phi(x,\rho) = O\left((x-1)^{\frac{-1}{2}} exp(\pm i\rho x)\right), x \to \infty,\ \rho \in \overline{\Pi}_{\pm}$. Thus $\lim_{x\to\infty}\phi(x,\rho) = 0$. The function $\phi(x,\rho)$ is called the Weyl solution of the boundary value problem L. Denote

$$M(\rho) = \phi(0,\rho). \tag{19}$$

We will call $M(\rho)$ the Weyl function for L. It follows from (18) and (19) that

$$M(\rho) = \frac{e(0,\rho)}{\Delta(\rho)}. \tag{20}$$

The function $M(\rho)$ is regular in $\Pi_{\pm} \setminus \Lambda'_{\pm}$ and continuous in $\overline{\Pi}_{\pm} \setminus \Lambda_{\pm}$. As $|\rho| \to \infty,\ \rho \in \overline{\Pi}_{\pm}$, we have

$$M(\rho) = \frac{1}{\rho(\beta_1 \pm 1)}[1]. \tag{21}$$

Using the initial conditions at the point $x = 0$, we get

$$\phi(x,\rho) = S(x,\rho) + M(\rho)\varphi(x,\rho), \tag{22}$$

$$\langle\varphi(x,\rho), \phi(x,\rho)\rangle = 1, \tag{23}$$

where $\langle y,z\rangle = yz' - y'z$.

Lemma 3.1. *Fix $\delta > 0$. Denote $G_\delta := \{\rho \in C : |\rho - \rho_k| \geqslant \delta, \rho_k \in \Lambda\}$. Then the following inequalities are valid:*

$$|e^{(m)}(x,\rho)| \leqslant C|\rho|^m exp(-|Im\rho|x), \quad x \geqslant a,$$
$$|e^{(m)}(x,\rho)| \leqslant C|\rho|^m exp(-|Im\rho|a)exp(|Re\rho|(a-x)), \quad x \in [0,a],$$
$$|\Delta(\rho)| \geqslant C|\rho|exp(-|Im\rho|a)exp(|Re\rho|a), \quad \rho \in G_\delta,$$
$$|\phi^{(m)}(x,\rho)| \leqslant C|\rho|^{m-1} exp(-|Re\rho|a)exp(-|Im\rho|(x-a)),\ \rho \in G_\delta, \quad x \geqslant a,$$
$$|\phi^{(m)}(x,\rho)| \leqslant C|\rho|^{m-1} exp(-|Re\rho|x),\ \rho \in G_\delta, \quad x \in [0,a],$$
$$|M(\rho)| \leqslant C|\rho|^{-1}, \quad \rho \in G_\delta.$$

Proof. It follows for $x \geqslant a$

$$\begin{aligned}|e^{(m)}(x,\rho)| &= \left|(\pm i\rho)^m R(x)^{m-\frac{1}{2}} exp(\pm(i\rho x - Q(x))[1]\right| \\ &= \left|(\pm i)^m R(x)^{m-\frac{1}{2}} exp(\pm(iRe\rho x - Q(x))) \times [1]\right| |\rho|^m exp(-|Im\rho|x) \\ &\leqslant C|\rho|^m exp(-|Im\rho|x),\end{aligned}$$

and using triangle equality for the function $e(x,\rho)$ in $[0,a]$, we have

$$\begin{aligned}|e^{(m)}(x,\rho)| =& \Big|\frac{\rho^m}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-1)^m(1 \mp i(a-1))exp(\rho a - iQ(a)) \\ & \times exp(-\rho x + iQ(x))[1] + (1 \pm i(a-1))exp(-\rho a + iQ(a))exp(\rho x - iQ(x))[1])\Big| \\ & \leqslant \left|\frac{\rho^m}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))\right| (|(-1)^m(1 \mp i(a-1))exp(\rho a - iQ(a)) \\ & \times exp(-\rho x + iQ(x))[1]| + |(1 \pm i(a-1))exp(-\rho a + iQ(a))exp(\rho x - iQ(x))[1]|) \\ & = \Big\{\left|\frac{1}{2}(a-1)^{\frac{-1}{2}} exp(\pm iRe\rho a)exp(\mp Q(a))\right| (|(-1)^m(1 \mp i(a-1))exp(iIm\rho a) \\ & \times exp(-iIm\rho x)exp(-iQ(a))exp(iQ(x))[1]| + |(1 \pm i(a-1))exp(-iIm\rho a) \\ & \times exp(iIm\rho x)exp(iQ(a))exp(-iQ(x))[1]|)\Big\} |\rho|^m exp(-|Im\rho|a)exp(|Re\rho|(a-x)) \\ & = C|\rho|^m exp(-|Im\rho|a)exp(|Re\rho|(a-x)).\end{aligned}$$

Since $|x \pm y| \geqslant ||x| - |y||$, we have for $\rho \in G_\delta$,

$$\begin{aligned}|\Delta(\rho)| =& \Big|\frac{\rho}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-1 \pm i(a-1))(1-\beta_1) \\ & \times exp(\rho a - iQ(a))[1] - (-1 \mp i(a-1))(1+\beta_1)exp(-\rho a + iQ(a))[1])\Big| \\ \geqslant & \left|\frac{\rho}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))\right| \times ||(-1 \pm i(a-1))(1-\beta_1) \\ & \times exp(\rho a - iQ(a))[1]| - |(-1 \mp i(a-1))(1+\beta_1) \\ & \times exp(-\rho a + iQ(a))[1]||\end{aligned}$$

$$= \left\{ \left| \frac{1}{2}(a-1)^{\frac{-1}{2}} exp(\pm iRe\rho a) exp(\mp Q(a)) \right| ||(-1 \pm i(a-1)) \times (1 - \beta_1) exp(Re\rho a) exp(-iQ(a))[1]| - |(-1 \mp i(a-1))(1+\beta_1) \times exp(-Re\rho a) \times exp(iQ(a))[1]|| \right\} |\rho| exp(-|Im\rho|a) exp(|Re\rho|a)$$

$$= C|\rho| exp(-|Im\rho|a) exp(|Re\rho|a).$$

Now, taking (18) and (20), we infer for $\rho \in G_\delta$,

$$|\phi^{(m)}(x,\rho)| \leqslant C|\rho|^{m-1} exp(-|Re\rho|a) exp(-|Im\rho|(x-a)), \ \ x \geqslant a,$$

$$|\phi^{(m)}(x,\rho)| \leqslant C|\rho|^{m-1} exp(-|Re\rho|x), \quad x \in [0,a],$$

$$|M(\rho)| \leqslant C|\rho|^{-1}.$$

Lemma 3.1 is proved. □

Now we will study the inverse problem for the boundary value problem L. The inverse problem is formulated as follows:

Inverse Problem 3.2. Given the Weyl function $M(\rho)$, construct the potentials $q_1(x)$, $q_0(x)$ and the coefficients β_1, β_0.

4. Uniqueness theorem

In this section, we prove the uniqueness theorem for the solution of the inverse problem from the given Weyl function. We agree that together with $L = L(q_1(x),q_0(x),\beta_1,\beta_0)$, consider a boundary value problem $\widetilde{L} = L(\tilde{q}_1(x), \tilde{q}_0(x), \tilde{\beta}_1, \tilde{\beta}_0)$ of the same form but with different coefficients. Also if a certain symbol denotes an object related to L, then the corresponding symbol with tilde will denote the analogs' object related to $\widetilde{L}$.

Theorem 4.1. *If $M(\rho) = \widetilde{M}(\rho)$ then $q_1(x) = \tilde{q}_1(x), q_0(x) = \tilde{q}_0(x)$ for $x > 0$, $\beta_1 = \tilde{\beta}_1$ and $\beta_0 = \tilde{\beta}_0$. Thus the specification of the Weyl function uniquely determines the coefficients.*

Proof. We consider the matrix $P(x,\rho) = [P_{j,k}(x,\rho)]_{j,k=1,2}$ defined by

$$P(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\phi}(x,\rho) \\ \tilde{\varphi}'(x,\rho) & \tilde{\phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \phi(x,\rho) \\ \varphi'(x,\rho) & \phi'(x,\rho) \end{bmatrix}. \tag{24}$$

By virtue of (23), this yields

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{\phi}'(x,\rho) - \phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = \phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\phi}(x,\rho). \end{cases} \tag{25}$$

Using (22) and (25), we calculate

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\widetilde{S}'(x,\rho) - S^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho) + \widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = S^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\widetilde{S}(x,\rho) - \widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho), \end{cases} \tag{26}$$

where $\widehat{M}(\rho) = \widetilde{M}(\rho) - M(\rho)$. Since $M(\rho) = \widetilde{M}(\rho)$, deduce $\widehat{M}(\rho) = 0$, and consequently the functions $P_{jk}(x,\rho)$, $k = 1,2$, are entire in ρ for each fixed $x \geqslant 0$. It follows from Corollary 2.3, Lemma 3.1 and (25) that for $x \geqslant 0$, $\rho \in G_\delta$,

$$|P_{11}(x,\rho)| \leqslant C, \quad |P_{12}(x,\rho)| \leqslant C|\rho|^{-1}.$$

Therefore $P_{11}(x,\rho) = P_1(x)$ and $P_{12}(x,\rho) = 0$ for each $x \geqslant 0$. Together with (24), we have for all x and ρ that

$$P_1(x)\tilde{\varphi}(x,\rho) = \varphi(x,\rho), \quad P_1(x)\tilde{\phi}(x,\rho) = \phi(x,\rho). \tag{27}$$

Since $M(\rho) = \widetilde{M}(\rho)$, it follows from (21) that $\beta_1 = \tilde{\beta}_1$.

First let $x \in [0,a]$. Taking the functions $e(x,\rho)$, $\varphi(x,\rho)$ in $[0,a]$, $\Delta(\rho)$, (18) and equality $\beta_1 = \tilde{\beta}_1$, we have as $\arg \rho \in \left(0, \frac{\pi}{2}\right)$, $|\rho| \to \infty$

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp(-i(Q(x) - \widetilde{Q}(x)))[1], \quad \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp(i(Q(x) - \widetilde{Q}(x)))[1]. \tag{28}$$

One has from (27) and (28) that

$$P_1(x) = exp(-i(Q(x) - \widetilde{Q}(x)))[1], \quad P_1(x) = exp(i(Q(x) - \widetilde{Q}(x)))[1], \tag{29}$$

and consequently, $Q(x) = \widetilde{Q}(x)$ and $P_1(x) = 1$ for $x \in [0,a]$.

Now let $x \geqslant a$. Taking the functions $e(x,\rho)$, $\varphi(x, \rho)$ in $[a,\infty)$, $\Delta(\rho)$, (18) and equalities $\beta_1 = \tilde{\beta}_1, Q(a) = \widetilde{Q}(a)$ into accounts, we get as $\arg \rho \in \left(0, \frac{\pi}{2}\right), |\rho| \to \infty$

$$\begin{cases} \frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp\Big((x-1)^{\frac{-1}{2}}(Q_a(x) - \widetilde{Q}_a(x))\Big)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp(-(Q_a(x) - \widetilde{Q}_a(x)))[1], \end{cases} \tag{30}$$

where $Q_a(x) = \frac{1}{2}\int_a^x q_1(t)dt$. It follows from (27) and (30) that

$$P_1(x) = exp\Big((x-1)^{\frac{-1}{2}}(Q_a(x) - \widetilde{Q}_a(x))\Big)[1], \quad P_1(x) = exp(-(Q_a(x) - \widetilde{Q}_a(x)))[1]. \tag{31}$$

Since $(x-1)^{\frac{-1}{2}} + 1 > 0$, deduce $Q_a(x) = \widetilde{Q}_a(x)$ and $P_1(x) = 1$ for $x \geqslant a$.

Thus $q_1(x) = \widetilde{q_1}(x), P_1(x) = 1$ for all $x \geqslant 0$. According to (27), we have

$$\varphi(x,\rho) = \tilde{\varphi}(x,\rho), \quad \phi(x,\rho) = \tilde{\phi}(x,\rho). \tag{32}$$

Hence $q_0(x) = \widetilde{q_0}(x)$ on $(0,\infty)$ and $\beta_0 = \tilde{\beta}_0$. Theorem 4.1 is proved. □

Remark 4.2. Using the method of spectral mappings (see, Yurko [9,10]) on the properties of the Weyl function obtained above, one can obtain a procedure for the solution of the inverse problem along with necessary and sufficient conditions for its solvability.

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References

[1] J.B. Conway, Functions of One Complex Variable, Springer, New York, 1995.
[2] W. Eberhard, G. Freiling, A. Schneider, Connection formulas for second-order differential equations with a complex parameter and having an arbitrary number of turning points, Math. Nachr. 165 (1994) 205–229.
[3] G. Freiling, V.A. Yurko, Inverse problems for differential equations with turning points, Inverse Problems 13 (1997) 1247–1263.
[4] G. Freiling, V.A. Yurko, Inverse spectral problems for differential equations on the half-line with turning points, J. Dif. Eqs. 154 (1999) 419–453.
[5] E.Y. Khruslov, D.G. Shepelsky, Inverse scattering method in electromagnetic sounding theory, Inverse Problems 10 (1994) 1–37.
[6] R. Mennicken, M. Moller, Non-Self-Adjoint Boundary Eigenvalue Problems, Elsevier, Amesterdam, 2003.
[7] J.D. Tamarkin, On Some Problems of the Theory of Ordinary Linear Differential Equations, Petrograd, 1917.
[8] V.A. Yurko, An inverse problem for pencils of differential operators, Sb. Math. 191 (2000) 1561–1586.
[9] V.A. Yurko, Inverse Spectral Problems for Differential Operators and Their Applications, Gordon and Breach, New York, 2000.
[10] V.A. Yurko, Method of Spectral Mappings in the Inverse Problem Theory, Inverse and III-Posed Problems Series, VSP, Utrecht, 2002.
[11] V.A. Yurko, Inverse spectral problems for differential pencils on the half-line with turning points, J. Math. Anal. Appl. 320 (2006) 439–463.
[12] V.A. Yurko, The inverse problem for pencils of differential operators on the half-line with turning points, J. Math. Sci. 150 (2008) 2620–2627.