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The Aleksandrov problem on non-Archimedean normed space

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Abstract Let X and Y be non-Archimedean normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with a non-Archimedean valuation. A mapping $f: X \to Y$ preserves distance n if for all $x, y \in X$ with $\|x - y\| = n$ it follows that $\|f(x) - f(y)\| = n$ and conversely. In this paper we shall study, instead of isometries, mappings satisfying the weaker assumption that they preserve distance in both directions. We shall prove that such mappings are not very far from being isometries.

1. Introduction

Alexandrov (1970) posed the question that: whether the existence of the single preserved distance implies that f is an isometry from the metric space E into itself.

Until now, the Alesandrov problem in linear normed spaces (Rassias and Semrl, 1993), linear 2-normed spaces (Chu et al., 2004) and linear n-normed spaces (Chu

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et al., 2004) has been studied, and some theorems were proved under some conditions. For more references on Alesandrov problem see Chen and Song, 2010; Ding, 2006; Ma, 2000. A natural question is that: Whether the Alesandrov problem can be proved in non-Archimedean normed under some conditions.

A non-Archimedean field is a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathcal{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leqslant max\{|r|,|s|\}$ for all $r,s \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in N$. An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.

If $\mathcal{K}$ is a field, for any $r \in \mathcal{K}$ then $r = r_1 + r_2$ is valid for any $r_1, r_2 \in \mathcal{K}$ with $r_1 \cdot r_2 > 0$. Another example of a non-Archimedean valuation is the mapping

$$|r|_1 = \begin{cases} 0, & \text{if} \quad r = 0, \\ \frac{1}{r}, & \text{if} \quad r > 0, \\ -\frac{1}{r}, & \text{if} \quad r < 0. \end{cases}$$

for any $r \in \mathcal{K}$.

Let X be a vector space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| (r \in \mathcal{K}, x \in X)$;
(iii) the strong triangle inequality

$$\|x + y\| \leqslant \max\{\|x\|, \|y\|\} (x, y \in X).$$

Then $(x, \|\cdot\|)$ is called a non-Archimedean space.

In this paper, we just consider only X and Y are non-Archimedean normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with non-Archimedean valuation $|\cdot|_1$ defined above, if without special statements.

2. Main results

Definition 2.1. Let X and Y be two non-Archimedean normed spaces. A mapping $f: X \to Y$ of X onto Y, is called an isometry if

$$\|f(x) - f(y)\| = \|x - y\|$$

for all x, $y \in X$.

If a mapping f of X onto Y is an isometry then the inverse mapping $f^{-1} : Y \to X$ is an isometry of Y onto X.

Definition 2.2. Let X and Y be two non-Archimedean normed spaces. A mapping $f: X \to Y$ satisfies the strong distance one preserving property (SDOPP) iff for all x, $y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

Definition 2.3. Let X and Y be two non-Archimedean normed spaces. A mapping $f: X \to Y$ satisfies the strong preserving distance n property (SDnPP) iff for all $x, y \in X$ with $\|x-y\| = n$ it follows that $\|f(x)-f(y)\| = n$ and conversely.

Definition 2.4. Let X and Y be two non-Archimedean normed spaces. We call a mapping $f: X \to Y$ Lipschitz mapping if there is a $K > 0$ such that

$$\|f(x) - f(y)\| \leqslant K\|x-y\|$$

for any $x, y \in X$.

In this paper we shall study, instead of isometries, mapping satisfying the weaker assumption that they preserve distance n in both direction. We shall see that such mappings are not far from being isometrics. More precisely we shall prove

Theorem 2.5. *Let X and Y be non-Archimedean normed spaces such that one of them has dimension greater than one. Suppose that $f: X \to Y$ is a surjective mapping satisfying (SDnPP) for any interger n. Then f is an injective mapping satisfying*

$$-1 < \|f(x) - f(y)\| - \|x-y\| < 1 \tag{1}$$

for all $x, y \in X$.

Proof. We shall show that both spaces have dimension greater than one. Let us first assume that $\dim Y \geqslant 2$. Because f satisfying SDnPP, so f satisfying SDOPP. It follows that there exist vectors $x, y, z \in Y$ such that

$$\|x-y\| = \|x-z\| = \|y-z\| = 1.$$

The mapping f is given to be surjective and to preserve distance one in both directions. Thus we can find $x_1, y_1, z_1 \in X$ satisfying

$$\|x_1 - y_1\| = \|x_1 - z_1\| = \|y_1 - z_1\| = 1.$$

This implies that $\dim X \geqslant 2$. Similarly, one can prove that if $\dim X \geqslant 2$ then $\dim Y \geqslant 2$. Claim that f is injective. Suppose on the contrary, that there are $x, y \in X$, $x \neq y$, such that $f(x) = f(y)$. We can find a vector $z \in X$ such that $\|x-z\| = 1$ and $\|z-y\| \neq 1$. Then $\|f(x)-f(z)\| = \|f(y)-f(z)\| = 1$. This implies that $\|y-z\| = 1$, which is a contradiction. Therefore f is a bijective mapping. Both f and f^{-1} preserve unit distance, more over f preserves distance n in both direction for any positive integer n.

In the sequel we shall need the following notations:

$$B(x,r) = \{z : \|z-x\| \leqslant r\},$$
$$B_0(x,r) = \{z : \|z-x\| < r\},$$
$$C_x(n, n+1] = \{z : n < \|z-x\| \leqslant n+1\}.$$

Let x be an arbitrary vector in X and n any positive integer, $n > 1$. Assume that $z \in B(x,n)$. Since $\dim X \geqslant 2$, we can find a sequence $x = x_0, x_1, \ldots, x_n = z$, such that

$$\|x_{i+1} - x_i\| = 1, \quad i = 0, 1, 2, \ldots, n-1.$$

Consequently, we have

$$\|f(x_n) - f(x_0)\| = \|f(z) - f(x)\| \leqslant n.$$

Therefore,

$$f(B(x,n)) \subset B(f(x),n)$$

The same result can be obtained for f^{-1}. Hence,

$$f(B(x,\, n)) = B(f(x),\, n), \quad x \in X, \quad n \in N \setminus \{1\}.$$

However, f is bijective and thus

$$f(C_x(n,n+1]) = C_{f(x)}(n,n+1], \quad x \in X, \quad n \in N \setminus \{1\}. \tag{2}$$

We fix an element $x \in X$ and choose $z \in C_x(1,2]$. We know that $f(z) \in B(f(x),2)$. The vector $u = x + \|z - x\|(z - x)$, then

$$\|u - x\| = \|\|z - x\|(z - x)\| = \frac{\|z - x\|}{\|z - x\|} = 1$$

so $\|f(u) - f(x)\| = 1$. Since

$$\|u - z\| \leqslant \max\{\|u - x\|, \|x - z\|\} = \|x - z\|$$

and

$$\|x - z\| \leqslant \max\{\|x - u\|, \|u - z\|\} = \|u - z\|.$$

So

$$\|u - z\| = \|x - z\|,$$

u is contained in $C_z(1,2]$, According to (2) we have $f(u) \in C_{f(z)}(1,2]$. Thus,

$$\|f(z) - f(u)\| > 1. \tag{3}$$

Let us assume that $\|f(z) - f(x)\| \leqslant 1$ holds, then

$$\|f(z) - f(u)\| \leqslant \max\{\|f(x) - f(z)\|, \|f(x) - f(u)\|\} = 1.$$

which contradicts (3).We have proved that

$$f(C_x(1,2]) = C_{f(x)}(1,2].$$

The same result holds for the mapping f^{-1}. Consequently, the relations

$$f(C_x(1,2]) = C_{f(x)}(1,2] \quad \text{and} \quad f(B_0(x,1]) = B_0(f(x),1)$$

hold for all $x \in X$. This together with (2) implies the inequality (1). $\square$

Theorem 2.6. *Let X and Y be non-Arichimedean normed spaces and one of them has dimension greater than one. Suppose that $f : X \to Y$ is a Lipschiz mapping with $K = 1$:*

$$\|f(x) - f(y)\| \leqslant \|x - y\| \quad \text{for any} \quad x, y \in X.$$

Assume also that f is a surjective mapping satisfying (SDnPP) and $\|f(x) - f(y)\| = \|x - y\|$ when $\|x - y\| \leqslant 1$ for all $x, y \in X$, then f is an isomety.

Proof. According to Theorem 2.5, f preserves distance n in both directions for any positive integer n. Choose $x, y \in X$ and find a positive integer $n_0 \geqslant 2$ satisfying $n_0 - 1 \leqslant \|x - y\| < n_0$. Assume that

$$\|f(x) - f(y)\| < \|x - y\|. \tag{4}$$

Set

$$z = x + \frac{\|x - y\|}{n_0}(x - y).$$

Clearly,

$$\|z - x\| = \left\|\frac{\|x - y\|}{n_0}(x - y)\right\| = n_0$$

and

$$\|z - y\| = \|x - y + \frac{\|x - y\|}{n_0}(x - y)\| = \left\|\left(1 + \frac{\|x - y\|}{n_0}\right)(x - y)\right\|$$
$$= \frac{n_0}{\|x - y\| + n_0}\|x - y\| < n_0.$$

It follows that

$$\|f(z) - f(x)\| = n_0 \quad \text{and} \quad \|f(z) - f(y)\| < n_0.$$

On the other hand,

$$\|f(z) - f(x))| \leqslant \max\{\|f(z) - f(y)\|, \|f(y) - f(x)\|\} < \max\{n_0, \|x - y\|\} = n_0,$$

which contradicts

$$\|f(x) - f(y)\| = n_0.$$

Therefore (4) is not valid. Hence

$$\|f(x) - f(y)\| = \|x - y\| \quad \textit{for any} \quad x, y \in X,$$

which implies that f is a isometry. □

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