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# The ($\leq$ 5)-hypomorphy of digraphs up to complementation

AYMEN BEN AMIRA[a], BECHIR CHAARI[a], JAMEL DAMMAK[a], HAMZA SI KADDOUR[b,*]

[a] Department of Mathematics, Faculty of Sciences of Sfax, B.P. 1171, 3000 Sfax, Tunisia
[b] Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France

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**Abstract.** Two digraphs $G = (V, E)$ and $G' = (V, E')$ are isomorphic up to complementation if $G'$ is isomorphic to $G$ or to the complement $\overline{G} := (V, \{(x, y) \in V^2 : x \neq y, (x, y) \notin E\})$ of $G$. The Boolean sum $G\dot{+}G'$ is the symmetric digraph $U = (V, E(U))$ defined by $\{x, y\} \in E(U)$ if and only if $(x, y) \in E$ and $(x, y) \notin E'$, or $(x, y) \notin E$ and $(x, y) \in E'$. Let $k$ be a nonnegative integer. The digraphs $G$ and $G'$ are ($\leq k$)-hypomorphic up to complementation if for every $t$-element subset $X$ of $V$, with $t \leq k$, the induced subdigraphs $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic up to complementation. The digraphs $G$ and $G'$ are hereditarily isomorphic (resp. hereditarily isomorphic up to complementation) if for each subset $X$ of $V$, the induced subdigraphs $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic (resp. isomorphic up to complementation). Here, we give the form of the pair $\{G, G'\}$ whenever $G$ and $G'$ are two digraphs, ($\leq$ 5)-hypomorphic up to complementation and such that the Boolean sum $U := G\dot{+}G'$ and the complement $\overline{U}$ are both connected, and thus we deduce that $G$ and $G'$ are hereditarily isomorphic up to complementation; we prove also that the value 5 is optimal. The case $U$ or $\overline{U}$ is not connected is studied in a forthcoming paper.

Keywords: Digraph; Isomorphism; $k$-hypomorphy up to complementation; Hereditary isomorphy up to complementation; Boolean sum; Symmetric digraph; Tournament; Indecomposability

* Corresponding author.
*E-mail addresses:* ayman.benamira@yahoo.fr (A. Ben Amira), chaari.bechir@ymail.com (B. Chaari), jdammak@yahoo.fr (J. Dammak), sikaddour@univ-lyon1.fr (H. Si Kaddour).
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## 1. Introduction

In this paper, we study the reconstruction of digraphs up to complementation (definitions and notations are given in Section 2). Ulam's reconstruction conjecture on digraphs [22], still unsolved for graphs, is well-known (see [2,3]). Fraïssé made a related conjecture about relational structures. The case of binary relations was solved by Lopez [14–16], he showed that all binary relations are $(\leq 6)$-reconstructible. The case of ternary relations was solved negatively by Pouzet [17]. On the other hand, Stockmeyer [20] showed that the tournaments are not, in general, $(-1)$-reconstructible, so invalidating the conjecture of Ulam for digraphs. In 1993, Hagendorf raised the $(\leq k)$-half-reconstruction problem for digraphs and solved it with Lopez [12,13], they showed that *the finite digraphs are* $(\leq 12)$*-half-reconstructible*. In 1995, Boudabbous and Lopez [6] showed that *the finite tournaments are* $(\leq 7)$*-half-reconstructible*. This motivated, in 2013, M. Alzohairi, M. Bouaziz and Y. Boudabbous to introduce the concept of $(\leq k)$-hereditary reconstructibility of posets [1]. In 2015, Y. Boudabbous proposed the problem of $(\leq k)$-hereditarily reconstruction of digraphs. He solved this problem for tournaments with A. Boussaïri, A. Chaïchaâ and N. El Amri [5].

We say that a symmetric digraph $G$ is *connected* if for any distinct vertices $a$ and $b$ of $G$, there are vertices $a = x_0, x_1, \dots, x_m = b$ of $G$, such that $x_i \underline{\quad}_G x_{i+1}$ for each $i \in \{0, \dots, m-1\}$. Otherwise $G$ is said *disconnected*. A *component* of $G$ is a maximal connected subdigraph of $G$. Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, 2-hypomorphic up to complementation. *The Boolean sum* $G \dot{+} G'$ of $G$ and $G'$ is the symmetric digraph $U = (V, E(U))$ defined by $\{x, y\} \in E(U)$ if and only if $(x, y) \in E$ and $(x, y) \notin E'$, or $(x, y) \notin E$ and $(x, y) \in E'$. Clearly $\overline{U} = \overline{G} \dot{+} G'$. Denote $\mathfrak{D}_{G,G'}$ the binary relation on $V$ such that: for $x \in V$, $x \ \mathfrak{D}_{G,G'} \ x$; and for $x \neq y \in V$, $x \ \mathfrak{D}_{G,G'} \ y$ if there exists a sequence $x = x_0, x_1, \dots, x_m = y$ of elements of $V$ satisfying $(x_i, x_{i+1}) \in E$ if and only if $(x_i, x_{i+1}) \notin E'$, for each $i$, $0 \leq i \leq m-1$. The relation $\mathfrak{D}_{G,G'}$ is an equivalence relation called *the difference relation*, its classes are called *difference classes*, this relation was introduced by Lopez [14]. Then clearly $C$ is a connected component of $U := G \dot{+} G'$ if and only if $C$ is an equivalence class of $\mathfrak{D}_{G,G'}$, and thus $\mathfrak{D}_{G,G'}$ and $\mathfrak{D}_{\overline{G},G'}$ have only one class if and only if $U$ and $\overline{U}$ are connected. In 2003, Dammak [8] proved the following result.

**Proposition 1.1** (*[8]*)**.** *Let $T$ and $T'$ be two finite tournaments,* $(\leq 5)$*-hypomorphic up to complementation, and $U := T \dot{+} T'$. If $U$ and $\overline{U}$ are connected, then $T$ and $T'$ are total orders.*

In 1999, Ille raised the problem of the $(\leq k)$-reconstruction up to complementation of digraphs. The case of symmetric digraphs was solved by Dammak, Lopez, Pouzet and Si Kaddour [9,10], they proved that, *the symmetric digraphs on $v$ vertices are $t$-reconstructible up to complementation* for every $4 \leq t \leq v-3$. In fact, the case $t = v-3$ was solved in [10] using the following result established by Pouzet, Si Kaddour and Trotignon [18].

**Theorem 1.2** (*[18]*)**.** *If $G$ and $G'$ are two symmetric digraphs,* 3*-hypomorphic up to complementation and* $|V(G)| \geq 10$*, then the connected components of $U := G \dot{+} G'$, or of its complement $\overline{U}$, are cycles of even length or paths.*

We define the symmetric digraph $P_n$ in the following manner, $V(P_n) = \{0, 1, \dots, n-1\}$, and for $i \neq j \in \{0, 1, \dots, n-1\}$, $\{i, j\}$ is an edge of $P_n$ when $|i - j| = 1$. Thus $P_n := 0 \underline{\quad} 1 \underline{\quad} \dots \underline{\quad} n-2 \underline{\quad} n-1$. A *path* is a symmetric digraph isomorphic to

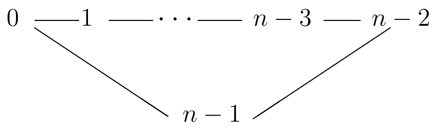



**Fig. 1.** $C_n$.

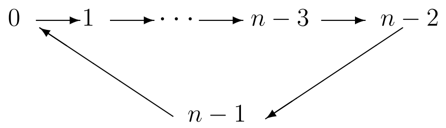



**Fig. 2.** $\overrightarrow{C_n}$.

$P_n$. A *cycle* is a symmetric digraph isomorphic to $C_n := (V(P_n), E(P_n) \cup \{\{0, n-1\}\})$ for some integer $n \geq 3$ (see Fig. 1).

We define the digraph $\overrightarrow{P_n}$ by, for $i \neq j \in \{0, 1, \ldots, n-1\}$, $i \longrightarrow_{\overrightarrow{P_n}} j$ when $j = i + 1$. Thus $\overrightarrow{P_n} := 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow n-1$. We call *directed path* or *oriented path* a digraph isomorphic to $\overrightarrow{P_n}$, and *directed cycle* or *oriented cycle* a digraph isomorphic to $\overrightarrow{C_n} := (V(\overrightarrow{P_n}), E(\overrightarrow{P_n}) \cup \{(n-1, 0)\})$ for some integer $n \geq 3$ (see Fig. 2).

We define $\overrightarrow{P_n^f}$ (resp. $\overrightarrow{C_n^f}$) obtained from $\overrightarrow{P_n}$ (resp. $\overrightarrow{C_n}$) by switching the void pairs by the full pairs. Thus $\overrightarrow{P_n^f} = \overline{\left(\overrightarrow{P_n}\right)^*}$ and $\overrightarrow{C_n^f} = \overline{\left(\overrightarrow{C_n}\right)^*}$.

A *total order* is a tournament $T$ such that for $x, y, z \in V(T)$, if $x \longrightarrow_T y$ and $y \longrightarrow_T z$ then $x \longrightarrow_T z$. Given a total order $O = (V, E)$, for $x, y \in V$, $x < y$ means $x \longrightarrow_O y$. Thus, a total order on $n$ vertices can be denoted by $v_0 < v_1 < \cdots < v_{n-1}$.

Our main result is the following.

**Theorem 1.3.** *Let $G$ and $G'$ be two digraphs on the same set $V$ of $n \geq 4$ vertices such that $G$ and $G'$ are ($\leq 5$)-hypomorphic up to complementation. Let $U := G \dot{+} G'$. If $U$ and $\overline{U}$ are connected, then $G'$ and $G$ are hereditarily isomorphic up to complementation; more precisely one of the following holds:*

(1) *$G$ and $G'$ are two total orders.*
(2) *$G \simeq \overrightarrow{P_n}$ or $G \simeq \overrightarrow{C_n}$, and $G' = G^*$.*
(3) *$G \simeq \overrightarrow{P_n}$ or $G \simeq \overrightarrow{C_n}$, and $G' = \overline{G^*}$.*
(4) *$G \simeq \overrightarrow{P_n^f}$ or $G \simeq \overrightarrow{C_n^f}$, and $G' = G^*$.*
(5) *$G \simeq \overrightarrow{P_n^f}$ or $G \simeq \overrightarrow{C_n^f}$, and $G' = \overline{G^*}$.*

In Proposition 3.5, we prove that the value 5 is optimal by giving two digraphs $G, G'$, on the same vertex set $V$ with $|V| \geq 5$, which are ($\leq 4$)-hypomorphic up to complementation and not ($\leq 5$)-hypomorphic up to complementation, $U := G \dot{+} G'$ and $\overline{U}$ are connected but $G$ and $G'$ are not isomorphic up to complementation, and thus not hereditarily isomorphic up to complementation.

From Theorem 1.3, we deduce trivially the following result for digraphs which is similar to Theorem 1.2.

**Corollary 1.4.** *Let $G$ and $G'$ be two digraphs on the same set $V$ of $n \geq 4$ vertices such that $G$ and $G'$ are $(\leq 5)$-hypomorphic up to complementation and $U := G \dot{+} G'$. If $U$ and $\overline{U}$ are connected and $G$ is not a total order, then $U$ or $\overline{U}$ is a cycle or a path.*

## 2. Definitions and notations

A *directed graph* or simply *digraph* $G$ consists of a finite and nonempty set $V$ of vertices together with a prescribed collection $E$ of ordered pairs of distinct vertices, called the set of the *edges* of $G$. Such a digraph is denoted by $(V(G), E(G))$ or simply $(V, E)$. Given a digraph $G = (V, E)$, to each nonempty subset $X$ of $V$ associate the *subdigraph* $(X, E \cap (X \times X))$ of $G$ induced by $X$ denoted by $G_{\restriction X}$. Given a proper subset $X$ of $V$, $G_{\restriction V \setminus X}$ is also denoted by $G - X$, and by $G - v$ whenever $X = \{v\}$. With each digraph $G = (V, E)$ associate its *dual* $G^* = (V, E^*)$ and its *complement* $\overline{G} = \left(V, \overline{E}\right)$ defined as follows. Given $x \neq y \in V$, $(x, y) \in E^*$ if $(y, x) \in E$, and $(x, y) \in \overline{E}$ if $(x, y) \notin E$.

Let $G = (V, E)$ be a digraph, for $x \neq y \in V$, $x \longrightarrow_G y$ or $y \longleftarrow_G x$ (or simply $x \longrightarrow y$ if there is no confusion) means $(x, y) \in E$ and $(y, x) \notin E$; $x \underline{\quad}_G y$ (or simply $x \underline{\quad} y$) means $(x, y) \in E$ and $(y, x) \in E$; $x \ldots_G y$ (or $x \ldots y$ or $x \quad_G y$) means $(x, y) \notin E$ and $(y, x) \notin E$. For $X, Y \subseteq V$, $X \underline{\quad}_G Y$ and $X \ldots_G Y$ (or $X \quad_G Y$) are defined in the same way. If $X = \{x\}$ or $Y = \{y\}$, we can replace $X$ by $x$ and $Y$ by $y$.

Given a digraph $G = (V, E)$, two distinct vertices $x$ and $y$ of $G$ form a *directed pair* or *oriented pair* if either $x \longrightarrow_G y$ or $x \longleftarrow_G y$. Otherwise, $\{x, y\}$ is a *neutral pair*; it is *full* if $x \underline{\quad}_G y$, and *void* if $x \ldots_G y$. Two interesting types of digraphs are symmetric digraphs and tournaments. A digraph $G = (V, E)$ is a *symmetric digraph* or *graph* (resp. *tournament*) whenever for $x \neq y \in V$, $x \underline{\quad}_G y$ or $x \ldots_G y$ (resp. $x \longrightarrow_G y$ or $y \longrightarrow_G x$). If $G = (V, E)$ is a graph, each edge $(x, y)$ of $G$ is identified with the pair $\{x, y\}$ and is called an *edge* of $G$. For instance, given a set $V$, $(V, \emptyset)$ is the *empty graph* on $V$ whereas $(V, [V]^2)$ is the *complete graph* on $V$, where $[V]^2$ is the set of pairs $\{x, y\}$ of distinct elements of $V$.

Given two digraphs $G = (V, E)$ and $G' = (V', E')$, a bijection $f$ from $V$ onto $V'$ is an *isomorphism* from $G$ onto $G'$ provided that for any $x, y \in V$, $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. The digraphs $G$ and $G'$ are *isomorphic*, which is denoted by $G \simeq G'$, if there exists an isomorphism from one onto the other, otherwise $G \not\simeq G'$. A digraph $H$ *embeds* into $G$, or $H$ is *embeddable* in $G$, if $H$ is isomorphic to an induced subdigraph of $G$.

Given two digraphs $G$ and $G'$ on the same vertex set $V$. They are *equal up to complementation* if $G' = G$ or $G' = \overline{G}$. Let $k$ be an integer with $0 < k < |V|$, the digraphs $G$ and $G'$ are $k$-*hypomorphic* (resp. $(-k)$-*hypomorphic*) if for every $k$-element (resp. $(|V| - k)$-element) subset $X$ of $V$, the induced subdigraphs $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic. The digraphs $G$ and $G'$ are $(\leq k)$-*hypomorphic* if they are $t$-hypomorphic for each integer $t \leq k$. A digraph $G$ is $k$-*reconstructible* (resp. $(-k)$-*reconstructible*) if any digraph $k$-hypomorphic (resp. $(-k)$-hypomorphic) to $G$ is isomorphic to $G$. A digraph $G$ is $(\leq k)$-*reconstructible* if any digraph $(\leq k)$-hypomorphic to $G$ is isomorphic to $G$. The digraphs $G$ and $G'$ are *isomorphic up to complementation* (resp. *hemimorphic*) if $G'$ is isomorphic to $G$ or $\overline{G}$ (resp. to $G$ or $G^*$). The digraphs $G'$ and $G$ are *hereditarily isomorphic* [19] if for each nonempty subset $X$ of $V$, the digraphs $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic. They are *hereditarily isomorphic up to complementation* [4] if they are hereditarily isomorphic, or $G'$

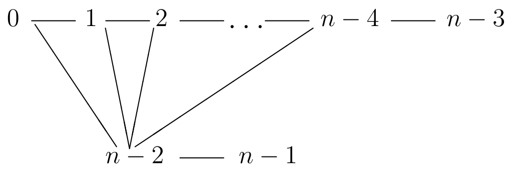



**Fig. 3.** $Q_n$.

and $\overline{G}$ are hereditarily isomorphic. Let $k$ be a positive integer, the digraphs $G$ and $G'$ are *$k$-hypomorphic up to complementation* (resp. *$k$-hemimorphic*) if for every $k$-element subset $X$ of $V$, the induced subdigraphs $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic up to complementation (resp. hemimorphic). The digraphs $G$ and $G'$ are *$(\leq k)$-hypomorphic up to complementation* (resp. *$(\leq k)$-hemimorphic*) if they are $t$-hypomorphic up to complementation (resp. *$t$-hemimorphic*) for each integer $t \leq k$. A digraph $G$ is *$k$-reconstructible up to complementation* (resp. *$k$-half-reconstructible*) if any digraph $k$-hypomorphic up to complementation (resp. $k$-hemimorphic) to $G$ is isomorphic up to complementation (resp. hemimorphic) to $G$. A digraph $G$ is *$(\leq k)$-reconstructible up to complementation* (resp. *$(\leq k)$-half-reconstructible*) if any digraph $(\leq k)$-hypomorphic up to complementation (resp. $(\leq k)$-hemimorphic) to $G$ is isomorphic up to complementation (resp. hemimorphic) to $G$.

A 3-*cycle* is a tournament isomorphic to $\overrightarrow{C_3} := (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$. A *flag* is a digraph hemimorphic to $(\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 1)\})$. A *peak* is a digraph hemimorphic to $(\{0, 1, 2\}, \{(0, 1), (0, 2), (1, 2), (2, 1)\})$ or to $(\{0, 1, 2\}, \{(0, 1), (0, 2)\})$. Let $G$ be a digraph, the *positive degree* (resp. *negative degree*) of a vertex $x$ of $G$, denoted $d^+_G(x)$ (resp. $d^-_G(x)$), is the number of $y \in V(G)$ such that $x \longrightarrow_G y$ (resp. $y \longrightarrow_G x$). Notice that, here, $d^+_G(x)$ (resp. $d^-_G(x)$) is not the *outdegree* (resp. *indegree*) of the vertex $x$. The *type* of $G$ is $(e, e')$ where $e$ and $e'$ are respectively the number of full pairs of $G$ and $\overline{G}$. Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs and $a, b \in V$. We say that $\{a, b\}$ *has the same character in $G$ and $G'$* if and only if $G_{\restriction\{a,b\}} \simeq G'_{\restriction\{a,b\}}$.

Let $G = (V, E)$ be a graph, the *degree* of a vertex $x$ of $G$, denoted $d_G(x)$, is the number of $y \in V(G)$ such that $x \underline{\quad}_G y$.

## 3. THE GALLAI DECOMPOSITION THEOREM

Given a digraph $G = (V, E)$, a subset $I$ of $V$ is an *interval* of $G$ if for every $x \in V \setminus I$ either $x \longrightarrow_G I$ or $x \longleftarrow_G I$ or $x \underline{\quad}_G I$ or $x \ldots_G I$. For instance, $\emptyset$, $V$ and $\{x\}$ (where $x \in V$) are intervals of $G$, called *trivial intervals*. A digraph is *indecomposable* if all its intervals are trivial, otherwise it is *decomposable*.

The graph $Q_n$ (see Fig. 3) is defined in the following manner. For $i \neq j \in \{0, 1, \ldots, n-1\}$, $\{i, j\}$ is an edge of $Q_n$ whenever either $i, j \in \{0, 1, \ldots, n-3\}$ and $|i - j| = 1$ or $\{i, j\} = \{n-2, \ell\}$, where $\ell \in \{0, 1, \ldots, n-4\} \cup \{n-1\}$.

**Theorem 3.1** ([7]). *Let $S = (V, E)$ be an indecomposable graph with $|V| \geq 4$. Let $W$ denote the set of $x \in V$ such that there is a subset $X$ of $V$ satisfying $S_{\restriction X}$ is isomorphic to $P_4$ and $x \in X$. We have: $|V \setminus W| \leq 1$. Furthermore, if $V \setminus W = \{x\}$, then there are a subset $X$ of $V$ containing $x$ and an isomorphism $f$ from $S_{\restriction X}$ onto $Q_5$ such that $f(x) = v_0$.*

**Theorem 3.2** ([7]). *Let $S = (V, E)$ be an indecomposable graph with $|V| \geq 5$. For $a \neq b \in V$, there is a subset $X$ of $V$ satisfying: $a, b \in X$ and there is an isomorphism $f$ from $S_{\restriction X}$ or $\overline{S}_{\restriction X}$ onto $P_k$ or $Q_k$, where $k \geq 5$, such that $f(\{a, b\}) = \{0, k-1\}$.*

We begin with a well-known property of the intervals. Given a digraph $G = (V, E)$, if X and Y are disjoint intervals of G, then $X \longrightarrow_G Y$, or $X \longleftarrow_G Y$, or $X \underline{\quad}_G Y$, or $X \ldots_G Y$. This property leads to consider interval partitions of $G$, that is, partitions of $V$, all the elements of which are intervals of $G$. The elements of such a partition $P$ become the vertices of the quotient $G/P = (P, E/P)$ of $G$ by $P$ defined as follows: given $X \neq Y \in P$, $(X, Y) \in E/P$ if $(x, y) \in E$ for $x \in X$ and $y \in Y$. Given a digraph $G = (V, E)$, a subset $X$ of $V$ is a *strong interval* [11] of $G$ provided that $X$ is an interval of $G$ and for each interval $Y$ of $G$, we have: if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. For $|V| \geq 2$, the family of the maximal strong intervals under inclusion which are distinct from $V$ is denoted by $P(G)$. The family $P(G)$ constitutes an interval partition of $V$. Now we state the Gallai decomposition theorem.

**Theorem 3.3** ([11]). *Given a digraph $G = (V, E)$, with $| V | \geq 2$. The corresponding quotient $G/P(G)$ is a complete digraph or an empty digraph or a total order or an indecomposable digraph with at least 3 vertices.*

The inverse operation of the quotient is the *lexicographic sum* defined as follows: let $m$ be an integer, $m \geq 1$, $S = (\{0, 1, \ldots, m-1\}, E)$ be a digraph and for $i < m$, $G_i = (V_i, E_i)$ be a digraph such that the $V_i$'s are nonempty and pairwise disjoint. The *lexicographic sum over $S$ of the $G_i$*'s or simply the S-*sum* of the $G_i$'s, is the digraph denoted by $S(G_0, G_1, \ldots, G_{m-1})$ and defined on the union of the $V_i$'s as follows: given $x \in V_i$ and $y \in V_j$, where $i, j \in \{0, 1, \ldots, m-1\}$, $(x, y)$ is an edge of $S(G_0, G_1, \ldots, G_{m-1})$ if either $i = j$ and $(x, y) \in E_i$ or $i \neq j$ and $(i, j) \in E$: this digraph replaces each vertex $i$ of $S$ by $G_i$. We say that the vertex $i$ of $S$ is *dilated by $G_i$*.

From Theorem 3.3, we have immediately this result.

**Corollary 3.4.** *Given a graph $G = (V, E)$. Then $G$ and $\overline{G}$ are connected if and only if $G = S(G_0, G_1, \ldots, G_{m-1})$, where $S$ is an indecomposable graph with at least 4 vertices and $G_i$ is a graph for each $i \in \{0, 1, \ldots, m-1\}$.*

The following result shows the optimality of the value 5 in Theorem 1.3.

**Proposition 3.5.** *Let $A_3 := \{\{a_0, b_0, c_0\}, \{(a_0, b_0), (b_0, c_0), (c_0, a_0)\}\}$. Let $G$ (resp. $G'$) be the digraph obtained from $\overrightarrow{P_n}$ $\left(\text{resp. } \left(\overrightarrow{P_n}\right)^*\right)$ by dilating the vertex 0 by $A_3$. Let $U := G\dot{+}G'$. Then $G$ and $G'$ are $(\leq 4)$-hypomorphic up to complementation, not $(\leq 5)$-hypomorphic up to complementation, $U$ and $\overline{U}$ are connected, but $G$ and $G'$ are not isomorphic up to complementation, and thus not hereditarily isomorphic up to complementation.*

**Proof.** Note that $A_3$ is an oriented cycle isomorphic to $C_3$. The graph $U$ is obtained from $P_n$ by dilating the vertex 0 by the empty graph with vertex set $\{a_0, b_0, c_0\}$. By Corollary 3.4, $U$ and $\overline{U}$ are connected. Clearly $G$ and $G'$ are $(\leq 4)$-hypomorphic up to complementation. The subdigraphs $G_{\restriction\{a_0,b_0,c_0,1,2\}}$ and $G'_{\restriction\{a_0,b_0,c_0,1,2\}}$ are not isomorphic because $d^+_{G'_{\restriction\{a_0,b_0,c_0,1,2\}}}(1) = 3$ but $d^+_{G_{\restriction\{a_0,b_0,c_0,1,2\}}}(x) \leq 2$ for all vertex $x$. The subdigraphs $\overline{G}_{\restriction\{a_0,b_0,c_0,1,2\}}$ and $G'_{\restriction\{a_0,b_0,c_0,1,2\}}$ are not isomorphic because there are full edges in $\overline{G}_{\restriction\{a_0,b_0,c_0,1,2\}}$ whereas there are none in $G'_{\restriction\{a_0,b_0,c_0,1,2\}}$. Thus $G$ and $G'$ are not ($\leq$ 5)-hypomorphic up to complementation. As $d^+_{G'}(1) = 3$ and there is no vertex $x$ in $G$ of degree 3, and there are full edges in $\overline{G}$ whereas there are none in $G'$, then $G$ and $G'$ are not isomorphic up to complementation. Thus $G$ and $G'$ are not hereditarily isomorphic up to complementation. □

## 4. Preliminary results

**Theorem 4.1** ([21]). *Let $G$ be a graph. If $G$ and $\overline{G}$ are connected then $G$ embeds a $P_4$.*

**Remark 4.2.** Let $G$ and $G'$ be two digraphs on the same set $V$ such that $G$ and $G'$ are ($\leq$ 3)-hypomorphic up to complementation. Let $U := G\dot{+}G'$ and $a, b, c \in V$. If $G_{\restriction\{a,b,c\}}$ is a peak or a flag, then $U_{\restriction\{a,b,c\}}$ is a complete or an empty graph.

**Lemma 4.3.** *Let $G$ and $G'$ be two digraphs on the same set $V$ such that $G$ and $G'$ are ($\leq$ 3)-hypomorphic up to complementation. Let $U := G\dot{+}G'$ and $a, b, c \in V$.*

(1) *If $E(U_{\restriction\{a,b,c\}})$ or $E(\overline{U}_{\restriction\{a,b,c\}})$ is the set $\{\{a,b\},\{b,c\}\}$, then $\{a,b\}$ is an oriented pair in $G$ if and only if $\{b,c\}$ is an oriented pair in $G$.*
(2) *If $E(U_{\restriction\{a,b,c\}})$ or $E(\overline{U}_{\restriction\{a,b,c\}})$ is the set $\{\{a,b\}\}$ and $\{a,b\}$ is an oriented pair in $G$, then $\{a,b\}$ is an interval of $G_{\restriction\{a,b,c\}}$ and $G'_{\restriction\{a,b,c\}}$.*
(3) *If $E(U_{\restriction\{a,b,c\}})$ or $E(\overline{U}_{\restriction\{a,b,c\}})$ is the set $\{\{a,b\}\}$ and $\{a,b\}$ is a neutral pair in $G$, then $\{a,b\}$ is not an interval of $G_{\restriction\{a,b,c\}}$, and $\{b,c\}$ is an oriented pair in $G$ if and only if $\{a,c\}$ is an oriented pair in $G$. Moreover if $c \longrightarrow_G a$ (resp. $c$ ___$_G a$) then $b \longrightarrow_G c$ (resp. $c \ldots_G b$).*

**Proof.** (1) By contradiction. Without loss of generality (W.l.o.g.), we assume that $a \longrightarrow_G b$ and $b$ ___$_G c$, then $a \longleftarrow_{G'} b$ and $b \ldots_{G'} c$. If $\{a,c\}$ is an oriented pair in $G$ not reversed in $G'$, then $G'_{\restriction\{a,b,c\}} \not\simeq G_{\restriction\{a,b,c\}}$ and $G'_{\restriction\{a,b,c\}} \not\simeq \overline{G}_{\restriction\{a,b,c\}}$ because exactly one of $G_{\restriction\{a,b,c\}}$ and $G'_{\restriction\{a,b,c\}}$ is a peak, which contradicts the 3-hypomorphy up to complementation. If $\{a,c\}$ is a neutral pair in $G$ not reversed in $G'$, then $G'_{\restriction\{a,b,c\}} \not\simeq G_{\restriction\{a,b,c\}}$ and $G'_{\restriction\{a,b,c\}} \not\simeq \overline{G}_{\restriction\{a,b,c\}}$ because exactly one of $G_{\restriction\{a,b,c\}}$ and $G'_{\restriction\{a,b,c\}}$ is a flag, which contradicts the 3-hypomorphy up to complementation.
(2) W.l.o.g., we assume that $E(U_{\restriction\{a,b,c\}}) = \{\{a,b\}\}$. Then $E(\overline{U}_{\restriction\{a,b,c\}}) = \{\{a,c\},\{b,c\}\}$ and $\overline{U} = G\dot{+}\overline{G'}$. We can assume that $a \longrightarrow_G b$, then $a \longrightarrow_{\overline{G'}} b$.

• Case 1. $\{b,c\}$ is an oriented pair in $G$.
W.l.o.g. we assume $b \longrightarrow_G c$, thus $b \longleftarrow_{\overline{G'}} c$. Since $a \longrightarrow_G b$, $b \longrightarrow_G c$ and $\{a,c\} \longrightarrow_{\overline{G'}} b$, from the 3-hypomorphy up to complementation we have $a \longrightarrow_G c$ and the conclusion follows.

• Case 2. $\{b,c\}$ is not an oriented pair in $G$.
W.l.o.g. we can assume $b$___$_G c$, thus $b \cdots_{\overline{G'}} c$. From (1) of this lemma, $\{a,c\}$ is a neutral pair in $G$. Since $G$ and $\overline{G'}$ are 3-hypomorphic up to complementation, $a$___$_G c$ and the conclusion follows.
(3) We have $E(U_{\restriction\{a,b,c\}})$ or $E(\overline{U}_{\restriction\{a,b,c\}}) = \{\{a,b\}\}$ and $\{a,b\}$ is a neutral pair in $\overline{G}$.
W.l.o.g., we can assume that $E(U_{\restriction\{a,b,c\}}) = \{\{a,b\}\}$ and $a$___$_G b$, so $a \ldots_{G'} b$.

• Case 1. $\{a,c\}$ is an oriented pair in $G$ not reversed in $G'$.

W.l.o.g., we assume that $a \longrightarrow_G c$, so $a \longrightarrow_{G'} c$. We have $U_{\restriction\{a,b,c\}}$ is neither a complete graph nor an empty graph, so from Remark 4.2, each of $G_{\restriction\{a,b,c\}}$ and $G'_{\restriction\{a,b,c\}}$ is neither a peak nor a flag, so $b \longleftarrow_G c$ and $b \longleftarrow_{G'} c$.
• Case 2. $\{a, c\}$ is a neutral pair in $G$ not reversed in $G'$.
W.l.o.g., we assume that $a \underline{\quad}_G c$, so $a \underline{\quad}_{G'} c$. As $a \underline{\quad}_G \{b, c\}$ and $a \underline{\quad}_{G'} c$ and $a \ldots_{G'} b$, then the 3-hypomorphy up to complementation applied to $G_{\restriction\{a,b,c\}}$ gives $b \ldots_G c$, so $b \ldots_{G'} c$.
In the two cases we have $\{a, b\}$ is not an interval of $G_{\restriction\{a,b,c\}}$. □

**Lemma 4.4.** *Let $G$ and $G'$ be two digraphs on the same vertex set $V$ such that $G$ and $G'$ are $(\leq 3)$-hypomorphic up to complementation, and $U := G \dot{+} G'$. Let $n \geq 3$ be an integer, $X := \{v_0, v_1, \ldots, v_{n-1}\} \subset V$ and $x \in V \setminus X$.*
*We assume that $U_{\restriction X\cup\{x\}} = x \underline{\quad} v_0 \underline{\quad} v_1 \underline{\quad} \ldots \underline{\quad} v_{n-1}$.*

(1) *If $G_{\restriction X} = v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{n-1}$, then $G_{\restriction X\cup\{x\}} = x \longrightarrow v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{n-1}$ and $G'_{\restriction X\cup\{x\}} = G^*_{\restriction X\cup\{x\}}$.*
(2) *If $G_{\restriction X} = \overrightarrow{P_n^f}$, then $G_{\restriction X\cup\{x\}}$ is isomorphic to $\overrightarrow{P_{n+1}^f}$ by an isomorphism $f$ such that $f(v_i) = i + 1$ for each $i \in \{0, \ldots, n-1\}$ and $f(x) = 0$, and $G'_{\restriction X\cup\{x\}} = G^*_{\restriction X\cup\{x\}}$.*

**Proof.** (1) We have $E(U_{\restriction\{x,v_0,v_1\}}) = \{\{x, v_0\}, \{v_0, v_1\}\}$ and $v_0 \longrightarrow_G v_1$, then (1) of Lemma 4.3 gives $\{x, v_0\}$ is an oriented pair in $G$, reversed in $G'$, let $j \in \{2, 3 \ldots n-1\}$, we have $E(U_{\restriction\{x,v_0,v_j\}}) = \{\{x, v_0\}\}$, then (2) of Lemma 4.3 applied to $\{x, v_0, v_j\}$ gives that $\{x, v_0\}$ is an interval of $G_{\restriction\{x,v_0,v_j\}}$. As $v_0 \ldots_G v_j$, thus $x \ldots_G v_j$ and $x \ldots_{G'} v_j$. We have $U_{\restriction\{x,v_1,v_2\}} = x \ldots v_1 \underline{\quad} v_2$, $x \ldots_G v_2$ and $v_1 \longrightarrow_G v_2$, then (2) of Lemma 4.3 gives $x \ldots_G v_1$ and $x \ldots_{G'} v_1$. As $v_0 \underline{\quad}_U v_1$ and $x \ldots_U v_1$ then, from Remark 4.2, $G_{\restriction\{x,v_0,v_1\}}$ is not a peak, thus $x \longrightarrow_G v_0$ and $x \longleftarrow_{G'} v_0$. Then, $G_{\restriction X\cup\{x\}} = x \longrightarrow v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{n-1}$ and $G'_{\restriction X\cup\{x\}} = G^*_{\restriction X\cup\{x\}}$.
(2) The proof is similar to that of first assertion. □

**Lemma 4.5.** *Let $G$ and $G'$ be two digraphs on the same vertex set $V$ such that $G$ and $G'$ are $(\leq 4)$-hypomorphic up to complementation, and $U := G \dot{+} G'$. Let $X := \{v_0, v_1, v_2, v_3\} \subset V$. If $U_{\restriction X} = v_0 \underline{\quad} v_1 \underline{\quad} v_2 \underline{\quad} v_3$, we have :*

(1) *If $\{v_0, v_1\}$ is a neutral pair in $G$, then*

$$\{G_{\restriction X}, G'_{\restriction X}\} = \left\{H, \overline{H^*}\right\} \text{ or } \{G_{\restriction X}, G'_{\restriction X}\} = \left\{H^*, \overline{H}\right\},$$

*where $H := v_1 \longrightarrow v_3 \longrightarrow v_0 \longrightarrow v_2$.*
(2) *If $(v_0 \longrightarrow_G v_1$ and $v_1 \longrightarrow_G v_2)$ or $(v_0 \longleftarrow_G v_1$ and $v_1 \longleftarrow_G v_2)$, then*

$$\{G_{\restriction X}, G'_{\restriction X}\} = \left\{\overrightarrow{P_4}, \left(\overrightarrow{P_4}\right)^*\right\} \text{ or } \{G_{\restriction X}, G'_{\restriction X}\} = \left\{\overrightarrow{P_4^f}, \left(\overrightarrow{P_4^f}\right)^*\right\}.$$

(3) *If $(v_0 \longrightarrow_G v_1, v_1 \longleftarrow_G v_2)$ or $(v_0 \longleftarrow_G v_1$ and $v_1 \longrightarrow_G v_2)$, then*

$$\{G_{\restriction X}, G'_{\restriction X}\} = \{v_0 < v_2 < v_1 < v_3,\ v_1 < v_0 < v_3 < v_2\}$$

$$\text{or } \{G_{\restriction X}, G'_{\restriction X}\} = \{v_2 < v_3 < v_0 < v_1,\ v_3 < v_1 < v_2 < v_0\}.$$

**Proof.** (1) As $\{v_0, v_1\}$ is a neutral pair in $G$, w.l.o.g., we assume that $v_0 \underline{\quad}_G v_1$. Then $v_0 \ldots_{G'} v_1$. We have $E(U_{\restriction\{v_0,v_1,v_2\}}) = \{\{v_0, v_1\}, \{v_1, v_2\}\}$ and $v_0 \underline{\quad}_G v_1$, so (1) of Lemma 4.3

applied to $\{v_0, v_1, v_2\}$ gives $\{v_1, v_2\}$ is a neutral pair in $G$ reversed in $G'$. We have $E(U_{\upharpoonright\{v_1,v_2,v_3\}}) = \{\{v_1, v_2\}, \{v_2, v_3\}\}$, so (1) of Lemma 4.3 applied to $\{v_1, v_2, v_3\}$ gives $\{v_2, v_3\}$ is a neutral pair in $G$ reversed in $G'$. According to the nature of the pair $\{v_1, v_2\}$, we have the following cases:

• Case 1. $v_1 \underline{\quad}_G v_2$.
Then $v_1 \cdots_{G'} v_2$. We have $v_0 \cdots_U v_2$, then the 3-hypomorphy up to complementation applied to $\{v_0, v_1, v_2\}$ gives $\{v_0, v_2\}$ is an oriented pair in $G$ not reversed in $G'$. We assume that $v_0 \longrightarrow_G v_2$ and $v_0 \longrightarrow_{G'} v_2$ (resp. $v_0 \longleftarrow_G v_2$ and $v_0 \longleftarrow_{G'} v_2$). As $U_{\upharpoonright\{v_0,v_2,v_3\}} = v_0 \ldots v_2 \underline{\quad} v_3$, then (3) of Lemma 4.3 gives $v_0 \longleftarrow_G v_3$ and $v_0 \longleftarrow_{G'} v_3$ (resp. $v_0 \longrightarrow_G v_3$ and $v_0 \longrightarrow_{G'} v_3$). As $U_{\upharpoonright\{v_0,v_1,v_3\}} = v_3 \ldots v_0 \underline{\quad} v_1$, then (3) of Lemma 4.3 gives $v_1 \longrightarrow_G v_3$ and $v_1 \longrightarrow_{G'} v_3$ (resp. $v_1 \longleftarrow_G v_3$ and $v_1 \longleftarrow_{G'} v_3$). Since $v_1 \underline{\quad}_U v_2$ and $v_1 \cdots_U v_3$, from Remark 4.2, $G_{\upharpoonright\{v_1,v_2,v_3\}}$ is not a flag, so $v_2 \underline{\quad}_G v_3$ and $v_2 \cdots_{G'} v_3$. Then $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = H$ and $G_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = \overline{H^*}$ (resp. $G_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = \overline{H}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = H^*$).

• Case 2. $v_1 \cdots_G v_2$.
Then $v_1 \underline{\quad}_{G'} v_2$. As $v_0 \cdots_U v_2$ and $v_1 \underline{\quad}_U v_2$ then, from Remark 4.2, $G_{\upharpoonright\{v_0,v_1,v_2\}}$ is not a flag, so $\{v_0, v_2\}$ is a neutral pair in $G$ not reversed in $G'$. W.l.o.g. we can assume that $v_0 \underline{\quad}_G v_2$, so $v_0 \underline{\quad}_{G'} v_2$. Since $E(U_{\upharpoonright\{v_0,v_2,v_3\}}) = \{\{v_2, v_3\}\}$ and $\{v_2, v_3\}$ is a neutral pair in $G$, then (3) of Lemma 4.3 gives $v_0 \cdots_G v_3$ and $v_0 \cdots_{G'} v_3$. We have $v_0 \cdots_{G'} \{v_1, v_3\}$, $v_0 \cdots_G v_3$ and $v_0 \underline{\quad}_G v_1$, so the 3-hypomorphy up to complementation applied to $\{v_0, v_1, v_3\}$ gives $v_1 \underline{\quad}_{G'} v_3$, so $v_1 \underline{\quad}_G v_3$. We have $v_1 \underline{\quad}_{G'} \{v_2, v_3\}$, $v_1 \cdots_G v_2$ and $v_1 \underline{\quad}_G v_3$, then the 3-hypomorphy up to complementation applied to $\{v_1, v_2, v_3\}$ gives $v_2 \cdots_{G'} v_3$, so $v_2 \underline{\quad}_G v_3$. Then $G_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$ have respectively the types (4,2) and (3,3), so $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}} \not\simeq G_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}} \not\simeq \overline{G}_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$, that contradict the 4-hypomorphy up to complementation.

(2) • Case 1. $v_0 \longrightarrow_G v_1$ and $v_1 \longrightarrow_G v_2$.
Then $v_1 \longrightarrow_{G'} v_0$ and $v_2 \longrightarrow_{G'} v_1$. We have $v_0 \cdots_U v_2$, if $\{v_0, v_2\}$ is an oriented pair in $G$, then one of the subdigraphs $G_{\upharpoonright\{v_0,v_1,v_2\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2\}}$ is a 3-cycle and the other is a total order of order 3, that contradict the 3-hypomorphy up to complementation, so $\{v_0, v_2\}$ is a neutral pair in $G$ not reversed in $G'$, thus $G_{\upharpoonright\{v_0,v_1,v_2\}} = \overrightarrow{P_3}$ or $\overrightarrow{P_3^f}$, and $G'_{\upharpoonright\{v_0,v_1,v_2\}} = G^*_{\upharpoonright\{v_0,v_1,v_2\}}$. As $U_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$ is a $P_4$ then, from Lemma 4.4, $G_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = \overrightarrow{P_4}$ or $\overrightarrow{P_4^f}$, and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = G^*_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$.

• Case 2. $v_0 \longleftarrow_G v_1$ and $v_1 \longleftarrow_G v_2$.
Then $v_0 \longrightarrow_{G'} v_1$ and $v_1 \longrightarrow_{G'} v_2$. From Case 1, by exchanging the roles of $G$ and $G'$, we have $G'_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = \overrightarrow{P_4}$ or $\overrightarrow{P_4^f}$, and $G_{\upharpoonright\{v_0,v_1,v_2,v_3\}} = (G')^*_{\upharpoonright\{v_0,v_1,v_2,v_3\}}$.

(3) • Case 1. $v_0 \longrightarrow_G v_1$ and $v_1 \longleftarrow_G v_2$.
Then $v_0 \longleftarrow_{G'} v_1$ and $v_1 \longrightarrow_{G'} v_2$. As $v_1 \underline{\quad}_U v_2$ and $v_0 \cdots_U v_2$ then, from Remark 4.2, $G_{\upharpoonright\{v_0,v_1,v_2\}}$ is not a peak, so $\{v_0, v_2\}$ is an oriented pair in $G$ not reversed in $G'$. We assume that $v_0 \longrightarrow_G v_2$ and $v_0 \longrightarrow_{G'} v_2$ (resp. $v_0 \longleftarrow_G v_2$ and $v_0 \longleftarrow_{G'} v_2$). We have $E(U_{\upharpoonright\{v_1,v_2,v_3\}}) = \{\{v_1, v_2\}, \{v_2, v_3\}\}$ and $v_1 \longleftarrow_G v_2$, then (1) of Lemma 4.3 gives $\{v_2, v_3\}$ is an oriented pair in $G$ reversed in $G'$, we have $E(U_{\upharpoonright\{v_0,v_2,v_3\}}) = \{\{v_2, v_3\}\}$, then (2) of Lemma 4.3 applied to $\{v_0, v_2, v_3\}$ gives $\{v_2, v_3\}$ is an interval of $G_{\upharpoonright\{v_0,v_2,v_3\}}$, so $v_0 \longrightarrow_G v_3$ and $v_0 \longrightarrow_{G'} v_3$ (resp. $v_0 \longleftarrow_G v_3$ and $v_0 \longleftarrow_{G'} v_3$). We have $E(U_{\upharpoonright\{v_0,v_1,v_3\}}) = \{\{v_0, v_1\}\}$ and $v_0 \longrightarrow_G v_1$, then (2) of Lemma 4.3 applied to $\{v_0, v_1, v_3\}$ gives $\{v_0, v_1\}$ is an interval of $G_{\upharpoonright\{v_0,v_1,v_3\}}$, so $v_1 \longrightarrow_G v_3$ and $v_1 \longrightarrow_{G'} v_3$ (resp. $v_1 \longleftarrow_G v_3$ and $v_1 \longleftarrow_{G'} v_3$). We have $v_1 \longleftarrow_G v_2$,

$v_1 \longrightarrow_G v_3$ and $v_1 \longrightarrow_{G'} \{v_2, v_3\}$ (resp. $v_1 \longrightarrow_{G'} v_2$, $v_1 \longleftarrow_{G'} v_3$ and $v_1 \longleftarrow_G \{v_2, v_3\}$), then the 3-hypomorphy up to complementation applied to $\{v_1, v_2, v_3\}$ gives $v_2 \longrightarrow_G v_3$, so $v_2 \longleftarrow_{G'} v_3$ (resp. $v_2 \longleftarrow_{G'} v_3$, so $v_2 \longrightarrow_G v_3$), thus $G_{\restriction X} = v_0 < v_2 < v_1 < v_3$ and $G'_{\restriction X} = v_1 < v_0 < v_3 < v_2$ (resp. $G_{\restriction X} = v_2 < v_3 < v_0 < v_1$ and $G'_{\restriction X} = v_3 < v_1 < v_2 < v_0$).

• Case 2. $v_0 \longleftarrow_G v_1$ and $v_1 \longrightarrow_G v_2$.
Then $v_0 \longrightarrow_{G'} v_1$ and $v_1 \longleftarrow_{G'} v_2$. From Case 1, by exchanging the roles of $G$ and $G'$, we have $G'_{\restriction X} = v_0 < v_2 < v_1 < v_3$ and $G_{\restriction X} = v_1 < v_0 < v_3 < v_2$ or $G'_{\restriction X} = v_2 < v_3 < v_0 < v_1$ and $G_{\restriction X} = v_3 < v_1 < v_2 < v_0$. □

**Proposition 4.6.** *Let $G$ and $G'$ be two digraphs on the same vertex set $V$, $(\leq 5)$-hypomorphic up to complementation. Let $U := G\dotplus G'$. If $U$ and $\overline{U}$ are connected and $G$ is not a tournament, then there exists $X \subset V$, such that $G_{\restriction X} \simeq \overrightarrow{P_4}$ or $\overrightarrow{P_4^f}$, and $G'_{\restriction X} = G^*_{\restriction X}$ or $G'_{\restriction X} = \overline{G^*}_{\restriction X}$.*

**Proof.** From Theorem 4.1, there exists $X := \{u_0, u_1, u_2, u_3\} \subset V$ such that $u_0$——$u_1$——$u_2$——$u_3$ is an induced $P_4$ of $U$. The hypotheses of Lemma 4.5 are satisfied. If we have (1) or (2) of Lemma 4.5, then we conclude.

Now we consider that only the situation (3) of Lemma 4.5 holds. (⋆)

That is if $X := \{u_0, u_1, u_2, u_3\} \subset V$ such that $u_0$——$u_1$——$u_2$——$u_3$ is an induced $P_4$ of $U$, then $\{G_{\restriction X}, G'_{\restriction X}\} = \{u_0 < u_2 < u_1 < u_3, u_1 < u_0 < u_3 < u_2\}$ or $\{u_2 < u_3 < u_0 < u_1, u_3 < u_1 < u_2 < u_0\}$. From this, if $u_i \longrightarrow_G u_{i+1}$ then $u_{i+1} \longleftarrow_G u_{i+2}$ for each $i \in \{0, 1\}$.

We will show that the situation (⋆) is impossible, which completes our proof.
As $G$ is not a tournament, there exist $a, b \in V(G)$ such that $\{a, b\}$ is a neutral pair in $G$.
From Corollary 3.4, $U = S(U_0, U_1, \ldots, U_{m-1})$, where $S$ is an indecomposable graph with at least 4 vertices and the $U_i$'s are graphs, for each $i \in \{0, 1, \ldots, m-1\}$.

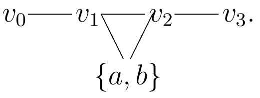
**Claim 4.7.** $\{a, b\} \nsubseteq V(U_i)$, *for each* $i \in \{0, 1, \ldots, m-1\}$.

**Proof.** We assume by contradiction, that there exists $i \in \{0, 1 \ldots, m-1\}$ such that $a, b \in V(U_i)$. Then from Theorem 3.1, there exist $v_0, v_1, v_2, v_3 \in V(U)$ such that one of the following cases holds.

• Case 1. In $U$, we have $v_0$——$v_1$——$v_2$——$\{a, b\}$.
Let $x \in \{a, b\}$. We have $U_{\restriction\{v_0,v_1,v_2,x\}} = v_0$——$v_1$——$v_2$——$x$, so from (⋆), $G_{\restriction\{v_0,v_1,v_2,x\}}$ and $G'_{\restriction\{v_0,v_1,v_2,x\}}$ are two total orders of order 4, w.l.o.g., we can assume that $G_{\restriction\{v_0,v_1,v_2,x\}} = v_0 < v_2 < v_1 < x$ and $G'_{\restriction\{v_0,v_1,v_2,x\}} = v_1 < v_0 < x < v_2$. Then $G_{\restriction\{v_0,v_1,v_2,a,b\}} = v_0 < v_2 < v_1 < \{a, b\}$ and $G'_{\restriction\{v_0,v_1,v_2,a,b\}} = v_1 < v_0 < \{a, b\} < v_2$. Clearly, since $G_{\restriction\{v_2,a,b\}}$ is a peak and $v_2$——$_U\{a, b\}$, from Remark 4.2, $a$——$_U b$. Since $G_{\restriction\{v_1,a,b\}}$ is a peak and $v_1 \ldots_U \{a, b\}$, from Remark 4.2, $a \ldots_U b$. A contradiction.

• Case 2. In $U$, we have $v_0$——$\{a, b\}$——$v_2$——$v_3$.
The proof is similar to that of Case 1.

• Case 3. In $U$, we have



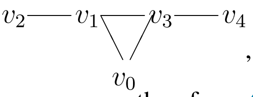
As $U_{\restriction\{v_0,v_1,v_2,v_3\}} = v_0 \text{—} v_1 \text{—} v_2 \text{—} v_3$, from ($\star$), $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$ are two total orders of order 4. W.l.o.g., we assume that $G_{\restriction\{v_0,v_1,v_2,v_3\}} = v_0 < v_2 < v_1 < v_3$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}} = v_1 < v_0 < v_3 < v_2$. Let $x \in \{a, b\}$. We have $E(U_{\restriction\{x,v_2,v_3\}}) = \{\{x, v_2\}, \{v_2, v_3\}\}$ (resp. $E(U_{\restriction\{x,v_0,v_1\}}) = \{\{v_0, v_1\}, \{x, v_1\}\}$) and $\{v_2, v_3\}$ (resp. $\{v_0, v_1\}$) is an oriented pair in $G$, then (1) of Lemma 4.3 applied to $\{x, v_2, v_3\}$ (resp. $\{x, v_0, v_1\}$) gives $\{x, v_2\}$ (resp. $\{x, v_1\}$) is an oriented pair in $G$ reversed in $G'$. Since $E(U_{\restriction\{x,v_1,v_3\}}) = \{\{x, v_1\}\}$, (2) of Lemma 4.3 applied to $\{x, v_1, v_3\}$ gives $\{x, v_1\}$ is an interval of $G_{\restriction\{x,v_1,v_3\}}$, we have $v_1 \longrightarrow_G v_3$, so $x \longrightarrow_G v_3$ and $x \longrightarrow_{G'} v_3$. We have $\{x, v_2\} \longrightarrow_G v_3$, $x \longrightarrow_{G'} v_3$, $v_3 \longrightarrow_{G'} v_2$, then the 3-hypomorphy up to complementation applied to $\{x, v_2, x_3\}$ gives $x \longrightarrow_{G'} v_2$, so $x \longleftarrow_G v_2$. We have $G_{\restriction\{v_2,a,b\}}$ is a peak and $v_2 \text{—}_U \{a, b\}$, then $a \text{—}_U b$. We have $G_{\restriction\{v_3,a,b\}}$ is a peak and $v_3 \ldots_U \{a, b\}$, then $a \ldots_U b$. A contradiction. □

From Claim 4.7, there are $i, j \in \{0, 1 \ldots, m-1\}$, $i \neq j$, such that $a \in V(U_i)$ and $b \in V(U_j)$. For each $X := \{v_0, v_1, v_2, v_3\} \subset V$, if $v_0 \text{—} v_1 \text{—} v_2 \text{—} v_3$ is an induced $P_4$ of $U$, then from ($\star$), $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$ are total orders, so $\{a, b\}$ is not a subset of $X$ and $m \geq 5$. From Theorem 3.2, there is a subset $Y := \{v_0, v_1, \ldots, v_{m-1}\}$ of $V(S)$ satisfying: $a, b \in Y$ and there is an isomorphism $f$ from $U_{\restriction Y}$ or $\overline{U}_{\restriction Y}$ onto $P_m$ or $Q_m$, such that $f(\{a, b\}) = \{v_0, v_{m-1}\}$.

• Case 1. $U_{\restriction\{v_0,v_1,\ldots,v_{m-1}\}} \simeq P_m$.
W.l.o.g., we can assume that $a = v_0$, $b = v_{m-1}$ and $U_{\restriction\{v_0,v_1,\ldots,v_{m-1}\}} = P_m$. We have for each $i \in \{0, 1 \ldots, m-4\}$, $U_{\restriction\{v_i,v_{i+1},v_{i+2},v_{i+3}\}} \simeq P_4$ then, from ($\star$), $G_{\restriction\{v_i,v_{i+1},v_{i+2},v_{i+3}\}}$ and $G'_{\restriction\{v_i,v_{i+1},v_{i+2},v_{i+3}\}}$ are total orders, thus $\{v_j, v_{j+1}\}$ is an oriented pair in $G$ reversed in $G'$ for each $j \in \{0, 1, \ldots, m-2\}$. For $i \in \{0, 1 \ldots, m-4\}$, we have $U_{\restriction\{v_{m-1},v_i,v_{i+1}\}} = v_{m-1} \ldots v_i \text{—} v_{i+1}$ and $\{v_i, v_{i+1}\}$ is an oriented pair in $G$, so (2) of Lemma 4.3 applied to $\{v_{m-1}, v_i, v_{i+1}\}$ gives $\{v_i, v_{i+1}\}$ is an interval of $G_{\restriction\{v_{m-1},v_i,v_{i+1}\}}$, then $\{v_0, v_1, \ldots, v_{m-3}\}$ is an interval of $G_{\restriction\{v_{m-1},v_0,v_1,\ldots,v_{m-3}\}}$. As $G_{\restriction\{v_{m-4},v_{m-3},v_{m-2},v_{m-1}\}}$ is a total order, $\{v_{m-3}, v_{m-1}\}$ is an oriented pair in $G$, then $\{v_0, v_{m-1}\} = \{a, b\}$ is oriented in $G$. A contradiction.

• Case 2. $U_{\restriction\{v_0,v_1,\ldots,v_{m-1}\}} \simeq Q_m$.
W.l.o.g., we can assume that $a = v_0$, $b = v_{m-1}$ and $U_{\restriction\{v_0,v_1,\ldots,v_{m-1}\}} = Q_m$.

Case 2.1. $m = 5$.

Then $U_{\restriction\{v_0,v_1,v_2,v_3,v_4\}} = Q_5 =$  , $a = v_0$ and $b = v_4$.

As $U_{\restriction\{v_2,v_1,v_3,v_4\}} = v_2 \text{—} v_1 \text{—} v_3 \text{—} v_4$, thus from ($\star$), $G_{\restriction\{v_2,v_1,v_3,v_4\}}$ and $G'_{\restriction\{v_2,v_1,v_3,v_4\}}$ are total orders. We have $E(U_{\restriction\{v_0,v_1,v_2\}}) = \{\{v_0, v_1\}, \{v_1, v_2\}\}$ and $\{v_1, v_2\}$ is an oriented pair in $G$, so (1) of Lemma 4.3 applied to $\{v_0, v_1, v_2\}$ gives $\{v_0, v_1\}$ is an oriented pair in $G$ reversed in $G'$, we have $E(U_{\restriction\{v_0,v_1,v_4\}}) = \{\{v_0, v_1\}\}$, thus (2) of Lemma 4.3 applied to $\{v_0, v_1, v_4\}$ gives $\{v_0, v_1\}$ is an interval of $G_{\restriction\{v_0,v_1,v_4\}}$. As $\{v_1, v_4\}$ is an oriented pair in $G$, then $\{v_0, v_4\} = \{a, b\}$ is an oriented pair in $G$. A contradiction.

Case 2.2. $m > 5$.

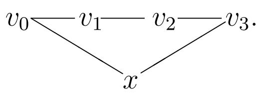
We have $U_{\restriction\{v_0,v_{m-1},v_{m-2},v_{m-3},v_{m-4}\}} = \{v_0, v_{m-1}\} \text{—} v_{m-2} \text{—} v_{m-4} \text{—} v_{m-3}$, where $\{v_0, v_{m-1}\}$ is an interval of $U_{\restriction\{v_0,v_{m-1},v_{m-2},v_{m-3},v_{m-4}\}}$. A contradiction is obtained by making a proof similar to that of Case 1 of the proof of Claim 4.7. □

## 5. Reconstruction up to complementation

**Lemma 5.1.** *Let $G$ and $G'$ be two digraphs on the same vertex set $V$ such that $G$ and $G'$ are $(\leq 5)$-hypomorphic up to complementation. Let $U := G\dot{+}G'$, $X := \{v_0, v_1, v_2, v_3\} \subset V$ and $x \in V \setminus X$. If $G_{\restriction X} = \overrightarrow{P_4}$ or $\overrightarrow{P_4^f}$, and $G'_{\restriction X} = G^*_{\restriction X}$ then,*

(1) $x \ldots_U \{v_1, v_2\}$.
(2) *Up to isomorphism, $U_{\restriction X\cup\{x\}}$ is one of the following graphs:*

$$v_0\underline{\quad}v_1\underline{\quad}v_2\underline{\quad}v_3 \qquad x$$
$$v_0\underline{\quad}v_1\underline{\quad}v_2\underline{\quad}v_3\underline{\quad}x$$



**Proof.** W.l.o.g., we can assume that $G_{\restriction X} = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3$, so $G'_{\restriction X} = G^*_{\restriction X} = v_3 \longrightarrow v_2 \longrightarrow v_1 \longrightarrow v_0$ and $U_{\restriction X} = v_0\underline{\quad}v_1\underline{\quad}v_2\underline{\quad}v_3$. It suffices to prove (1) because (2) is a consequence of (1). By contradiction $x\underline{\quad}_U v_1$ or $x\underline{\quad}_U v_2$.

• Case 1. $x\underline{\quad}_U v_1$ and $x \ldots_U v_2$, or $x\underline{\quad}_U v_2$ and $x \ldots_U v_1$.
W.l.o.g., we can assume that $x\underline{\quad}_U v_1$ and $x \ldots_U v_2$.

Case 1.1. $x \ldots_U v_3$.
Since $U_{\restriction\{x,v_1,v_2,v_3\}} = x\underline{\quad}v_1\underline{\quad}v_2\underline{\quad}v_3$ and $G_{\restriction\{v_1,v_2,v_3\}} = v_1 \longrightarrow v_2 \longrightarrow v_3$, by Lemma 4.4, $G_{\restriction\{x,v_1,v_2,v_3\}} = x \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3$ and $G'_{\restriction\{x,v_1,v_2,v_3\}} = G^*_{\restriction\{x,v_1,v_2,v_3\}}$. Thus in $G$ we have $\{v_0, x\} \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3$, and in $G'$ we have $\{v_0, x\} \longleftarrow v_1 \longleftarrow v_2 \longleftarrow v_3$. So $G'_{\restriction X\cup\{x\}} \not\simeq G_{\restriction X\cup\{x\}}$. On the other hand, $G'_{\restriction X\cup\{x\}}$ and $\overline{G}_{\restriction X\cup\{x\}}$ are not isomorphic because their types are distinct. Then $G'_{\restriction X\cup\{x\}}$ and $G_{\restriction X\cup\{x\}}$ are not 5-hypomorphic up to complementation, a contradiction.

Case 1.2. $x\underline{\quad}_U v_3$ and $x \ldots_U v_0$.
As $E(\overline{U}_{\restriction\{x,v_0,v_3\}}) = \{\{x, v_0\}, \{v_0, v_3\}\}$ and $v_0 \ldots_G v_3$, (1) of Lemma 4.3 gives $\{x, v_0\}$ is a neutral pair in $G$. We have $E(\overline{U}_{\restriction\{x,v_0,v_1\}}) = \{\{x, v_0\}\}$ and $v_0 \longrightarrow_G v_1$, then (3) of Lemma 4.3 gives $v_1 \longrightarrow_G x$ and $v_1 \longleftarrow_{G'} x$. As $E(\overline{U}_{\restriction\{x,v_1,v_3\}}) = \{\{v_1, v_3\}\}$ and $v_1 \ldots_G v_3$, (3) of Lemma 4.3 gives $x \longrightarrow_G v_3$ and $x \longleftarrow_{G'} v_3$. Since $E(U_{\restriction\{x,v_0,v_3\}}) = \{\{x, v_3\}\}$, thus (2) of Lemma 4.3 applied to $\{v_0, x, v_3\}$ gives $\{x, v_3\}$ is an interval for $G_{\restriction\{x,v_0,v_3\}}$, since $v_0 \ldots_G v_3$, so $x \ldots_G v_0$ and $x \ldots_{G'} v_0$.
In $G$ we have $v_0 \longrightarrow v_1 \longrightarrow \{x, v_2\} \longrightarrow v_3$ and in $G'$ we have $v_0 \longleftarrow v_1 \longleftarrow \{x, v_2\} \longleftarrow v_3$, so $G'_{\restriction X\cup\{x\}} \not\simeq G_{\restriction X\cup\{x\}}$. By types, $G'_{\restriction X\cup\{x\}} \not\simeq \overline{G}_{\restriction X\cup\{x\}}$. We get a contradiction with the 5-hypomorphy up to complementation.

Case 1.3. $x\underline{\quad}_U \{v_0, v_3\}$.
As $E(\overline{U}_{\restriction\{x,v_0,v_2\}}) = \{\{v_0, v_2\}, \{v_2, x\}\}$ and $v_0 \ldots_G v_2$, (1) of Lemma 4.3 gives $\{x, v_2\}$ is a neutral pair in $G$. Since $E(\overline{U}_{\restriction\{x,v_2,v_3\}}) = \{\{x, v_2\}\}$ (resp. $E(\overline{U}_{\restriction\{x,v_1,v_2\}}) = \{\{x, v_2\}\}$ and $v_1 \longrightarrow_G v_2$), (3) of Lemma 4.3 applied to $\{x, v_3, v_2\}$ (resp. $\{x, v_1, v_2\}$) gives $x \longleftarrow_G v_3$ and $x \longrightarrow_{G'} v_3$ (resp. $x \longrightarrow_G v_1$ and $x \longleftarrow_{G'} v_1$). We have $E(\overline{U}_{\restriction\{x,v_0,v_3\}}) = \{\{v_0, v_3\}\}$, thus (3) of Lemma 4.3 applied to $\{x, v_0, v_3\}$ gives $x \longrightarrow_G v_0$ and $x \longleftarrow_{G'} v_0$. As $E(U_{\restriction\{x,v_0,v_2\}}) = \{\{x, v_0\}\}$, (2) of Lemma 4.3 applied to $\{x, v_0, v_2\}$ gives $\{x, v_0\}$ is an interval of $G_{\restriction\{x,v_0,v_2\}}$, then $x \ldots_G v_2$ and $x \ldots_{G'} v_2$. The induced digraphs $G_{\restriction\{v_0,v_1,v_2,x\}}$ and $G'_{\restriction\{v_0,v_1,v_2,x\}}$

are not isomorphic up to complementation, which gives a contradiction with the hypothesis $G$ and $G'$ are ($\leq 5$)-hypomorphic up to complementation.

• Case 2. $x \underline{\quad}_U \{v_1, v_2\}$.

Case 2.1. $x \ldots_U \{v_0, v_3\}$.

We have $E(\overline{U}_{\restriction\{x,v_1,v_3\}}) = \{\{x, v_3\}, \{v_1, v_3\}\}$ and $v_1 \ldots_G v_3$ (resp. $E(\overline{U}_{\restriction\{x,v_0,v_2\}}) = \{\{x, v_0\}, \{v_0, v_2\}\}$ and $v_0 \ldots_G v_2$), then (1) of Lemma 4.3 gives $\{x, v_3\}$ (resp. $\{x, v_0\}$) is a neutral pair in $G$. We have $E(\overline{U}_{\restriction\{x,v_2,v_3\}}) = \{\{x, v_3\}\}$ and $v_2 \longrightarrow_G v_3$ (resp. $E(\overline{U}_{\restriction\{x,v_0,v_1\}}) = \{\{x, v_0\}, \{x, v_1\}\}$ and $v_0 \longrightarrow_G v_1$), then (3) of Lemma 4.3 applied to $\{x, v_2, v_3\}$ (resp. $\{v_0, v_1, x\}$) gives $v_2 \longleftarrow_G x$ and $v_2 \longrightarrow_{G'} x$ (resp. $v_1 \longrightarrow_G x$ and $v_1 \longleftarrow_{G'} x$). We have $E(U_{\restriction\{x,v_1,v_3\}}) = \{\{x, v_1\}\}$ (resp. $E(U_{\restriction\{x,v_0,v_2\}}) = \{\{x, v_2\}\}$), then (2) of Lemma 4.3 applied to $\{x, v_1, v_3\}$ (resp. $\{v_0, v_2, x\}$) gives $\{x, v_1\}$ (resp. $\{x, v_2\}$) is an interval of $G_{\restriction\{x,v_1,v_3\}}$ (resp. $G_{\restriction\{v_0,v_2,x\}}$) then $x \ldots_G v_3$ and $x \ldots_{G'} v_3$ (resp. $x \ldots_G v_0$ and $x \ldots_{G'} v_0$). By types, $G'_{\restriction\{v_0,v_1,v_2,x\}} \not\simeq \overline{G}_{\restriction\{v_0,v_1,v_2,x\}}$. If $\sigma$ is an isomorphism from $G_{\restriction\{v_0,v_1,v_2,x\}}$ onto $G'_{\restriction\{v_0,v_1,v_2,x\}}$, then $\sigma(v_1) = v_1$ because $v_1$ is the only vertex in $\{v_0, v_1, v_2, x\}$, whose not adjacent to any neutral pair in $G_{\restriction\{v_0,v_1,v_2,x\}}$; now since $d^+_{G_{\restriction\{v_0,v_1,v_2,x\}}}(v_1) = 2$ and $d^+_{G'_{\restriction\{v_0,v_1,v_2,x\}}}(v_1) = 1$, we deduce that $G'_{\restriction\{v_0,v_1,v_2,x\}} \not\simeq G_{\restriction\{v_0,v_1,v_2,x\}}$. A contradiction.

Case 2.2. $x \underline{\quad}_U v_3$ and $x \ldots_U v_0$ or $x \underline{\quad}_U v_0$ and $x \ldots_U v_3$.

W.l.o.g., we can assume that $x \underline{\quad}_U v_3$ and $x \ldots_U v_0$.

We do the same proof as Case 1.2. In $G$ we have $v_0 \longrightarrow v_1 \longrightarrow \{x, v_2\} \longrightarrow v_3$ and in $G'$ we have $v_0 \longleftarrow v_1 \longleftarrow \{x, v_2\} \longleftarrow v_3$, so $G'_{\restriction X\cup\{x\}}$ and $G_{\restriction X\cup\{x\}}$ are not 5-hypomorphic up to complementation, a contradiction.

Case 2.3. $x \underline{\quad}_U \{v_0, v_3\}$.

According to the nature of $\{x, v_2\}$ in $G$, we can distinguish the following subcases.

Case 2.3.1. $x \longrightarrow_G v_2$ or $x \longleftarrow_G v_2$.

W.l.o.g. we can suppose $x \longrightarrow_G v_2$. As $x \underline{\quad}_U v_2$ then $x \longleftarrow_{G'} v_2$. We have $E(\overline{U}_{\restriction\{x,v_0,v_2\}}) = \{\{v_0, v_2\}\}$, $v_0 \ldots_G v_2$ and $x \longrightarrow_G v_2$. So, (3) of Lemma 4.3 applied to $\{x, v_0, v_2\}$ gives $x \longleftarrow_G v_0$ and $x \longrightarrow_{G'} v_0$. We have $E(\overline{U}_{\restriction\{x,v_0,v_3\}}) = \{\{v_0, v_3\}\}$ and $v_0 \ldots_G v_3$, then (3) of Lemma 4.3 applied to $\{x, v_0, v_3\}$ gives $x \longrightarrow_G v_3$ and $x \longleftarrow_{G'} v_3$. We have $E(\overline{U}_{\restriction\{x,v_1,v_3\}}) = \{\{v_1, v_3\}\}$ and $v_1 \ldots_G v_3$. So (3) of Lemma 4.3 applied to $\{x, v_1, v_3\}$ gives $x \longleftarrow_G v_1$ and $x \longrightarrow_{G'} v_1$. We have that $x$ is the only vertex in $\{v_0, v_2, v_3, x\}$, which is not adjacent to any neutral pair in $G_{\restriction\{v_0,v_2,v_3,x\}}$. As $d^+_{G_{\restriction\{v_0,v_2,v_3,x\}}}(x) \neq d^+_{G'_{\restriction\{v_0,v_2,v_3,x\}}}(x)$, then $G'_{\restriction\{v_0,v_2,v_3,x\}} \not\simeq G_{\restriction\{v_0,v_2,v_3,x\}}$. Moreover $G'_{\restriction\{v_0,v_2,v_3,x\}} \not\simeq \overline{G}_{\restriction\{v_0,v_2,v_3,x\}}$ because their types are distinct. We get a contradiction with the 4-hypomorphy up to complementation.

Case 2.3.2. $x \ldots_G v_2$.

Then $x \underline{\quad}_{G'} v_2$. As $v_2 \ldots_G \{x, v_0\}$, $v_0 \ldots_{G'} v_2$ and $x \underline{\quad}_{G'} v_2$, thus $x \underline{\quad}_G v_0$, so $x \ldots_{G'} v_0$. As $v_0 \ldots_{G'} \{x, v_3\}$, $v_0 \ldots_G v_3$ and $x \underline{\quad}_G v_0$, then $x \underline{\quad}_{G'} v_3$, so $x \ldots_G v_3$. As $v_3 \ldots_G \{x, v_1\}$, $v_1 \ldots_{G'} v_3$ and $x \underline{\quad}_{G'} v_3$, then $x \underline{\quad}_G v_1$, so $x \ldots_{G'} v_1$. Since $G_{\restriction\{v_0,v_2,v_3,x\}}$ and $G'_{\restriction\{v_0,v_2,v_3,x\}}$ have respectively the types $(1, 4)$ and $(2, 3)$, $G'_{\restriction\{v_0,v_2,v_3,x\}}$ and $G_{\restriction\{v_0,v_2,v_3,x\}}$ are not isomorphic up to complementation, a contradiction.

Case 2.3.3. $x \underline{\quad}_G v_2$.

We do the same proof as Case 2.3.2. □

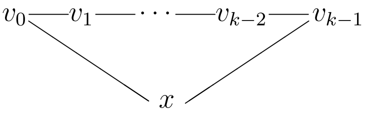
From Lemma 5.1, we obtain the following result.

**Corollary 5.2.** *Let $G$ and $G'$ be two digraphs on the same vertex set $V$ such that $G$ and $G'$ are $(\leq 5)$-hypomorphic up to complementation. Let $U := G\dot{+}G'$, $X := \{v_0, v_1, \ldots, v_{k-1}\} \subset V$ and $x \in V \setminus X$. If $G_{\restriction X} = \overrightarrow{P_k}$ or $\overrightarrow{P_k^f}$, and $G'_{\restriction X} = G^*_{\restriction X}$ then,*

(1) $x \ldots_U \{v_1, \ldots, v_{k-2}\}$.
(2) *Up to isomorphism, $U_{\restriction X \cup \{x\}}$ is one of the following graphs:*

$$x \qquad v_0 — v_1 — \cdots — v_{k-2} — v_{k-1}$$

$$x — v_0 — v_1 — \cdots — v_{k-2} — v_{k-1}$$

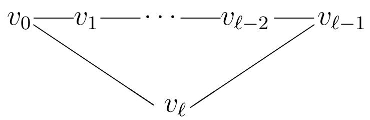



**Proposition 5.3.** *Let $G$ and $G'$ be two digraphs on the same set $V$ of $n \geq 4$ vertices, such that $G$ and $G'$ are $(\leq 5)$-hypomorphic up to complementation and $U := G\dot{+}G'$ is connected. Let $X \subset V$.*

(1) *If $G_{\restriction X} \simeq \overrightarrow{P_4}$ and $G'_{\restriction X} = G^*_{\restriction X}$, then $G \simeq \overrightarrow{P_n}$ or $G \simeq \overrightarrow{C_n}$, and $G' = G^*$.*
(2) *If $G_{\restriction X} \simeq \overrightarrow{P_4}$ and $G'_{\restriction X} = \overline{G^*}_{\restriction X}$, then $G \simeq \overrightarrow{P_n}$ or $G \simeq \overrightarrow{C_n}$, and $G' = \overline{G^*}$.*
(3) *If $G_{\restriction X} \simeq \overrightarrow{P_4^f}$ and $G'_{\restriction X} = G^*_{\restriction X}$, then $G \simeq \overrightarrow{P_n^f}$ or $G \simeq \overrightarrow{C_n^f}$, and $G' = G^*$.*
(4) *If $G_{\restriction X} \simeq \overrightarrow{P_4^f}$ and $G'_{\restriction X} = \overline{G^*}_{\restriction X}$, then $G \simeq \overrightarrow{P_n^f}$ or $G \simeq \overrightarrow{C_n^f}$, and $G' = \overline{G^*}$.*

**Proof.** It suffices to prove (1) because (2), (3) and (4) are consequences of (1). As $G_{\restriction X} \simeq \overrightarrow{P_4}$, let $\overrightarrow{P_\ell}$ be a largest induced oriented path in $G$ reversed in $G'$. Clearly, $\ell \geq 4$. W.l.o.g. we can assume $\overrightarrow{P_\ell} = v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{\ell-1}$ and $G'_{\restriction V(\overrightarrow{P_\ell})} = v_0 \longleftarrow v_1 \longleftarrow \cdots \longleftarrow v_{\ell-1}$. So $U_{\restriction V(\overrightarrow{P_\ell})} = v_0 — v_1 — \ldots — v_{\ell-2} — v_{\ell-1}$. If $V(\overrightarrow{P_\ell}) = V$, then $G = \overrightarrow{P_\ell}$ and $G' = G^*$. In the rest of this proof, we assume that $V \setminus V(\overrightarrow{P_\ell}) \neq \emptyset$. As $U$ is connected, there exists $v_\ell \in V \setminus V(\overrightarrow{P_\ell})$, such that $U_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}}$ is connected. From (2) of Corollary 5.2, up to isomorphism, $U_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}}$ is one of the following graphs:

$$v_\ell — v_0 — v_1 — \cdots — v_{\ell-2} — v_{\ell-1}$$




If $U_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}}$ is the graph $v_\ell — v_0 — v_1 — \ldots — v_{\ell-2} — v_{\ell-1}$ then, from Lemma 4.4, we have $G_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}} = v_\ell \longrightarrow v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{\ell-1}$ and $G'_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}} = G^*_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}}$, that contradict the fact that $\overrightarrow{P_\ell}$ is the largest induced oriented path in $G$ reversed in $G'$. Then $U_{\restriction V(\overrightarrow{P_\ell}) \cup \{v_\ell\}}$ is the second graph.

If there is $x$ in $V \setminus (V(\overrightarrow{P_\ell}) \cup \{v_\ell\})$, we have $v_{i-1}$——$v_i$——$v_{i+1}$——$v_{i+2}$ for each $i \in \{1, \dots, \ell-2\}$, then from (1) of Lemma 5.1, $x \dots_U v_i$ for each $i \in \{1, \dots, \ell-1\}$, also we have $v_{\ell-1}$——$v_\ell$——$v_0$——$v_1$, so from (1) of Lemma 5.1, $x \dots_U \{v_0, v_\ell\}$. Thus $x \dots_U (V(\overrightarrow{P_\ell}) \cup \{v_\ell\})$. As $U$ is connected, $V = V(\overrightarrow{P_\ell}) \cup \{v_\ell\}$.

We have $G_{\restriction\{v_0,v_1,\dots,v_{\ell-2}\}} = v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{\ell-2}$ and $U_{\restriction\{v_\ell,v_0,v_1,\dots,v_{\ell-2}\}} = v_\ell$——$v_0$——$v_1$——$\dots$——$v_{\ell-3}$——$v_{\ell-2}$, then from Lemma 4.4, $G_{\restriction\{v_\ell,v_0,v_1,\dots,v_{\ell-2}\}} = v_\ell \longrightarrow v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{\ell-2}$ and $G'_{\restriction\{v_\ell,v_0,v_1,\dots,v_{\ell-2}\}} = G^*_{\restriction\{v_\ell,v_0,v_1,\dots,v_{\ell-2}\}}$.

We have $\overline{U}_{\restriction\{v_{\ell-2},v_{\ell-1},v_\ell\}} = v_{\ell-2}$——$v_\ell \dots v_{\ell-1}$, $v_{\ell-2} \longrightarrow_G v_{\ell-1}$ and $v_{\ell-2} \dots_G v_\ell$, then (3) Lemma 4.3 applied to $\{v_{\ell-2}, v_{\ell-1}, v_\ell\}$ gives $v_{\ell-1} \longrightarrow_G v_\ell$ and $v_{\ell-1} \longleftarrow_{G'} v_\ell$. Then $G = \overrightarrow{C_{\ell+1}}$ and $G' = G^*$.

**Proof of Theorem 1.3.** If $G$ is a tournament then from Proposition 1.1, $G$ and $G'$ are total orders. If $G$ is not a tournament, then using Proposition 4.6, there exists a subset $X$ of $V(G)$ such that, $G_{\restriction X} \simeq \overrightarrow{P_4}$ or $\overrightarrow{P_4^f}$, and $G'_{\restriction X} = G^*_{\restriction X}$ or $\overline{G^*}_{\restriction X}$; then we conclude using Proposition 5.3.

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