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Symmetric duality for left and right Riemann–Liouville and Caputo fractional differences

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Abstract. A discrete version of the symmetric duality of Caputo–Torres, to relate left and right Riemann–Liouville and Caputo fractional differences, is considered. As a corollary, we provide an evidence to the fact that in case of right fractional differences, one has to mix between nabla and delta operators. As an application, we derive right fractional summation by parts formulas and left fractional difference Euler–Lagrange equations for discrete fractional variational problems whose Lagrangians depend on right fractional differences.

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1. Introduction

The study of differences of fractional order is a subject with a long and rich history [26,30–33]. The topic has attracted the attention of a very active community of researchers in the 21st century. In [7], the Q-operator connection between delay-type and advanced-type equations is established and its discrete version is used in [1–3]. In [14], the fundamental elements of a theory of difference operators and difference equations of fractional order are presented, while [11] discusses basic properties of nabla fractional sums and differences;

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the validity of a power rule and a law of exponents. Using such properties, a discrete Laplace transform is studied and applied to initial value problems [11]. In [13], the simplest discrete fractional problem of the calculus of variations is defined and necessary optimality conditions of Euler–Lagrange type derived. Moreover, a Gompertz fractional difference model for tumor growth is introduced and solved. In [16,17], the study of fractional discrete-time variational problems of order α, $0 < \alpha \leq 1$, involving discrete analogues of Riemann–Liouville fractional-order derivatives on time scales, is introduced. A fractional formula for summation by parts is proved, and then used to obtain Euler–Lagrange and Legendre type necessary optimality conditions. The theoretical results are supported by several illustrative examples [16,17]. More generally, it is also possible to investigate fractional calculus on an arbitrary time scale (that is, on an arbitrary nonempty closed set of the real numbers) [18–21,35]. The literature on the discrete fractional calculus is now vast: see [9,10,37–39] and references therein. For a comprehensive treatment, related topics of current interest and an extensive list of references, we refer the interested readers to the book [28].

In the recent article [25], Caputo and Torres introduced and developed a duality theory for left and right fractional derivatives, that we call here *symmetric duality*, defined by $f^*(t) = f(-t)$, where f is defined on $[a, b]$. They used this symmetric duality to relate left and right fractional integrals and left and right fractional Riemann–Liouville and Caputo derivatives. Here we show that the theory of [25] can also be extended to the discrete fractional calculus. As we prove here, the symmetric duality is very interesting because it confirms and provides a solid foundation to the right discrete fractional calculus, as done with the Q-operator in [2]. Indeed, in his articles [2,3], Abdeljawad used the well-known Q-operator, $Qf(t) = f(a + b - t)$, to relate left and right fractional sums and left and right fractional differences within the delta and nabla operators. Here we show that Abdeljawad's definitions [2,3] for right Riemann–Liouville and Caputo fractional differences are in some sense a consequence of symmetric duality.

The paper is organized as follows. In Section 2, we recall necessary notions and results from the discrete fractional calculus. Main results are then given in Section 3, where several identities for delta and nabla fractional sums and differences are proved from symmetric duality. We end with applications in Sections 4 and 5: in Section 4 we prove summation by parts formulas for right fractional differences (Theorems 34 and 35), which are then used in Section 5 to obtain left versions of the fractional difference Euler–Lagrange equations for discrete right fractional variational problems.

2. Preliminaries

In this section, we review well-known definitions and essential results from the literature of discrete fractional calculus, and we fix notations. For a natural number n, the factorial polynomial is defined by $t^{(n)} = \prod_{j=0}^{n-1}(t - j) = \frac{\Gamma(t+1)}{\Gamma(t+1-j)}$, where Γ denotes the special gamma function and the product is zero when $t + 1 - j = 0$ for some j. More generally, for arbitrary real α, we define $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, where one uses the convention that division at a pole yields zero. Given that the forward and backward difference operators are defined by $\Delta f(t) = f(t + 1) - f(t)$ and $\nabla f(t) = f(t) - f(t - 1)$, respectively, we define iteratively the operators $\Delta^m = \Delta(\Delta^{m-1})$ and $\nabla^m = \nabla(\nabla^{m-1})$, where m is a natural number. Follows some properties of the factorial function.

Lemma 1 (*See [15]*). *Let $\alpha \in \mathbb{R}$. Assume the following factorial functions are well defined. Then, $\Delta t^{(\alpha)} = \alpha t^{(\alpha-1)}$; $(t-\alpha)t^{(\alpha)} = t^{(\alpha+1)}$; $\alpha^{(\alpha)} = \Gamma(\alpha+1)$; if $t \le r$, then $t^{(\alpha)} \le r^{(\alpha)}$ for any $\alpha > r$; if $0 < \alpha < 1$, then $t^{(\alpha\nu)} \ge (t^{(\nu)})^{\alpha}$; $t^{(\alpha+\beta)} = (t-\beta)^{(\alpha)} t^{(\beta)}$.*

The next two relations, the proofs of which are straightforward, are also useful for our purposes: $\nabla_s (s-t)^{(\alpha-1)} = (\alpha-1)(\rho(s)-t)^{(\alpha-2)}$, $\nabla_t (\rho(s)-t)^{(\alpha-1)} = -(\alpha-1)(\rho(s)-t)^{(\alpha-2)}$, where $\rho(s) = s-1$ is the backward jump operator. With respect to the nabla fractional calculus, we have the following definition.

Definition 2 (*See [22,23,29,36]*). Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$. The m rising (ascending) factorial of t is defined by $t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k)$, $t^{\overline{0}} = 1$; the α rising function by $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$, $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, with $0^{\overline{\alpha}} = 0$.

Remark 3. For the rising factorial function, observe that $\nabla(t^{\overline{\alpha}}) = \alpha t^{\overline{\alpha-1}}$, $(t^{\overline{\alpha}}) = (t+\alpha-1)^{(\alpha)}$, and $\Delta_t (s-\rho(t))^{\overline{\alpha}} = -\alpha (s-\rho(t))^{\overline{\alpha-1}}$.

Notation 4. *Along the text, we use the following notations.*

(i) *For a real $\alpha > 0$, we set $n = [\alpha] + 1$, where $[\alpha]$ is the greatest integer less than α.*
(ii) *For real numbers a and b, we denote $\mathbb{N}_a = \{a, a+1, \ldots\}$ and ${}_b\mathbb{N} = \{b, b-1, \ldots\}$.*
(iii) *For $n \in \mathbb{N}$ and a real a, we denote ${}_{\ominus}\Delta^n f(t) = (-1)^n \Delta^n f(t)$, $t \in \mathbb{N}_a$.*
(iv) *For $n \in \mathbb{N}$ and a real b, we denote $\nabla_{\ominus}^n f(t) = (-1)^n \nabla^n f(t)$, $t \in {}_b\mathbb{N}$.*

Follows the definitions of delta/nabla left/right fractional sums.

Definition 5 (*See [3]*). Let $\sigma(t) = t+1$ and $\rho(t) = t-1$ be the forward and backward jump operators, respectively. The delta left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s), \quad t \in \mathbb{N}_{a+\alpha};$$

the delta right fractional sum of order $\alpha > 0$ (ending at b) by

$$\begin{aligned} {}_b\Delta^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (s-\sigma(t))^{(\alpha-1)} f(s) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s)-t)^{(\alpha-1)} f(s), \quad t \in {}_{b-\alpha}\mathbb{N}; \end{aligned}$$

the nabla left fractional sum of order $\alpha > 0$ (starting from a) by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1};$$

and the nabla right fractional sum of order $\alpha > 0$ (ending at b) by

$$\begin{aligned} {}_b\nabla^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{\alpha-1}} f(s) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s)-t)^{\overline{\alpha-1}} f(s), \quad t \in {}_{b-1}\mathbb{N}. \end{aligned}$$

Regarding fractional sums, the next remarks are important.

Remark 6. Operator $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$; operator ${}_b\Delta^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-\alpha}\mathbb{N}$; $\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$; while ${}_b\nabla^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions on ${}_b\mathbb{N}$.

Remark 7. Let $n \in \mathbb{N}$. Function $u(t) = \Delta_a^{-n} f(t)$ is solution to the initial value problem $\Delta^n u(t) = f(t)$, $u(a+j-1) = 0$, $t \in \mathbb{N}_a$, $j = 1, 2, \ldots, n$; function $u(t) = {}_b\Delta^{-n} f(t)$ is solution to the initial value problem $\nabla_{\ominus}^n u(t) = f(t), u(b-j+1) = 0, t \in {}_b\mathbb{N}, j = 1, 2, \ldots, n$; $\nabla_a^{-n} f(t)$ satisfies the nth order discrete initial value problem $\nabla^n y(t) = f(t)$, $\nabla^i y(a) = 0$, $i = 0, 1, \ldots, n-1$; while ${}_b\nabla^{-n} f(t)$ satisfies the nth order discrete initial value problem ${}_{\ominus}\Delta^n y(t) = f(t)$, ${}_{\ominus}\Delta^i y(b) = 0$, $i = 0, 1, \ldots, n-1$.

Remark 8. Consider the Cauchy functions $f(t) = \frac{(t-\sigma(s))^{(n-1)}}{(n-1)!}$; $g(t) = \frac{(\rho(s)-t)^{(n-1)}}{(n-1)!}$; $h(t) = \frac{(t-\rho(s))^{\overline{n-1}}}{\Gamma(n)}$; and $i(t) = \frac{(s-\rho(t))^{\overline{n-1}}}{\Gamma(n)}$. Then, $f(t)$ vanishes at $s = t-(n-1), \ldots, t-1$; $g(t)$ vanishes at $s = t+1, t+2, \ldots, t+(n-1)$; $h(t)$ satisfies $\nabla^n y(t) = 0$; and $i(t)$ satisfies ${}_{\ominus}\Delta^n y(t) = 0$.

Now we recall the definitions of delta/nabla left/right fractional differences in the sense of Riemann–Liouville. The definitions of Caputo fractional differences, denoted by ${}^C\Delta$ and ${}^C\nabla$ instead of Δ and ∇, respectively, are not given here, and we refer the reader to, e.g., [1,2,34].

Definition 9 (*See [6,33]*)**.** The delta left fractional difference of order $\alpha > 0$ (starting from a) is defined by

$$\begin{aligned} \Delta_a^{\alpha} f(t) &= \Delta^n \Delta_a^{-(n-\alpha)} f(t) \\ &= \frac{\Delta^n}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t-\sigma(s))^{(n-\alpha-1)} f(s), \quad t \in \mathbb{N}_{a+(n-\alpha)}; \end{aligned}$$

the delta right fractional difference of order $\alpha > 0$ (ending at b) by

$$\begin{aligned} {}_b\Delta^{\alpha} f(t) &= \nabla_{\ominus}^n {}_b\Delta^{-(n-\alpha)} f(t) \\ &= \frac{(-1)^n \nabla^n}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (s-\sigma(t))^{(n-\alpha-1)} f(s), \quad t \in {}_{b-(n-\alpha)}\mathbb{N}; \end{aligned}$$

the nabla left fractional difference of order $\alpha > 0$ (starting from a) by

$$\nabla_a^\alpha f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{n-\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1},$$

and the nabla right fractional difference of order $\alpha > 0$ (ending at b) is defined by

$${}_b\nabla^\alpha f(t) = {}_\ominus\Delta^n {}_b\nabla^{-(n-\alpha)} f(t) = \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{n-\alpha-1}} f(s), \quad t \in {}_{b-1}\mathbb{N}.$$

Regarding the domains of the fractional differences, we observe the following.

Remark 10. The delta left fractional difference Δ_a^α maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(n-\alpha)}$; the delta right fractional difference ${}_b\Delta^\alpha$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-(n-\alpha)}\mathbb{N}$; the nabla left fractional difference ∇_a^α maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n}$; and the nabla right fractional difference ${}_b\nabla^\alpha$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-n}\mathbb{N}$.

Lemma 11 (*See [15]*). *If* $\alpha > 0$, *then* $\Delta_a^{-\alpha} \Delta f(t) = \Delta \Delta_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)$.

Lemma 12 (*See [6]*). *If* $\alpha > 0$, *then* ${}_b\Delta^{-\alpha} \nabla_\ominus f(t) = \nabla_\ominus {}_b\Delta^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b)$.

Lemma 13 (*See [12]*). *If* $\alpha > 0$, *then* $\nabla_{a+1}^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)$.

The result of Lemma 13 was obtained in [12] by applying the nabla left fractional sum starting from a and not from $a+1$. Lemma 14 provides a version of Lemma 13 proved in [3]. Actually, the nabla fractional sums defined in the articles [12] and [3] are related [5].

Lemma 14 (*See [3]*). *For any* $\alpha > 0$, *the equality*

$$\nabla_a^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) \tag{1}$$

holds.

Remark 15. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, with the help of Lemma 14, we have

$$\nabla \nabla_a^\alpha f(t) = \nabla \nabla^n (\nabla_a^{-(n-\alpha)} f(t)) = \nabla^n (\nabla \nabla_a^{-(n-\alpha)} f(t))$$

or

$$\nabla \nabla_a^\alpha f(t) = \nabla^n \left[\nabla_a^{-(n-\alpha)} \nabla f(t) + \frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(a) \right].$$

Then, using the identity $\nabla^n \frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}$, we infer that (1) is valid for any real α.

With the help of Lemma 14, Remark 15, and the identity $\nabla(t-a)^{\overline{\alpha-1}} = (\alpha-1)(t-a)^{\overline{\alpha-2}}$, we arrive inductively to the following generalization.

Theorem 16 (*See [2,5]*). *For any real number α and any positive integer p, the equality*

$$\nabla_{a+p-1}^{-\alpha}\, \nabla^p f(t) = \nabla^p \nabla_{a+p-1}^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-(a+p-1))^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a+p-1)$$

holds, where f is defined on $\mathbb{N}_a$.

Lemma 17 (*See [3]*). *For any $\alpha > 0$, the equality*

$$_b\nabla^{-\alpha}{}_{\ominus}\Delta f(t) = {}_{\ominus}\Delta_b \nabla^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b) \tag{2}$$

holds.

Remark 18. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, with the help of Lemma 17, we have

$$_a\Delta_b \nabla^{\alpha} f(t) = {}_a\Delta_{\ominus}\Delta^n ({}_b\nabla^{-(n-\alpha)} f(t)) = {}_{\ominus}\Delta^n ({}_{\ominus}\Delta_b \nabla^{-(n-\alpha)} f(t))$$

or

$$_{\ominus}\Delta_b \nabla^{\alpha} f(t) = {}_{\ominus}\Delta^n \left[{}_b\nabla^{-(n-\alpha)}{}_{\ominus}\Delta f(t) + \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(b) \right].$$

Then, using the identity ${}_{\ominus}\Delta^n \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}$, we infer that (2) is valid for any real α.

With the help of Lemma 17, Remark 18, and the identity $\Delta(b-t)^{\overline{\alpha-1}} = -(\alpha-1)(b-t)^{\overline{\alpha-2}}$, we arrive by induction to the following generalization.

Theorem 19 (*See [2,5]*). *For any real number α and any positive integer p, the equality*

$$\begin{aligned} {}_{b-p+1}\nabla^{-\alpha}{}_{\ominus}\Delta^p f(t) &= {}_{\ominus}\Delta^p\, {}_{b-p+1}\nabla^{-\alpha} f(t) \\ &\quad - \sum_{k=0}^{p-1} \frac{(b-p+1-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\, {}_{\ominus}\Delta^k f(b-p+1) \end{aligned}$$

holds, where f is defined on ${}_b\mathbb{N}$.

3. Symmetric duality for left and right Riemann–Liouville and Caputo fractional differences

In this section, we use the recent notion of duality for the continuous fractional calculus, as introduced by Caputo and Torres in [25], to prove symmetric duality identities for delta and nabla fractional sums and differences. The next result (as well as Theorem 27), shows that the left fractional sum of a given function f is the right fractional sum of the dual of f.

Theorem 20 (*Symmetric Duality of Nabla Fractional Sums*)**.** *Let $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$ be a given function and $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual, that is, $f^*(t) = f(-t)$. Then,*

$$(\nabla_a^{-\alpha} f)(t) = ({}_{-a}\nabla^{-\alpha} f^*)(-t), \tag{3}$$

where on the right-hand side of (3) we have the nabla right fractional sum of f^* ending at $-a$ and evaluated at $-t$.

Proof. Using Definition 5, and the change of variable $s = -u$, we have

$$\begin{aligned}(\nabla_a^{-\alpha} f)(t) &= \frac{1}{\Gamma(\alpha)} \sum_{u=a+1}^{t} (t-\rho(u))^{\overline{\alpha-1}} f(u)\\ &= -\frac{1}{\Gamma(\alpha)} \sum_{s=-a-1}^{-t} (t+s+1)^{\overline{\alpha-1}} f(-s)\\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=-t}^{-a-1} (s-\rho(-t))^{\overline{\alpha-1}} f^*(s)\\ &= ({}_{-a}\nabla^{-\alpha} f^*)(-t).\end{aligned}$$

This concludes the proof. □

While in the fractional case symmetric duality relates left and right operators, the next two results (Lemma 21 and Theorem 22) show that in the integer-order case the symmetric duality relates delta and nabla operators. This is a consequence of the general duality on time scales [24].

Lemma 21 (*Symmetric Duality of Forward and Backward Difference Operators*)**.** *Let $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$ and $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual function. Then,*

$$-(\nabla f)^*(t) = \Delta f^*(t) \tag{4}$$

and

$$-(\Delta f)^*(t) = \nabla f^*(t). \tag{5}$$

Proof. Let $g(t) = \nabla f(t) = f(t) - f(t-1)$. Then,

$$-g^*(t) = -g(-t) = f(-t-1) - f(-t) = f(-(t+1)) - f(-t) = \Delta f^*(t)$$

and (4) is proved. The proof of (5) is similar by defining $h(t) = \Delta f(t) = f(t+1) - f(t)$. □

Relations (4) and (5) are easily generalized to the higher-order case.

Theorem 22 (*Symmetric Duality of Integer-Order Difference Operators*)**.** *Let $n \in \mathbb{N}$, $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$ be a given function and $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual. Then,*

$$(\nabla^n f)^*(t) = (-1)^n \Delta^n f^*(t) = {}_{\ominus}\Delta^n f^*(t)$$

and

$$(\Delta^n f)^*(t) = (-1)^n \nabla^n f^*(t) = \nabla_{\ominus}^n f^*(t).$$

Proof. The case $n = 1$ is true from Lemma 21. We obtain the intended result by induction. □

Our previous results allow us to relate left and right nabla fractional differences.

Theorem 23 (*Symmetric Duality of Riemann–Liouville Nabla Fractional Difference Operators*)**.** *Assume $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$ and let $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual. Then, for each $n-1 < \alpha \le n$, $n \in \mathbb{N}_1$, we have that the left fractional difference of f starting at a and evaluated at t is the right fractional difference of f^* ending at $-a$ and evaluated at $-t$:*

$$(\nabla_a^\alpha f)(t) = ({}_{-a}\nabla^\alpha f^*)(-t).$$

Proof. By Definition 9, and the help of Theorems 20 and 22, it follows that

$$\begin{aligned}
(\nabla_a^\alpha f)(t) &= \nabla^n (\nabla_a^{-(n-\alpha)} f)(t) \\
&= \nabla^n (-a\nabla^{-(n-\alpha)} f^*)(-t) \\
&= \nabla^n ({}_{-a}\nabla^{-(n-\alpha)} f^*)^*(t) \\
&= {}_{\ominus}\Delta^n ({}_{-a}\nabla^{-(n-\alpha)} f^*)^*(t) \\
&= {}_{\ominus}\Delta^n ({}_{-a}\nabla^{-(n-\alpha)} f^*)(-t) \\
&= ({}_{-a}\nabla^\alpha f^*)(-t).
\end{aligned}$$

This completes the proof. □

An analogous result to Theorem 23 also holds for fractional differences in the sense of Caputo.

Theorem 24 (*Symmetric Duality of Caputo Nabla Fractional Difference Operators*)**.** *Given a function $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$, let $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual. Then, for each $n-1 < \alpha \le n$, $n \in \mathbb{N}_1$, we have*

$$({}^C\nabla_{a(\alpha)}^\alpha f)(t) = ({}_{-a(\alpha)}^{C}\nabla^\alpha f^*)(-t),$$

where $a(\alpha) = a + n - 1$.

Proof. Using the definition of Caputo fractional differences, Theorems 20 and 22, we have

$$\begin{aligned}
({}^C\nabla_{a(\alpha)}^\alpha f)(t) &= (\nabla_{a(\alpha)}^{-(n-\alpha)} \nabla^n f)(t) \\
&= ({}_{a(\alpha)}\nabla^{-(n-\alpha)} (\nabla^n f(t))^*)(-t) \\
&= ({}_{a(\alpha)}\nabla^{-(n-\alpha)} ({}_{\ominus}\Delta^n f^*(t))^*)(-t) \\
&= ({}_{-a(\alpha)}^{C}\nabla^\alpha f^*)(-t).
\end{aligned}$$

The proof is complete. □

We now show that symmetric duality results for delta fractional sums and delta fractional differences can be achieved from our previous duality results on nabla fractional operators by using the approach in [8]. For delta fractional sums and differences we make use of the next two lemmas summarized and cited accurately in [8].

Lemma 25 (*See [8]*)**.** *Let $0 \leq n-1 < \alpha \leq n$ and let $y(t)$ be defined on $\mathbb{N}_a$. Then, the following statements are valid:*
(i) $(\Delta_a^{\alpha})y(t-\alpha) = \nabla_{a-1}^{\alpha} y(t)$ *for* $t \in \mathbb{N}_{n+a}$*;*
(ii) $(\Delta_a^{-\alpha})y(t+\alpha) = \nabla_{a-1}^{-\alpha} y(t)$ *for* $t \in \mathbb{N}_a$*.*

Lemma 26 (*See [8]*)**.** *Let $y(t)$ be defined on ${}_{b+1}\mathbb{N}$. The following statements are valid:*
(i) $({}_b\Delta^{\alpha})y(t+\alpha) = {}_b\nabla^{\alpha} y(t)$ *for* $t \in {}_{b-n}\mathbb{N}$*;*
(ii) $({}_b\Delta^{-\alpha})y(t-\alpha) = {}_{b+1}\nabla^{-\alpha} y(t)$ *for* $t \in {}_b\mathbb{N}$*.*

Our next result is the delta analogous of Theorem 20.

Theorem 27 (*Symmetric Duality of Delta Fractional Sums*)**.** *Let $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$, $a < b$, and $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual function. Then, for each $n-1 < \alpha \leq n$, $n \in \mathbb{N}_1$, and $t \in \mathbb{N}_a$, we have*

$$\Delta_a^{-\alpha} f(\sigma^{\alpha}(t)) = ({}_{-a}\Delta^{-\alpha} f^*)(-\sigma^{\alpha}(t)),$$

where the fractional forward jump operator σ^{α} is defined by $\sigma^{\alpha}(t) = t+\alpha$.

Proof. The following relations hold:

$$\begin{aligned}\Delta_a^{-\alpha} f(t+\alpha) &= \nabla_{a-1}^{-\alpha} f(t)\\ &= ({}_{-a+1}\nabla^{-\alpha} f^*)(-t)\\ &= ({}_{-a}\Delta^{-\alpha} f^*)(-t-\alpha).\end{aligned}$$

The above equalities follow by Lemma 25(ii), Theorem 20, and Lemma 26(ii) (with $b = -a$). □

We now prove symmetric duality for delta fractional differences.

Theorem 28 (*Symmetric Duality of Riemann–Liouville Delta Fractional Difference Operators*)**.** *Let $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$, $a < b$, and $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual function. Then, for each $n-1 < \alpha \leq n$, $n \in \mathbb{N}_1$, and $t \in \mathbb{N}_{a+n}$, we have*

$$\Delta_a^{\alpha} f(\rho^{\alpha}(t)) = ({}_{-a}\Delta^{\alpha} f^*)(-\rho^{\alpha}(t)),$$

where the fractional backward jump operator ρ^{α} is defined by $\rho^{\alpha}(t) = t-\alpha$.

Proof. By Lemma 25(i), Theorem 23, and Lemma 26(i) (with $b = -a$), it follows that

$$\begin{aligned}\Delta_a^{\alpha} f(t-\alpha) &= \nabla_{a-1}^{\alpha} f(t)\\ &= {}_{-a+1}\nabla^{\alpha} f^*(-t)\\ &= ({}_{-a}\Delta^{\alpha} f^*)(-t+\alpha).\end{aligned}$$

The result is proved. □

The next proposition states a relation between left delta Caputo fractional differences and left nabla Caputo fractional differences.

Proposition 29 (*See [2]*). *For $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$, $n = [\alpha] + 1$, $a(\alpha) = a + n - 1$, we have*

$$({}^C\Delta_a^\alpha f)(t - \alpha) = ({}^C\nabla_{a(\alpha)}^\alpha f)(t), \quad t \in \mathbb{N}_{a+n}.$$

Analogously, the following proposition relates right delta Caputo fractional differences and right nabla Caputo fractional differences.

Proposition 30 (*See [2]*). *For $f : {}_b\mathbb{N} \to \mathbb{R}$, $\alpha > 0$, $n = [\alpha] + 1$, $b(\alpha) = b - n + 1$, we have*

$$({}_b^C\Delta^\alpha f)(t + \alpha) = ({}_{b(\alpha)}^C\nabla^\alpha f)(t), \quad t \in {}_{b-n}\mathbb{N}.$$

Using Propositions 29 and 30, as well as our Theorem 24 on the symmetric duality of the Caputo nabla fractional difference operators, we prove a symmetric duality result for the delta fractional differences.

Theorem 31 (*Symmetric Duality of Caputo Delta Fractional Difference Operators*). *Let $f : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$, $a < b$, and $f^* : \mathbb{N}_{-b} \cap {}_{-a}\mathbb{N} \to \mathbb{R}$ be its symmetric dual. Then, for each $n - 1 < \alpha \leq n$, $n \in \mathbb{N}_1$, and $t \in \mathbb{N}_{a+n}$, we have*

$${}^C\Delta_a^\alpha f(\rho^\alpha(t)) = ({}_{-a}^C\Delta^\alpha f^*)(-\rho^\alpha(t)),$$

where $\rho^\alpha(t) = t - \alpha$.

Proof. From Proposition 29, Theorem 24, and Proposition 30, we have

$$\begin{aligned} {}^C\Delta_a^\alpha f(t - \alpha) &= {}^C\nabla_{a(\alpha)}^\alpha f(t) \\ &= {}_{-a(\alpha)}^C\nabla^\alpha f^*(-t) \\ &= ({}_{-a}^C\Delta^\alpha f^*)(-t + \alpha). \end{aligned}$$

The result follows by using the definition of the fractional backward jump operator $\rho^\alpha(t)$. $\square$

4. Summation by Parts for Fractional Differences

The next version of fractional "integration by parts" with boundary conditions was proved in [2, Theorem 43]. It has then been used in [4] to obtain a discrete fractional variational principle.

Theorem 32 (*See [2]*). *Let $0 < \alpha < 1$ and f and g be two functions defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, where $a \equiv b \bmod 1$. Then,*

$$\sum_{s=a+1}^{b-1} g(s) {}^C\nabla_a^\alpha f(s) = f(s) {}_b\nabla^{-(1-\alpha)} g(s) \mid_a^{b-1} + \sum_{s=a+1}^{b-1} f(s-1)({}_b\nabla^\alpha g)(s-1),$$

where $({}_b\nabla^{-(1-\alpha)} g)(b-1) = g(b-1)$.

If we interchange the role of Caputo and Riemann–Liouville operators, we obtain the following version of summation by parts for fractional differences. It was proved in [4] and used there to obtain a discrete fractional Euler–Lagrange equation.

Theorem 33 (*See [4]*). *Let $0 < \alpha < 1$ and f and g be functions defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, where $a \equiv b \bmod 1$. Then,*

$$\sum_{s=a+1}^{b-1} f(s-1)\nabla_a^{\alpha} g(s) = f(s)\nabla_a^{-(1-\alpha)} g(s)\,|_a^{b-1} + \sum_{s=a}^{b-2} g(s+1)\,{}_b^C\nabla^{\alpha} f(s)$$

$$= f(s)\nabla_a^{-(1-\alpha)} g(s)\,|_a^{b-1} + \sum_{s=a+1}^{b-1} g(s)\,({}_b^C\nabla^{\alpha} f)(s-1),$$

where $\nabla_a^{-(1-\alpha)} g(a) = 0$.

The above formulas of fractional summation by parts were only obtained for left fractional differences. In this section, we use the symmetric duality results of Section 3 to obtain new summation by parts formulas for the right fractional differences.

Theorem 34. *Let $0 < \alpha < 1$ and f and g be functions defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, where $a \equiv b \bmod 1$. Then,*

$$\sum_{s=a+1}^{b-1} g(s)\,{}_b^C\nabla^{\alpha} f(s) = f(s)\nabla_a^{-(1-\alpha)} g(s)\,|_b^{a+1} + \sum_{s=a+1}^{b-1} f(s+1)(\nabla_a^{\alpha} g)(s+1),$$

where $\left(\nabla_a^{-(1-\alpha)} g\right)(a+1) = g(a+1)$.

Proof. First, we note that if we apply Theorem 24 to f^* starting at $-b$, $a(\alpha) = a$, $n = 1$, and using the fact that $f^{**} = f$, we conclude that

$$({}^C\nabla_{-b}^{\alpha} f^*)(t) = ({}_b^C\nabla^{\alpha} f^{**})(-t) = ({}_b^C\nabla^{\alpha} f)(-t). \tag{6}$$

Then, by the change of variable $s = -t$, it follows from (6) that

$$\sum_{s=a+1}^{b-1} g(s)\,{}_b^C\nabla^{\alpha} f(s) = -\sum_{t=-a-1}^{-b+1} g(-t)\,{}_b^C\nabla^{\alpha} f(-t)$$

$$= \sum_{t=-b+1}^{-a-1} g^*(t)({}_b^C\nabla^{\alpha} f)(-t)$$

$$= \sum_{t=-b+1}^{-a-1} g^*(t)({}^C\nabla_{-b}^{\alpha} f^*)(t).$$

Applying Theorem 32 to the pair (f^*, g^*) with $a \to -b$ and $b \to -a$, we reach at

$$\begin{aligned}\sum_{s=a+1}^{b-1} g(s)\,{}_b^C\nabla^\alpha f(s) &= f^*(s)({}_{-a}\nabla^{-(1-\alpha)}g^*)(s)|_{-b}^{-a-1} \\ &\quad + \sum_{s=-b+1}^{-a-1} f^*(s-1)({}_{-a}\nabla^\alpha g^*)(s-1) \\ &= f^*(s)({}_{-a}\nabla^{-(1-\alpha)}g^*)(s)|_{-b}^{-a-1} \\ &\quad + \sum_{s=-b}^{-a-2} f^*(s)({}_{-a}\nabla^\alpha g^*)(s) \\ &= f(s)({}_{-a}\nabla^{-(1-\alpha)}g^*)(-s)|_b^{a+1} \\ &\quad + \sum_{s=a+2}^{b} f(s)({}_{-a}\nabla^\alpha g^*)(-s).\end{aligned}$$

Then, by Theorems 20 and 23, we have

$$\begin{aligned}\sum_{s=a+1}^{b-1} g(s)\,{}_b^C\nabla^\alpha f(s) &= f(s)(\nabla_a^{-(1-\alpha)}g)(s)|_b^{a+1} + \sum_{s=a+2}^{b} f(s)(\nabla_a^\alpha g)(s) \\ &= f(s)(\nabla_a^{-(1-\alpha)}g)(s)|_b^{a+1} + \sum_{s=a+1}^{b} f(s+1)(\nabla_a^\alpha g)(s+1)\end{aligned}$$

and the result is proved. □

Theorem 34 relates a right Caputo nabla difference with a left Riemann–Liouville nabla difference. In contrast, the next theorem relates a right Riemann–Liouville nabla difference with a left Caputo nabla difference.

Theorem 35. *Let* $0 < \alpha < 1$ *and* f *and* g *be functions defined on* $\mathbb{N}_a \cap {}_b\mathbb{N}$*, where* $a \equiv b \bmod 1$*. Then,*

$$\sum_{s=a+1}^{b-1} f(s-1)_b\nabla^\alpha g(s) = f(s)_b\nabla^{-(1-\alpha)}g(s)\,|_b^{a+1} + \sum_{s=a+1}^{b-1} g(s)({}^C\nabla_a^\alpha f)(s+1),$$

where $({}_b\nabla^{-(1-\alpha)}g)(b) = 0$.

Proof. First, we apply Theorem 23 for f^* starting at $-b$, $n = 1$. Using the relation $f^{**} = f$, we see that

$$(\nabla_{-b}^\alpha f^*)(t) = ({}_b\nabla^\alpha f^{**})(-t) = ({}_b\nabla^\alpha f)(-t). \tag{7}$$

Then, by the change of variable $s = -t$ and the help of (7), we have

$$\begin{aligned}
\sum_{s=a+1}^{b-1} f(s-1)_b\nabla^\alpha g(s) &= -\sum_{t=-a-1}^{-b+1} f(-t-1)_b\nabla^\alpha g(-t) \\
&= \sum_{t=-b+1}^{-a-1} f^*(t-1)(_b\nabla^\alpha g^*)(-t) \\
&= \sum_{t=-b+1}^{-a-1} f^*(t-1)(\nabla^\alpha_{-b} g^*)(t).
\end{aligned}$$

Applying Theorem 33 to the pair (f^*, g^*) with $a \to -b$ and $b \to -a$, we reach at

$$\begin{aligned}
\sum_{s=a+1}^{b-1} f(s-1)_b\nabla^\alpha g(s) &= f^*(s)(\nabla_{-b}^{-(1-\alpha)} g^*)(s)|_{-b}^{-a-1} \\
&\quad + \sum_{s=-b}^{-a-2} g^*(s-1)({}_{-a}^{C}\nabla^\alpha f^*)(s) \\
&= f(s)(\nabla_{-b}^{-(1-\alpha)} g^*)(-s)|_b^{a+1} \\
&\quad + \sum_{s=a+2}^{b} g^*(-s-1)({}_{-a}^{C}\nabla^\alpha f^*)(-s).
\end{aligned}$$

Then, by Theorems 20 and 24, we have

$$\begin{aligned}
\sum_{s=a+1}^{b-1} f(s-1)_b\nabla^\alpha g(s) &= f(s)({}_b\nabla^{-(1-\alpha)} g)(s)|_b^{a+1} + \sum_{s=a+2}^{b} g(s-1)({}^C\nabla_a^\alpha f)(s) \\
&= f(s)({}_b\nabla^{-(1-\alpha)} g)(s)|_b^{a+1} + \sum_{s=a+1}^{b-1} g(s)({}^C\nabla_a^\alpha f)(s+1).
\end{aligned}$$

This concludes the proof. □

5. Application to the discrete fractional variational calculus

The study of discrete fractional variational problems is mainly concentrated on obtaining Euler–Lagrange equations for a minimizer of a problem containing left fractional differences [16,17,27]. Roughly speaking, after applying a fractional summation by parts formula, one is able to express Euler–Lagrange equations by means of right fractional differences. In this section, we reverse the order and we start by minimizing a discrete functional containing right fractional differences. We will then use the summation by parts formulas obtained in Section 4 to prove left versions of the fractional difference Euler–Lagrange equations obtained in [4] (cf. Theorems 3.2 and 3.3 in [4]).

Theorem 36. *Let* $0 < \alpha < 1$ *be noninteger,* $a, b \in \mathbb{R}$*, and* f *be defined on* $\mathbb{N}_a \cap {}_b\mathbb{N}$*, where* $a \equiv b \bmod 1$*. Assume that the discrete functional*

$$J(f) = \sum_{t=a+1}^{b-1} L\left(t, f(t), {}_b\nabla^\alpha f(t)\right)$$

has a local extremizer in $S = \{y : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$ *is bounded*$\}$ *at some* $f \in S$*, where* $L : (\mathbb{N}_a \cap {}_b\mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*. Further, assume that either* ${}_b\nabla^{-(1-\alpha)} f(a+1) = B$ *or* $L_2^\sigma(a+1) = 0$*. Then,*

$$\left[L_1(s) + ({}^C\nabla_a^\alpha L_2^\sigma)(s+1)\right] = 0 \quad \textit{for all} \quad s \in \mathbb{N}_{a+1} \cap {}_{b-1}\mathbb{N},$$

where $L_1(s) = \frac{\partial L}{\partial f}(s, f(s), {}_b\nabla^\alpha f(s))$*,* $L_2(s) = \frac{\partial L}{\partial_b \nabla^\alpha f}(s, f(s), {}_b\nabla^\alpha f(s))$ *and* $L_2^\sigma(t) = L_2(\sigma(t))$*.*

Proof. Similar to the proof of Theorem 3.2 in [4] by making use of our Theorem 35. □

Finally, we obtain the Euler–Lagrange equation for a Lagrangian depending on the Caputo right fractional difference and by making use of the summation by parts formula in Theorem 34.

Theorem 37. *Let* $0 < \alpha < 1$ *be noninteger,* $a, b \in \mathbb{R}$*, and* f *be defined on* $\mathbb{N}_a \cap {}_b\mathbb{N}$*, where* $a \equiv b \bmod 1$*. Assume that the discrete functional*

$$J(f) = \sum_{t=a+1}^{b-1} L\left(t, f^\sigma(t), {}_b^C\nabla^\alpha f(t)\right)$$

has a local extremizer in $S = \{y : \mathbb{N}_a \cap {}_b\mathbb{N} \to \mathbb{R}$ *is bounded*$\}$ *at some* $f \in S$*, where* $L : (\mathbb{N}_a \cap {}_b\mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*. Further, assume that either* $f(a+1) = C$ *and* $f(b) = D$ *or the natural boundary conditions* $\nabla_a^{-(1-\alpha)} L_2(a+1) = \nabla_a^{-(1-\alpha)} L_2(b) = 0$ *hold. Then,*

$$\left[L_1(s) + (\nabla_a^\alpha L_2)(\sigma(s))\right] = 0 \quad \textit{for all} \quad s \in \mathbb{N}_{a+2} \cap {}_{b-1}\mathbb{N}.$$

Proof. Similar to the proof of Theorem 3.3 in [4] by making use of our Theorem 34. □

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