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ORIGINAL ARTICLE

Subsets of a projective variety $X \subset \mathbb{P}^n$ spanning a given $P \in \mathbb{P}^n$

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Abstract Fix an integral variety $X \subset \mathbb{P}^n$, $P \in \mathbb{P}^n$, and an integer $k > 0$. Let $\mathcal{S}(X, P, k)$ be the set of all subsets $S \subset X$ such that $\sharp(S) = k$, $P \in \langle S \rangle$ and $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$. Here we study $\mathcal{S}(X, P, k)$ (non-emptiness and dimension) in the extremal case $k = n - \dim(X) + 1$.

Introduction

Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. Set $m := \dim(X)$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\,\rangle$ denotes the linear span. In the applications when X is a Veronese embedding of $\mathbb{P}^m$ the X-rank is also called the "structured rank" or "symmetric tensor rank" (this is related to the virtual array concept considered in sensor array processing [1,6,10]).

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For an arbitrary X up to now the only general result is due to Landsberg and Teitler, who proved the following result [9, Proposition 4.1].

Theorem 1 [9]. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional subvariety. Then $r_X(P) \leqslant n - m + 1$ for all $P \in \mathbb{P}^n$.*

Theorem 1 is known to be sharp for rational normal curves [9, Theorem 5.1]; [3, §3, and references therein] and for a few scattered examples (some non-linearly normal smooth rational curves with a tangent with very high order of contact, many space curves [11], a degree $n + 1$ linearly normal curve with an ordinary node or an ordinary cusps). But all these examples, except rational normal curves, have a small set of points with X-rank $n - m + 1$. For any X, P any finite set $S \subset X$ computing $r_X(P)$ is linearly independent, (i.e. $\dim(\langle S \rangle) = \sharp(S) - 1$) and $P \notin \langle S' \rangle$ for all $S' \subsetneqq S$. Here we consider the cases of S as above with $P \in \langle S \rangle$, but dropping the assumption "$\sharp(S) = r_X(P)$)" and see that we obtain in this way a characterization of certain minimal degree subvarieties. We think that the sets $\mathcal{S}(X, P, k)$ may be useful also because to prove if $\mathcal{S}(X, P, k)$ is empty or not we do not need to compute $r_X(P)$.

Since $r_X(P) \leqslant n - m + 1$ by the quoted theorem of Landsberg and Teitler, there is $A \subset X$ such that $\sharp(A) = r_X(P) \leqslant n - m + 1$, $P \in \langle A \rangle$ and $P \notin \langle A' \rangle$ for any $A' \subsetneqq A$. If $r_X(P) < n - m + 1$, then adding to A any $(n - m + 1 - r_X(P))$ general points we obtain $B \subset X$ such that $\sharp(B) = n - m + 1$, B is linearly independent and $P \in \langle B \rangle$. But of course, there are smaller subsets of B spanning P. It is natural to ask if we may find some B (obtained in a different way) with this additional property. Our answer is that this is possible for almost all, but not all, pairs (X, P) (see Theorems 2 and 3). Only part (iii) of Theorem 3 is not a complete and explicit description. For every integer $k > 0$ let $\mathcal{S}(X, P, k)$ be the set of all subsets $S \subset X$ such that $\sharp(S) = k$, $P \in \langle S \rangle$ and $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$. Take any $S \in \mathcal{S}(X, P, k)$. The latter condition in the definition of these sets implies $\dim(\langle S \rangle) = k - 1$, i.e. S is linearly independent. Thus $\mathcal{S}(X, P, k) = \emptyset$ for all $k \geqslant n + 2$. Obviously $\mathcal{S}(X, P, k) = \emptyset$ if $k < r_X(P)$, and $\mathcal{S}(X, P, r_X(P)) \neq \emptyset$. Obviously $\mathcal{S}(X, P, n+1)$ contains a non-empty open subset of the symmetric product of $n + 1$ copies of X. Thus $\dim(\mathcal{S}(X, P, n+1)) = (n+1)m$ and every subset of X with cardinality $n + 1$ is a limit of a family of elements of $\mathcal{S}(X, P, n+1)$.

To state our results we need to introduce two definitions and the following notation.

For any subset U of a projective space $\mathbb{P}^r$ such that $b := \dim(\langle U \rangle) \leqslant r - 1$, let $\ell_U : \mathbb{P}^r \setminus \langle U \rangle \to \mathbb{P}^{r-b-1}$ denote the linear projection from the linear space $\langle U \rangle$.

Definition 1. Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate subvariety. Set $x := \dim(Y)$. We say that Y belongs to $\mathcal{A}(x, r)$ if it has minimal degree (i.e. $\deg(Y) = r - x + 1$) and Y belongs to one of the following classes:

(a) Y is a (cone over a) rational normal curve and $r \geqslant 3x - 1 + \eta$, where $\eta = 0$ if $r - x$ is even and $\eta = 1$ if $r - x$ is odd.
(b) $(x, r) = (2, 5)$ and Y is the Veronese surface.

In case (a) we allow the case $x = 1$, i.e. $\mathcal{A}(1, r),\ r \geqslant 2$, is the set of all rational normal curves of $\mathbb{P}^r$. See [7] for the complete classification of all minimal degree subvarieties of $\mathbb{P}^r$.

We will prove that $Y \in \mathcal{A}(x, r)$ if and only if there is no $S \subset Y$ such $\sharp(S) = r - x + 2$, $\dim(\langle S \rangle) = r - x$ and every proper subset of Y is linearly independent (Proposition 4). If we only assume $\sharp(S) \leqslant r - x + 2$, then the rational normal curve of $\mathbb{P}^r$ is the only example (Corollary 1).

Theorem 2. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $P \in \mathbb{P}^n \setminus X$. Set $Y := \ell_P(X)$.*

(i) *If either $n = m + 1$ or $\deg(Y) \geqslant n - m + 1$, then $\mathcal{S}(X, P, n - m + 1)$ contains an $m(n - m)$-dimensional family of subsets of X.*
(ii) *Assume $n \geqslant m + 2$. We have $\mathcal{S}(X, P, n - m + 1) = \emptyset$ if and only if $Y \in \mathcal{A}(m, n - 1)$.*
(iii) *Let $\Sigma \subsetneq X$ be any proper closed subset. Fix a general $S' \subset X$ such that $\sharp(S') = n - m - 1$. If either $n = m + 1$ or $n \geqslant m + 2$ and $\deg(Y) \geqslant n - m + 1$, then there is $S \in \mathcal{S}(X, P, n - m + 1)$ such that $S \cap \Sigma = \emptyset$ and $S' \subset S$.*

We have $\deg(X) = a \cdot \deg(Y)$, where $a := \deg(\ell_P| X)$. Since $\deg(X) \geqslant n - m + 1$, in case (ii) we have $a \geqslant 2$. For strong restrictions on the set of all $P \in \mathbb{P}^n \setminus X$ such that $\ell_P| X$ has degree > 1, see [4].

Definition 2. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional subvariety. Fix $P \in X$. We assume that X is not a cone with vertex containing P and the following holds. Let $Y \subset \mathbb{P}^{n-1}$ be the closure of $\ell_P(X \backslash \{P\})$. Since X is not a cone with vertex containing P, we have $\dim(Y) = m$. Let a be the degree of the morphism $\ell_P| (X \backslash \{P\}) \to Y$. We say that $(X, P) \in \mathcal{B}(m, n)$ if $a = 1, Y$ has degree $n - m$, $Y \notin \mathcal{A}(m, n - 1)$ if $m \leqslant n - 2$ and there is an $(n - m - 1)$-linear subspace $V \subset \mathbb{P}^n$ such that $P \in V$, $V \cap X$ has positive dimension and the set-theoretic intersection $V \cap (X \backslash \{P\})_{red}$ spans V.

Theorem 3. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Let $Y \subset \mathbb{P}^{n-1}$ be the closure of $\ell_P(X \backslash \{P\})$ in $\mathbb{P}^{n-1}$. Set ψ: $\ell_P| (X \backslash \{P\})$. If X is not a cone with vertex containing P let a be the degree of the morphism ψ: $X \backslash \{P\} \to Y$. Let $\mathcal{F}$ denotes the set of all lines contained in X and containing P.*

(i) *Assume* $n = m + 1$. *We have* $\mathcal{S}(X, P, 2) \neq \emptyset$ *if and only if either* $\mathcal{F} \neq \emptyset$ *or* $a \geqslant 2$.

(ii) *Assume* $n \geqslant m + 2$ *and that* X *is not a cone with vertex containing* P. *If* $a \geqslant 2$, *then* $\mathcal{S}(X, P, n - m + 1) = \emptyset$ *if and only if* $Y \in \mathcal{A}(m, n - 1)$. *If* $a = 1$ *and* $deg(Y) \geqslant n - m + 1$, *then* $\mathcal{S}(X, P, n - m + 1) \neq \emptyset$.

(iii) *Assume* $n \geqslant m + 2$, *that* X *is not a cone with vertex containing* P, $a = 1$ *and* $deg(Y) = n - m$. *If* $Y \in \mathcal{A}(m, n - 1)$, *then* $\mathcal{S}(X, P, n - m + 1) = \emptyset$. *If* $\mathcal{S}(X, P, n - m + 1) \neq \emptyset$, *then* $(X, P) \in \mathcal{B}(m, n)$.

(iv) *Assume* $n \geqslant m + 2$ *and that* X *is a cone with vertex containing* P. *Then* $\mathcal{S}(X, P, n - m + 1) = \emptyset$ *if and only if* $deg(X) = n - m$ *and one of the following two cases occurs:*

  (iv1) $m = 3$ *and* X *is a cone over a Veronese surface;*

  (iv2) $m \geqslant 2$, X *is a cone over a rational normal curve and* $n \leqslant 3m - 6 + \eta$, *where* $\eta = 0$ *if* $n - m$ *is even and* $\eta = 1$ *if* $n - m$ *is odd.*

Take the set-up of parts (ii) and (iii) of Theorem 3, i.e. assume $m \geqslant n + 2$, $P \in X$ and X not a cone with vertex containing P. Let μ be the multiplicity of X at P. We have $\deg(Y) = \mu + a \cdot \deg(Y)$, where μ is the multiplicity of μ at P. Thus if we know μ we get a very strong restriction for the possible integers $a \geqslant 1$ and $\deg(Y) \geqslant n - m$. If $Y \in \mathcal{A}(m, n - 1)$, then $\deg(Y) = n - m$. Take the set-up of part (iv) of Theorem 3. The two exceptional cases just mean $Y \in \mathcal{A}(m - 1, n - 1)$.

Part (iii) of Theorem 2 is a mildly interesting base-point-free-theorem for the family of sets $\mathcal{S}(X, P, n - m + 1)$. The same dimensional count which gives the expected dimension of secant varieties gives the expectation that usually $\mathcal{S}(X, P, n - m + 1)$ is very large. The surprising fact is that sometimes $\mathcal{S}(X, P, n - m + 1)$ is empty and that all cases in which $\mathcal{S}(X, P, n - m + 1) = \emptyset$ may be described in terms of minimal degree subvarieties. Fix $Q \in \mathbb{P}^n \setminus \{P\}$. Statements like part (iii) of Theorem 2 should be useful to handle inner projections from $P \in X$ and the delicate relations between the sets $\{S \in \mathcal{S}(X, Q, k) : P \in S\}$ and $\mathcal{S}(Y, \ell_P(Q), k - 1)$. For the corresponding statement for Theorem 3, see Remark 2.

The proofs

Lemma 1. *Let* $Y \subset \mathbb{P}^r$ *be an integral and non-degenerate subvariety. Set* $x := dim(Y)$ *and assume* $deg(Y) \geqslant r - x + 2$. *Fix any proper closed subset* $\Delta \subsetneq Y$ *and a general* $A \subset Y \backslash \Delta$ *such that* $\sharp(A) = r - x + 1$. *Let* Γ *be the set of all* $B \subset Y$ *such that* $\sharp(B) = r - x + 2$, $dim(\langle B \rangle) = r - x$ *and every proper subset of* B *is linearly independent*. *Then* $\Gamma \neq \emptyset$ *and there is* $B \in \Gamma$ *such that* $A \subset B$ *and* $B \cap \Delta = \emptyset$.

Proof. By Bertini's theorem and the linearly general position lemma [2, p. 109] a general $(r-x)$-dimensional linear subspace H of $\mathbb{P}^r$ intersects Y in a reduced set of $\deg(Y)$ points in linearly general position in H, i.e. every $E \subseteq Y \cap H$ spans a linear subspace of dimension $\min\{r-x, \sharp(E)-1\}$. Since A is chosen general, the same is true for the $(r-x)$-dimensional linear space $\langle A\rangle$. Since we fix A after fixing Δ and $\dim(\Delta) \leqslant x-1$, we may assume $\langle A\rangle \cap \Delta = \emptyset$. Since $\deg(Y) \geqslant r-x+2$, we have $(Y \cap \langle A\rangle)\backslash A \neq \emptyset$. Fix any $O \in (Y \cap \langle A\rangle)\backslash A$. Since $\{O\} \cup A$ is in linearly general position in $\langle A\rangle$, we have $\{O\} \cup A \in \Gamma$. $\square$

Lemma 2. *Let $Y \subset \mathbb{P}^r$, $r \geqslant 4$, be a non-degenerate and smooth degree $r-1$ surface. If $r = 5$, then assume that Y is not the Veronese surface. Then there is $B \subset Y$ such that $\sharp(B) = r$, $\dim(\langle B\rangle) = r-2$ and $\dim(\langle B'\rangle) = \sharp(B')-1$ for every proper subset B' of B.*

Proof. There is an integer e such that $0 \leqslant e \leqslant (r-1)/2$, $e \equiv r-1 \pmod 2$ and Y is isomorphic to the Hirzebruch surface F_e, i.e. to the rational ruled surface with a section h of the ruling with self-intersection $-e$ see [8, V.2.17]. We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}^{\oplus 2}$ and we may take as a basis of $\mathrm{Pic}(F_e)$ the section h and a fiber f of the corresponding ruling. Thus $h^2 = -e, h \cdot f = 1$ and $f^2 = 0$. The embedding $j : F_e \hookrightarrow \mathbb{P}^r$ with Y as its image is given by the complete linear system $|\mathcal{O}_{F_e}(h + ((r+e-1)/2)f)|$. Since $(r+e-3)/2 \geqslant e$, the linear system $|\mathcal{O}_{F_e}(h + ((r+e-3)/2)f)|$ is spanned. Thus its general element is a smooth curve and j maps each smooth element of it into a smooth rational curve $D \subset Y$ such that $\dim(\langle D\rangle) = r-2$ and D is a rational normal curve in $\langle D\rangle$ [8, V.2.17]. Take as B any r points of D. $\square$

Lemma 3. *Let $Y \subset \mathbb{P}^5$ be a Veronese surface. Fix $S \subset X$ such that $\sharp(S) = 5$ and $\dim(\langle S\rangle) \leqslant 3$. Then there exists $S' \subset S$ such that $\sharp(S') = 4$ and $\dim(\langle S'\rangle) = 2$.*

Proof. Let $j : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese embedding with $Y = j(\mathbb{P}^2)$. Take $A \subset \mathbb{P}^2$ such that $j(A) = S$. Thus $\sharp(A) = 5$. Since $\dim(\langle S\rangle) \leqslant 3$, we have $h^1(\mathbb{P}^2, \mathcal{I}_A(2)) > 0$. There is a line $L \subset \mathbb{P}^2$ such that $\sharp(A \cap L) \geqslant 4$ (e.g. use [3, Lemma 4.6]. Take $A' \subseteq A \cap L$ such that $\sharp(A') = 4$ and set $S' := j(A')$. $\square$

Lemma 4. *Let $Y \subset \mathbb{P}^6$, be a three-dimensional cone over a Veronese surface of $\mathbb{P}^5$. Then there is $S \subset Y$ such that $\sharp(S) = 5$, $\dim(\langle S\rangle) = 3$ and each proper subset of S is linearly independent.*

Proof. Let O be the vertex of Y. Fix a hyperplane $H \subset \mathbb{P}^6$ such that $O \notin H$. Thus $H \cap Y$ is isomorphic to a Veronese surface. Fix a smooth conic $D \subset Y \cap H$. Let W be the quadric cone of $\langle \{O\} \cup D\rangle \cong \mathbb{P}^3$ with vertex O and D as a basis. Let $S_1 \subset W\backslash\{O\}$ be a general subset such that $\sharp(S_1) = 4$. Since S_1 is general, it spans

$\langle\{O\} \cup D\rangle$ and $\ell_O(S_1)$ are 4 points of S. Set $S := \langle\{O\} \cup S_1\rangle$. By construction S is linearly dependent, while S_1 is linearly independent. Since any 3 points of D are linear independent and $O \notin S_1$ we get $\dim(\langle\{O\} \cup S_2\rangle = \sharp(S_2)$ for every $S_2 \subsetneqq S_1$. $\square$

Remark 1. Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate subvariety. Set $k := \dim(Y)$ and assume $k \geqslant 2$. Let $H \subset \mathbb{P}^r$ be a hyperplane such that $Y_H := Y \cap H$ is integral (e.g. take as H a general hyperplane). Fix an integer y such that $3 \leqslant y \leqslant r + 1$. Let $\Gamma(y)$ (resp. $\Gamma_H(y)$) be the set of all $B \subset Y$ (resp. $B \subset Y_H$) such that $\sharp(S) = y, \dim(\langle B\rangle) = y - 2$ and every proper subset of B is linearly independent. Since $Y_H \subset Y$, we have $\Gamma_H(y) \subseteqq \Gamma(y)$.

Proposition 1. *Fix integers $r > x \geqslant 2$. Let $Y \subset \mathbb{P}^r$ be an x-dimensional cone over the rational normal curve of $\mathbb{P}^{r-x+1}$. Let Γ be the set of all $S \subset Y$ such that $\sharp(S) = r - x + 1, \dim(\langle S\rangle) = r - x - 1$ and $\dim(\langle S'\rangle) = \sharp(S') - 1$ for all $S' \subsetneqq S$. Then $\Gamma \neq \emptyset$ if and only if $r \leqslant 3x - 2 + \eta$, where $\eta = 0$ if $r - x$ is even and $\eta = 1$ if $r - x$ is odd.*

Proof. Let V be the vertex of Y. We have $\dim(V) = x - 2 \geqslant 0$. Fix an integer s such that $0 \leqslant s \leqslant \min\{x, r - x + 1\}$. Set $\Gamma_s := \{S \in \Gamma : \sharp(S \cap V) = s\}$. It is sufficient to check for which pairs (r, x) there is $s \in \{0, \ldots, \min\{x, r - x + 2\}\}$ such that $\Gamma_s \neq \emptyset$. If $r - x + 1 \leqslant x$ (i.e. if $r \leqslant 2x - 1$), then Γ_{r-x+1} is defined and non-empty.

Now assume $r = 2x$. Fix any $O \in Y \backslash V$. Since Y contains the $(x-1)$-dimensional linear space $\langle V \cup \{O\}\rangle$, a general $S \subset \langle V \cup \{O\}\rangle$ with cardinality $x + 1$ belongs to Γ_{r-x}. Thus $\Gamma_{r-x} \neq \emptyset$ if $r = 2x$. Hence from now on we always assume $r \geqslant 2x + 1$ and $s \leqslant r - x + 1$. Let $M \subset \mathbb{P}^r$ be a general $(r - x + 1)$-dimensional linear subspace. Since M is general, $V \cap M = \emptyset$ and $C := Y \cap M$ is a rational normal curve. See ℓ_V as a linear projection from $\mathbb{P}^r \setminus V$ onto M. Thus $u := \ell_V|(Y \backslash V) : Y \backslash V \to C$ is a submersion with as fibers the $(x-1)$-dimensional affine spaces $u^{-1}(Q) = \langle\{Q\} \cup V\rangle \backslash V$ for all $Q \in C$. Assume the existence of $S \in \Gamma_s$ and set $\{Q_1, \ldots, Q_h, Q_{h+1}, \ldots, Q_{h+k}\} := u(S \backslash S \cap V)$, with h, k non-negative integers, the sequence $\{a_i := \sharp(u^{-1}(Q_i) \cap S)\}_{1 \leqslant i \leqslant h+k}$ non-decreasing and $a_i = 1$ if and only if $h + 1 \leqslant i \leqslant h + k$. Notice that $\sum_{i=1}^{h+k} a_i = r - x + 1 - s$.

Set $M_i := \langle u^{-1}(Q_i) \cap S\rangle$ and $D_i := M_i \cap V$. Since $r \geqslant 2x + 1$, the set S is not contained in any $(x-1)$-dimensional projective space $\langle V \cup u^{-1}(Q)\rangle$, $Q \in C$. Thus each set $u^{-1}(Q_i) \cap S$ is a proper subset of S. Thus each set $u^{-1}(Q_i) \cap S$ is linearly independent. Thus M_i is an $(a_i - 1)$-dimensional linear subspace of $\langle u^{-1}(Q_i)\rangle$ not contained in the hyperplane V of $\langle u^{-1}(Q_i)\rangle$. Thus D_i is an $(a_i - 2)$-dimensional linear subspace of V (with $D_i = \emptyset$ if and only if $h + 1 \leqslant i \leqslant h + k$). Set $D_0 := \langle S \cap V\rangle$. Since $s \leqslant x - 1$, we have $\dim(D_0) = s - 1$, with the convention

$\dim(\emptyset) = -1$. Since $\dim(\langle S\rangle) = \sharp(S) - 2$, the linear subspaces $D_0, M_1, \cdots, M_{h+k}$ fail to be linearly independent just by 1. Since $\dim(V) = x - 2$, we have

$$s + a_1 + \cdots + a_h \leqslant x + h - \epsilon \tag{1}$$

where $\epsilon = 0$ if the linear subspaces $D_0, \ldots, D_h$ are not linearly independent and $\epsilon = 1$ otherwise.

(a) Here we assume $k = 0$, i.e. $a_i \geqslant 2$ for all i. Thus $s + a_1 + \cdots + a_h = r - x + 1$. Since $a_i \geqslant 2$ for all $i \in \{1, \ldots, h\}$, the maximal value of the right hand side of (1) with $\epsilon = 0$ (i.e. the maximal value of h) is obtained taking $s = 0$, $h = x$ and $a_i = 2$ for all i. Since $s + a_1 + \cdots + a_h = r - x + 1$, if $r - x + 1$ is odd we also need either $s \geqslant 1$ or $a_i \geqslant 3$ for some i. Thus no S with $k = 0$ exists if either $r \geqslant 3x$ and $r - x + 1$ is even or $r \geqslant 3x - 1$ and $r - x + 1$ is odd. Equivalently, for the existence part with $k = 0$ it is necessary to assume $r \leqslant 3x - 1$, because if $r = 3x - 1$, then $r - x + 1$ is even.

(b) Here we assume $k > 0$. Hence $S' := (S \cap V) \cup \bigcup_{i=1}^{h} (u^{-1}(Q_i) \cap S) \subsetneq S$. Thus $\dim(\langle S'\rangle) = \sharp(S') - 1$. Hence $\langle S'\rangle$ is the direct sum of the linear subspaces D_0 and M_i, $1 \leqslant i \leqslant h$, while the sum $D_0 + \cdots + D_h$ is a direct sum, i.e. in (1) we take $\epsilon = 1$. Since C is a rational normal curve of $\mathbb{P}^{r-x+1}$, any $r - x + 2$ of its points are linearly independent. Since $\ell_V(S \backslash S \cap V) = Q_1 + \cdots + Q_{h+k}$ and $s + 2h + k \leqslant r - x + 1$, the set $\ell_V(S \backslash S \cap V) = Q_1 + \cdots + Q_{h+k}$ is linearly independent. Hence $Q_{h+1}, \ldots, Q_{h+k}$ give k independent conditions to the linear system $|\mathcal{I}_{D_0 \cup D_1 \cup \cdots \cup D_h}(1)|$. Thus S is linearly independent, contradiction.

(c) Here we assume $r \leqslant 3x - 2 + \eta$, where $\eta = 0$ if $r - x$ is even and $\eta = 1$ if $r - x$ is odd. Here we make a construction which proves the "if" part of the lemma.

(c1) Here we also assume $r - x + 1$ even. Fix a linear subspace $W \subseteq V$ such that $\dim(W) = (r - x + 1)/2 - 2$. W exists, because $r \leqslant 3x - 1$, i.e. $(r - x + 1)/2 - 2 \leqslant x - 2$. Fix $(r - x + 1)/2$ general points $O_1, \ldots, O_{(r-x+1)/2} \in W$. For each $i \in \{1, \ldots, (r - x + 2)/2\}$ take a general line $D_i \subset Y$ containing O_i. Take a general $S_i \subset D_i \backslash \{O_i\}$ such that $\sharp(S_i) = 2$. Set $S := S_1 \cup \cdots \cup S_{(r-x+1)/2}$. Since $\dim(\langle D_1 \cup \cdots \cup D_{(r-x+1)/2}\rangle) = (r - x + 1)/2 - 2 + (r - x + 1)/2 = \sharp(S) - 2$ and each S_i is general in D_i, we get $S \in \Gamma_0$.

(c2) Now assume $r - x + 1$ odd. Hence $r \geqslant x + 2$. Fix a linear subspace $W \subseteq V$ such that $\dim(W) = (r - x)/2 - 1$. W exists, because $r \leqslant 3x - 2$, i.e. $(r - x)/2 - 1 \leqslant x - 2$. Fix $(r - x)/2 + 1$ general points $O_0, O_1, \ldots, O_{(r-x)/2} \in W$. For each $i \in \{1, \ldots, (r - x)/2\}$ take a general line $D_i \subset Y$ containing O_i. Take a general $S_i \subset D_i \backslash \{O_i\}$ such that $\sharp(S_i) = 2$. Set $S := \{O_0\} \cup S_1 \cup \cdot \cup S_{(r-x)/2}$. Since O_0 is general in W and each S_i is general in D_i, we get $S \in \Gamma_1$. This construction proves the "if" part of the lemma. $\square$

Proposition 2. *Fix integers $r > x \geqslant 2$ and $y \in \{r-x, r-x+2\}$. Let $Y \subset \mathbb{P}^r$ be an x-dimensional cone over the rational normal curve of $\mathbb{P}^{r-x+1}$. Let $\Gamma(y)$ be the set of all $S \subset Y$ such that $\sharp(S) = y$, $\dim(\langle S\rangle) = y-2$ and $\dim(\langle S'\rangle) = \sharp(S')-1$ for all $S' \subsetneqq S$. Then $\Gamma(y) \neq \emptyset$ if and only if $y \leqslant 2x-2+\eta$, where $\eta = 0$ if $r-x$ is even and $\eta = 1$ if $r-x$ is odd.*

Proof. We modify the proof of Proposition 1 in the following way. We have $r-x \equiv y \pmod 2$. For the non-existence part we use (1) with $\epsilon := r-x+1-y+\alpha$ and $\alpha = 0$ if the linear subspaces $D_0, D_1, \ldots, D_h$ are linearly dependent, $\alpha = 1$ otherwise. For the existence part we take $s = 0$ if $r-x$ is even and $s = 1$ if $r-x$ is odd. If $r-x$ is even, then we take $W \subseteq V$ such that $\dim(W) = y/2-2$. Hence we need $y/2-2 \leqslant x-2$. We take $y/2$ general points $O_i \in W$, $1 \leqslant i \leqslant y/2+2$. If $r-x$ is odd we take $W \subseteq V$ such that $\dim(W) = (y+1)/2-2$. Hence we need $(y+1)/2-2 \leqslant x-2$. We take $(y-1)/2+1$ general points $O_i \in W$, $0 \leqslant i \leqslant (y-1)/2$. We use step (c) of the proof of Proposition 1 with these new data. □

Proposition 3. *Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate subvariety such that $\deg(Y) = r-x+1$, where $x := \dim(Y)$. There is no $S \subset Y$ such that $\sharp(S) = r-x+1$, $\dim(\langle S\rangle) = r-x-1$, and every proper subset of S is linearly independent if and only if Y is in the following list:*

(i) *$x = r-1$, i.e. Y is a quadric hypersurface;*
(ii) *$x = 1$, i.e. Y is a rational normal curve;*
(iii) *$x \geqslant 2$, Y is a cone over a rational normal curve and $r \geqslant 3x-1+\eta$, where $\eta = 0$ if $r-x$ is even and $\eta = 1$ if $r-x$ is odd.*

Proof. Any two points of $\mathbb{P}^r$ are linearly independent. Thus S does not exists if $r-x+1 = 2$, i.e. if Y is a quadric hypersurface. If Y is a rational normal curve, then every subset of it with cardinality $\leqslant \deg(Y)+1$ is linearly independent. Proposition 1 gives that the one listed in (iii) are exactly the cones over a rational normal curve with no S as in the statement. Hence the "if" part is true.

Now we check the "only if" part. Thus we may assume $r \geqslant x+2$ and that Y is not a cone over a rational normal curve. First assume $x = 2$ and Y smooth. If Y is a Veronese surface (and hence $(r,x) = (5,2)$), then it is sufficient to take 4 points in a conic $C \subset Y$. Now assume that Y is a Hirzebruch surface. There is an integer e such that $0 \leqslant e \leqslant (r-1)/2$, $e \equiv r-1 \pmod 2$ and Y is isomorphic to the Hirzebruch surface F_e, i.e. to the rational ruled surface with a section h of the ruling with self-intersection $-e$ (see [8, V.2.17]). We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}^{\oplus 2}$ and we may take as a basis of $\mathrm{Pic}(F_e)$ the section h and a fiber f of the corresponding ruling. Thus $h^2 = -e, h\cdot f = 1$ and $f^2 = 0$. First assume $r \geqslant e+5$, i.e $(r+e-5)/2 \geqslant e$. The embedding $j: F_e \hookrightarrow \mathbb{P}^r$ with Y as its image is given by the complete linear system $|\mathcal{O}_{F_e}(h + ((r+e-1)/2)f)|$. Since $(r + e - 5)/2 \geqslant e$, the linear system $|\mathcal{O}_{F_e}(h + ((r+e-5)/2)f)|$ is spanned. Thus its general element is a smooth curve and j maps each smooth element of it into a smooth rational curve $D \subset Y$ such that $\dim(\langle D \rangle) = r - 3$ and D is a rational normal curve in $\langle D \rangle$ [8, V.2.17]. Take as B any $r - 1$ points of D. Now assume $r \leqslant e + 4$. Since $r \geqslant x + 2 = 4$, $1 \leqslant e \leqslant (r-1)/2$ and $e \equiv r - 1 \pmod 2$ we get $(r,e) \in \{(4,1),(5,0),(5,2)\}$.

If $(r,e) = (4,1)$, then take as S any 3 points of a line of the ruling of Y.

If $(r,s) = (5,0)$, then use that $j(h)$ is a smooth conic, because $(h + 2f) \cdot h = 2$; take 4 points of $j(h)$.

Now assume $(r,e) = (5,2)$; $j(h)$ is a line; take any $F \in |f|$; $j(h \cup F)$ is a reducible conic and we may take as S the union of two points of $j(h) \backslash j(h \cap F)$ and two points of $j(h) \backslash j(h \cap F)$.

Now assume $x \geqslant 3$. Let $M \subset \mathbb{P}^r$ be a general linear subspace of codimension $x - 2$. Since Y is not a cone over a rational normal curve, the scheme $Y \cap V$ is a smooth minimal degree surface of V. Apply what we just proved for the case $x = 2$ to $Y \cap V$ and then apply $(x - 2)$ times Remark 1. □

Proposition 4. *Fix integers $r > x \geqslant 2$. Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate x-dimensional subvariety such that $\deg(Y) = r - x + 1$. Let Γ be the set of all $S \subset Y$ such that $\sharp(S) = r - x + 2$, $\dim(\langle S \rangle) = r - x$, and $\dim(\langle S' \rangle) = \sharp(S') - 1$ for all $S' \subsetneq S$. Then $\Gamma = \emptyset$ if and only if either Y is a Veronese surface or Y is a cone over a rational normal curve and $r \leqslant 3x - 4 + \eta$, where $\eta = 0$ if $r - x$ is even and $\eta = 1$ if $r - x$ is odd.*

Proof. If Y is a cone over a rational normal curve, then we use the case $y = r - x + 2$ of Proposition 2. If Y is a Veronese surface, then we use Lemma 3. If $x \geqslant 3$ and Y is a cone over a Veronese surface, then we use Lemma 4. In all other cases a general two-dimensional linear section Y_1 of Y is a minimal degree smooth Hirzebruch surface. Apply Lemma 2 to Y_1 and then apply $(x - 2)$ times Remark 1. □

Corollary 1. *Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate subvariety. Set $x := \dim(Y)$. There is no $S \subset Y$ such that $\sharp(S) \leqslant r - x + 2$ and S is linearly dependent if and only if Y is a rational normal curve.*

Proof. Assume that there is no $S \subset Y$ such that $\sharp(S) \leqslant r - x + 2$ and linearly dependent. Lemma 1 gives $\deg(Y) = r - x + 1$. Since any 3 points on a line are linearly dependent, Y cannot contain a line. Thus the list of all minimal degree subvarieties [7] gives that either Y is a rational normal curve or $(x,r) = (2,5)$ and Y is a Veronese surface. Let Y be a Veronese surface. There is a smooth conic $C \subset \mathbb{P}^2$. Any 4 points of C are linearly dependent. Any $r + 1$ points of a rational normal curve of $\mathbb{P}^r$ are linearly independent. □

Lemma 5. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $P \in \mathbb{P}^n \setminus X$. Let $V \subset \mathbb{P}^n$ be a general $(n-m)$-dimensional linear subspace passing through P. Then the scheme $V \cap X$ is a reduced union of deg(X) points and $V \cap X$ spans V.*

Proof. Since $P \notin V$, Bertini's theorem gives that $V \cap X$ is a reduced set of deg(X) points. To see the last assertion we use induction on m. Let $H \subset \mathbb{P}^n$ be a general hyperplane containing P. Look at the exact sequence of coherent sheaves on $\mathbb{P}^n$:

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H}(1) \to 0 \tag{2}$$

Since X is integral, we have $h^0(X, \mathcal{O}_X) = 1$. Thus $h^1(\mathcal{I}_X) = 0$. From (2) we get that H is spanned by the scheme $X \cap H$. If $m = 1$, then we are done, because in this case $H = V$. Now assume $m \geqslant 2$ and that the lemma is true for $(m-1)$-dimensional subvarieties of $\mathbb{P}^{n-1}$. Bertini's theorem gives that $X \cap H$ is integral. The inductive assumption gives that $X \cap V = (X \cap H) \cap V$ spans V. $\square$

The proof of Theorem 2 (resp. Theorem 3) is divided into two steps, called (a) and (b) (resp. six steps called (a), (b), (c), (d), (e) and (f)). These steps concern pairs (X,P) for which the description of the sets $\mathcal{S}(X, P, n-m+1)$ is different. When $\mathcal{S}(X, P, n-m+1) \neq \emptyset$ the step of the proof corresponding to the pair (X,P) gives a more detailed description and/or construction of the set $\mathcal{S}(X, P, n-m+1)$ than the one claimed in the statement of Theorems 2 and 3.

Proof of Theorem 2. If $m = n-1$, then the result is obvious (even part (iiii)), because (in characteristic zero) a general line through any $P \notin X$ intersects the hypersurface X in deg(X) $\geqslant 2$ points. Hence we may assume $n \geqslant m + 2$. Since $P \notin X, \phi := \ell_P | X$ is a morphism. Since $P \notin X$, no line through P is contained in X. Thus ϕ is a finite morphism. Set $a := \deg(\phi)$.

(a) Here we assume deg(Y) $\geqslant n - m + 1$ and prove parts (i) and (iii) in this case. Let W be a general linear subspace of $\mathbb{P}^{n-1}$ with codimension m. Hence the scheme $Y \cap W$ is reduced and $\sharp(Y \cap W) = \deg(Y) \geqslant n - m + 1$. In the set-up of (iii) we have $\ell_P(\Sigma) \cap W = \emptyset$. Since W is general and we work in characteristic zero, the set $Y \cap W$ is in linearly general position in W, i.e. any $B \subseteq Y \cap W$ spans a linear subspace of dimension $\min\{dim(W), \sharp(B) - 1\}$ [2, p. 109]. Since $\sharp(Y \cap W) \geqslant \dim(W) + 2$, there is $B \subseteq Y \cap W$ such that $\sharp(B) = \dim(W) + 2$. Since B is in linearly general position in W, we have $W = \langle B' \rangle$ for every subset B' of B such that $\sharp(B') = \sharp(B) - 1$. Let V be the only codimension m linear subspace of $\mathbb{P}^n$ such that $P \in V$ and $\ell_P(V \setminus \{P\}) = W$. Since W is general, V may be considered as a general codimension m linear subspace of $\mathbb{P}^n$ passing through P. Since $P \notin X$, Lemma 5 gives that $X \cap V$ is a reduced set of deg(X) points and $\langle V \cap X \rangle = V$. In the set-up of (iii) we have $V \cap \Sigma = \emptyset$. Take $S \subseteq X \cap V$ such that

$\sharp(S) = n-m+1, V = \langle S\rangle$ and $\phi| S$ is injective. Since $\langle S\rangle = V$, we have $P \in \langle S\rangle$. Fix any $S' \subsetneq S$. Since $\phi| S'$ is injective and $\phi(S')$ is linearly independent, $P \notin \langle S'\rangle$. Thus $S \in \mathcal{S}(X, P, n-m+1)$.

(b) Now assume $\deg(Y) = n-m$. Since $\deg(X) \geqslant n-m+1$ and $P \notin X$, we get $a \geqslant 2$. Assume the existence of $S \in \mathcal{S}(X, P, n-m+1)$. Since $n \geqslant m+2$, we have $n-m+1 \geqslant 3$. Thus the definition of $\mathcal{S}(X, P, n-m+1)$ shows that P is not contained in a line spanned by two of the points of S. Thus $\sharp(\phi(S)) = n-m+1$. Since $P \in \langle S\rangle$, $\dim(\langle\phi(S)\rangle) = \dim(\langle S\rangle) - 1 = \sharp(\phi(S)) - 2$. Hence $\phi(S)$ is not linearly independent. Since $P \notin \langle S'\rangle$ for any $S' \subsetneq S$, each proper subset of $\ell_P(S)$ is linearly independent. Proposition 4 gives $Y \notin \mathcal{A}(m, n-1)$. Now assume $Y \notin \mathcal{A}(m, n-1)$. Let $\Sigma_1 \subset Y$ be the union of $\phi(\Sigma)$ and all points Q of Y such that $\phi^{-1}(Q)_{red}$ is a unique point. Fix a general $B' \subset Y\backslash\Sigma_1$ such that $\sharp(B') = n-m$. Proposition 4 gives the existence of $B \subset Y$ such that $B \cap \Sigma_1 = \emptyset, B' \subset B, \sharp(B) = n-m+1$, $\dim(\langle B\rangle) = n-m+1$ and every proper subset of B is linearly independent. Set $\{O\} := B\backslash B'$. Take $A' \subset X$ such that $\sharp(A') = n-m$ and $\phi(A') = B'$. Since B' is general in Y, A' may be seen as a general union of $n-m$ points of X. Take $O_1, O_2 \in \phi^{-1}(O)$. We will check that at least one of the sets $A' \cup \{O_1\}$ and $A' \cup \{O_i\}$ belongs to $\mathcal{S}(X, P, n-m+1)$. Since B is linearly dependent and $\ell_P(A' \cup \{O_i\}) = B$, each set $A' \cup \{O_i, P\}$ is linearly dependent. Since A' is linearly independent, we get that $A' \cup \{O_i\} \notin \mathcal{S}(X, P, n-m+1)$ if and only if $O_i \in \langle A'\rangle$. Assume $O_i \in \langle A'\rangle$ for all $i \in \{1,2\}$. Thus $\langle\{O_1, O_2\}\rangle \subseteq \langle A'\rangle$. Since $\ell_P(O_1) = \ell_P(O_2)$ and $O_1 \neq O_2$, we have $P \in \langle\{O_1, O_2\}\rangle$. Thus $\ell_P(A')$ is linearly dependent, contradiction. □

Proof of Theorem 3. Fix a hyperplane $H \subset \mathbb{P}^n$ such that $P \notin H$. We see ℓ_P as a linear projection onto H. Thus we see Y as an integral and non-degenerate subvariety of H. If X is a cone with vertex containing P, then $\dim(Y) = m-1$. If X is not a cone with vertex containing P, then $\dim(Y) = m$ and ψ is generically finite. If X is not a cone we call μ the multiplicity of X at P. We have $a \geqslant 1, \mu \geqslant 1$ and $\deg(X) = \mu + a \cdot \deg(Y)$. We divide the proof into six steps (a)–(f). In step (a) we prove the case $n = m+1$, while in the other steps we assume $n \geqslant m+2$. In step (b) we prove that $\mathcal{S}(X, P, n-m+1) = \emptyset$ in all cases listed in parts (ii)–(iv) of Theorem 3. In step (f) we handle the case in which X is a cone with vertex containing P, while in steps (c)–(e) we assume that X is not a cone with vertex containing P; in step (c) we describe the case $a \geqslant 2$, while in steps (d) and (e) we describe the case $a = 1$.

(a) Here we assume $n = m+1$. Hence we are looking at pairs of distinct points of $X\backslash\{P\}$ spanning a line containing P. For any $D \in \mathcal{F}$ any two points of $D\backslash\{P\}$ give an element of $\mathcal{S}(X, P, 2)$. Call $\mathcal{S}(X, P, 2)'$ the subset of $\mathcal{S}(X, P, 2)$ formed by the sets spanning a line not contained in X. Obviously $\mathcal{S}(X, P, 2)' = \emptyset$ if X is a cone with vertex containing P. Now assume that X is not a cone with vertex containing P. Bezout's theorem gives $\mathcal{S}(X, P, 2)' = \emptyset$ if $a = 1$. If $a \geqslant 2$, then for a general line $T \subset \mathbb{P}^n$ through P there is $S \in \mathcal{S}(X, P, 2)'$ contained in T. Thus $\dim(\mathcal{S}(X, 2, P)') \geqslant n - 1 = m$ and there is $S \in \mathcal{S}(X, P, 2)'$ such that $S \cap \Sigma = \emptyset$ and containing a general point of X.

(b) From now on we assume $n \geqslant m + 2$. We have $\dim(Y) = m - 1$ (resp. $\dim(Y) = m$) if X is (resp. is not) a cone with vertex containing P. Assume $\mathcal{S}(X, P, n - m + 1) \neq \emptyset$ and fix $S \in \mathcal{S}(X, P, n - m + 1)$. Thus S is linearly independent, $P \in \langle S \rangle$ and $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$. Taking $\sharp(S') = 1$ we get $P \notin S$. Since $n - m + 1 \geqslant 3$ and $P \notin \langle S' \rangle$ for all $S' \subset S$ such that $\sharp(S') = 2$, we get that no line spanned by two of the points of S contains P. Thus $\sharp(\ell_P(S)) = \sharp(S) = n - m + 1$. Since S is linearly independent and $P \in \langle S \rangle$, we get $\dim(\langle \ell_P(S) \rangle) = n - m - 1$. Since $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$, any proper subset of $\ell_P(S)$ is linearly independent. Thus $Y \notin \mathcal{A}(m, n - 1)$ if $\dim(Y) = m$, while Y is not as in (iv1) or (iv2) of the statement of Theorem 3 if $\dim(Y) = m - 1$ (Proposition 4 with $x := m - 1$). Thus we proved that $\mathcal{S}(X, P, n - m + 1) = \emptyset$ in all cases claimed in the statement of Theorem 3.

(c) Here we assume that X is not a cone with vertex containing P and $a \geqslant 2$. Our standing assumptions say $n \geqslant m + 2$ and $Y \notin \mathcal{A}(m, n - 1)$. Let $\Sigma_2 \subset Y$ be any finite union of proper subvarieties such that $\sharp(\psi^{-1}(Q)) = a$ for all $Q \in Y \backslash \Sigma_2$. Fix a general $A' \subset X$ such that $\sharp(A') = n - m$. Since A' is general, we have $P \notin \langle A' \rangle, \psi(A') \subset Y \backslash \Sigma_2, \sharp(\psi(A')) = n - m$ and $\psi(A')$ is general in Y. Since $Y \notin \mathcal{A}(m, n - 1)$, there is $B \subset Y$ such that $\sharp(B) = n - m + 1$, $B \cap \Sigma_2 = \emptyset, \dim(\langle B \rangle) = n - m - 1$ and every proper subset of B is linearly independent. Set $\{O\} := B \backslash \psi(A')$. Since $O \notin \Sigma_2$, we may find $O_1, O_2 \in X \backslash \{P\}$ such that $\psi(O_i) = O$, $i = 1, 2$, and $O_1 \neq O_2$. Step (b) of the proof of Theorem 2 gives that at least one of the sets $A' \cup \{O_i\}$, $i = 1, 2$, belongs to $\mathcal{S}(X, P, n - m + 1)$.

(d) Here we assume that X is not a cone with vertex containing P, $a = 1$ and $\deg(Y) \geqslant n - m + 1$. Thus $\dim(Y) = m$ and $\deg(X) = \mu + \deg(Y)$. Let $\Sigma_1 \subsetneqq Y$ be any proper closed subset such that $\psi | (X \backslash (\{P\} \cup \psi^{-1}(\Sigma_1)))$ is a bijection over $Y \backslash \Sigma_1$. Let $V \subset \mathbb{P}^n$ be a general $(n - m)$-dimensional linear subspace containing P. Since V is general and $\dim(\Sigma_1) \leqslant m - 1$, $(V \backslash \{P\}) \cap \psi^{-1}(\Sigma_1) = \emptyset$. Thus the scheme $V \cap X$ is a disjoint union of a connected degree μ scheme V_P with P as its support and the union E of $\deg(Y)$ points such that $\psi(E) = Y \cap W$. Since $Y \cap W$ is in linearly general position in W and ψ is induced by ℓ_P, any $S' \subset E$ such that $\sharp(S') \leqslant n - m$ is linearly independent and $P \notin \langle S' \rangle$. Fix one such S'. Since $\deg(Y) \geqslant n - m + 1$, there is $Q \in Y \cap W \backslash \psi(S')$. Take $O \in V \cap X \backslash \{P\}$ such that $\psi(O) = Q$. Since $\psi(S' \cup \{O\})$ is linearly dependent and $\ell_P(O) \notin \ell_P(S')$, we get that either $P \in \langle S' \cup \{O\} \rangle$ or $O \in \langle S' \rangle$. Since any proper subsets of $\psi(A) \cup \{Q\}$ is linearly independent, in the former case we get $S' \cup \{O\} \in \mathcal{S}(X, P, n - m + 1)$. Now assume $O \in \langle S' \rangle$. Since W is general, S' may be sees as a general subset of X with cardinality $n - m$. Hence $\langle S' \rangle \cap X = S'$ [5, Proposition 2.6], contradicting the assumption $O \in \langle S' \rangle$.

(e) Assume that X is not a cone with vertex containing P, $a = 1, \deg(Y) = n - m$. By part (b) we may assume $Y \notin \mathcal{A}(m, n-1)$. Assume $\mathcal{S}(X, P, n-m+1) \neq \emptyset$ and fix $S \in \mathcal{S}(X, P, n-m+1)$. If $\langle S \rangle \cap X$ has a connected component supported by Q, then this component has degree at least μ. Since $\deg(X) = \mu + n - m$, $\sharp(S) = n - m + 1$ and $P \notin S$, Bezout's theorem shows that the scheme $V \cap \langle S \rangle$ has positive dimension. Hence $(X, P) \in \mathcal{B}(m, n)$.

(f) Here we assume that X is a cone with vertex containing P. In this case $Y = X \cap H$ is a basis of the cone X and $\deg(X) = \deg(Y)$. By part (b) we may assume that $Y \notin \mathcal{A}(m-1, n-1)$. Since a general fiber of ψ contains at least two points, the proof of step (c) gives the non-emptiness of $\mathcal{S}(X, P, n-m+1)$. $\square$

Remark 2. Take the set-up of Theorem 3. Fix any proper closed subset $\Sigma \subsetneq X$ and a general $A \subset X$ such that $\sharp(A) = n - m$. First assume $n = m + 1$. In step (a) of the proof of Theorem 3 we proved that if $a \geqslant 2$ there is $S \in \mathcal{S}(X, P, 2)$ containing the point A and disjoint from Σ. Now assume $n \geqslant m + 2$ and $\deg(X) \geqslant n - m + 1$. In steps (c) and (d) of the proof of Theorem 3 we obtained $\dim(\mathcal{S}(X, P, n-m+1)) \geqslant m(n-m)$ and the existence $S \in \mathcal{S}(X, P, n-m+1)$ such that $S \cap \Sigma = \emptyset$ and $A \subset S$. The condition $S \cap \Sigma = \emptyset$ (for arbitrary Σ) is not satisfied in the case $n = m + 1$ and $a = 1$, unless X is a cone with vertex containing P.

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