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Submersions of generic submanifolds of a Kaehler manifold

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Abstract. Kobayashi has shown that for the submersion $\pi : M \to B$ of a $\boldsymbol{CR}$-submanifold of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B, B is necessarily a Kaehler manifold. Since generic submanifolds are more general class of $\boldsymbol{CR}$-submanifolds, in this present article we study the submersions of generic submanifolds $\boldsymbol{M}$ of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $\boldsymbol{B}$ and prove that $\boldsymbol{B}$ is necessarily a Kaehler manifold in this case too. We also obtain the decomposition theorems for such submersions and derive the relation between the holomorphic sectional curvatures of $\overline{M}$ restricted to $\mathcal{D}$ and that of $\boldsymbol{B}$. Also the geometry of fibres is discussed.

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1. Introduction

Let $\overline{M}$ be an almost Hermitian manifold with almost complex structure J and M a Riemannian manifold isometrically immersed in $\overline{M}$. We note that submanifolds of a Kaehler manifold are determined by the behavior of tangent bundle of the submanifold under the action of the almost complex structure of the ambient manifold. A submanifold M is called holomorphic (complex) if $J(T_p(M)) \subset T_p(M)$, for every $p \in M$, where $T_p(M)$ denotes the tangent space to M at the point p. M is called totally real if $J(T_p(M)) \subset T_p^{\perp}(M)$, for every $p \in M$, where $T_p^{\perp}(M)$ denotes the normal space to M at the point p. As a generalization of holomorphic and totally real submanifolds, CR-submanifolds were introduced by Bejancu [1]. A CR-submanifold M of an almost

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Hermitian manifold $\overline{M}$ with an almost complex structure J requires two orthogonal complementry distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ defined on M such that $\mathcal{D}$ is invariant under J and $\mathcal{D}^{\perp}$ is totally real (cf. [1,2]). There is yet another generalization of CR-submanifolds known as generic submanifolds [3]. These submanifolds are defined by relaxing the condition on the complementary distribution of holomorphic distribution. Let M be a real submanifold of an almost Hermitian manifold $\overline{M}$, and let $D_p = T_pM \cap JT_pM$ be the maximal holomorphic subspace of $T_p(M)$. If $\mathcal{D} : p \to \mathcal{D}_p$ defines a smooth holomorphic distribution M, then M is called a generic submanifold of $\overline{M}$. The complementary distribution $\mathcal{D}^{\perp}$ of $\mathcal{D}$ is called purely real distribution on M. A generic submanifold is a CR-submanifold if the purely real distribution on M is totally real. A purely real distribution $\mathcal{D}^{\perp}$ on a generic submanifold M is called proper if it is not totally real. A generic submanifold is called proper if purely real distribution is proper.

On the other hand the study of the Riemannian submersion $\pi : M \to B$ of a Riemannian manifold M onto a Riemannian manifold B was initiated by O'Neill [7]. A submersion π naturally gives rise to two distributions on M called the horizontal and vertical distributions respectively, of which the vertical distribution is always integrable giving rise to the fibers of the submersion which are closed submanifolds of M.

For a CR-submanifold M of a Kaehler manifold $\overline{M}$, the distribution $\mathcal{D}^{\perp}$ is integrable [2]. Kobayashi [6] observed the similarity between the total space of submersion $\pi : M \to B$ and the CR-submanifold M of a Kaehler manifold $\overline{M}$ in terms of the distributions. Thus he considered submersion $\pi : M \to B$ of a CR-submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B such that the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ of M become respectively the horizontal and vertical distributions required by the submersion π and π restricted to $\mathcal{D}$ become an isometry which preserves the complex structures that is $J' o \pi_* = \pi_* o J$ on $\mathcal{D}$ where J and J' are the almost complex structures of $\overline{M}$ and B respectively. He has shown that under this situation B is necessarily a Kaehler manifold and obtained the relation between holomorphic sectional curvatures of $\overline{M}$ restricted to $\mathcal{D}$ and that of B. Further this study has been extended by Deshmukh et al. [4], in which they obtained the relations between the Ricci curvatures and the scalar curvatures of a Kaehler manifold and the base manifold.

To deal with the similar question for the generic submanifold of a Kaehler manifold, one has the difficulty that the distribution $\mathcal{D}^{\perp}$ for generic submanifold of a Kaehler manifold is not necessarily integrable to match the requirement of the submersion. To overcome this difficulty we consider the submersion $\pi : M \to B$ of generic submanifolds M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B with the assumption that $\mathcal{D}^{\perp}$ is integrable. In the present paper, we study the submersions of generic submanifolds M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B with integrable purely real distribution $\mathcal{D}^{\perp}$ and prove that for the submersion of a generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B, B is necessarily a Kaehler manifold and obtain the decomposition theorems for the generic submanifold M. Also we have obtained the relation between the holomorphic sectional curvatures of $\overline{M}$ restricted to $\mathcal{D}$ and that of B.

2. Preliminaries

In this section we give terminology and notations used throughout this paper. We recall the notion of an almost complex structure and some necessary facts and formulae from the theory of almost Hermitian manifolds and their submanifolds.

An almost complex structure on a smooth manifold $\overline{M}$ is a smooth tensor field J of type $(1,1)$ with the property that $J^2 = -I$. A smooth manifold equipped with such an almost complex structure is called an almost complex manifold. An almost complex manifold $(\overline{M}, J)$ endowed with a chosen Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y) \tag{2.1}$$

for all $X, Y \in \Gamma(T\overline{M})$, is called an almost Hermitian manifold.

The Levi-Civita connection $\bar{\nabla}$ of an almost Hermitian manifold $\overline{M}$ can be extended to the whole tensor algebra on $\overline{M}$ and in this way we obtain tensor fields like $(\bar{\nabla}_X J)$ and that

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y, \tag{2.2}$$

for all $X, Y \in \Gamma(T\overline{M})$.

An almost Hermitian manifold $\overline{M}$ is called a Kaehler manifold if

$$(\bar{\nabla}_X J)Y = 0, \tag{2.3}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Let M be a Riemannian manifold isometrically immersed in $\overline{M}$. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

and

$$\bar{\nabla}_X N = -\tilde{\mathcal{A}}_N X + \nabla^{\perp}_X N \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$, where ∇ is the induced Riemannian connection on M, N is a vector field normal to M, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and $\tilde{\mathcal{A}}_N$ is the shape operator.

The second fundamental form and the shape operator are related by the following relation:

$$g(\tilde{\mathcal{A}}_N X, Y) = g(h(X, Y), N), \tag{2.6}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $\mathrm{N} \in \Gamma(\mathrm{T}^{\perp}\mathrm{M})$.

For any vector field X tangent to M, we put

$$JX = PX + FX, \tag{2.7}$$

where PX and FX are the tangential and normal component of JX respectively. Then P is an endomorphism of the tangent bundle TM and F a normal bundle-valued 1-form on M.

For any vector field N normal to M, we put

$$JN = tN + fN, \tag{2.8}$$

where tN and fN are the tangential and normal components of JN respectively. f is an endomorphism of the normal bundle $T^{\perp}M$ and t a tangent bundle valued 1-form on M.

Now, suppose M is a real submanifold of an almost Hermitian manifold $\overline{M}$ with almost complex structure J. Let $\mathcal{D}_p = T_pM \cap JT_pM,\ p \in M$ be the maximal complex subspace of the tangent space T_pM which is contained in $T_p\overline{M}$. If the dimension of $\mathcal{D}_p$ is constant at each point $p \in M$, and it defines a differentiable distribution on M, then M is called a generic submanifold of $\overline{M}$ [3]. We call $\mathcal{D}$ as the holomorphic distribution and the orthogonal complementary distribution $\mathcal{D}^{\perp}$ of $\mathcal{D}$ in TM is purely real distribution which satisfy the following;

$$\mathcal{D} \perp \mathcal{D}^{\perp}, \quad \mathcal{D}^{\perp} \cap J\mathcal{D}^{\perp} = \{0\}.$$

Let v defines a differentiable vector sub-bundle of $T^{\perp}M$ satisfying;

$$T^{\perp}M = F\mathcal{D}^{\perp} \oplus v,\ \ t(T^{\perp}M) = \mathcal{D}^{\perp}.$$

For a generic submanifold M we have

$$P\mathcal{D} = \mathcal{D} \quad \text{and} \quad P\mathcal{D}^{\perp} \subset \mathcal{D}^{\perp}.$$

It is known that the horizontal distribution $\mathcal{D}$ of a generic submanifold M of a Kaehler manifold $\overline{M}$ is integrable if and only if

$$g(h(X, JY), FZ) = g(h(Y, JX), FZ), \tag{2.9}$$

for any $X,\ Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$ and the vertical distribution $\mathcal{D}^{\perp}$ is integrable if and only if

$$\nabla_Z PW - \nabla_W PZ + \tilde{\mathcal{A}}_{FZ}W - \tilde{\mathcal{A}}_{FW}Z \in \mathcal{D}^{\perp}, \tag{2.10}$$

for any vector fields $Z,\ W \in \mathcal{D}^{\perp}$.

A generic submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be a generic product submanifold if it is locally a Riemannian product of the leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$. In this case $\nabla_U X \in \mathcal{D}$, or equivalently $\nabla_U Z \in \mathcal{D}^{\perp}$ for all $U \in \Gamma(TM),\ \ X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Let $\bar{R}$ and R be the curvature tensor corresponding to the connection $\bar{\nabla}$ and ∇ respectively. Then the equation of Gauss is

$$\begin{aligned}\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) &- g(h(X, W), h(Y, Z)) \\ &+ g(h(X, Z), h(Y, W)),\end{aligned} \tag{2.11}$$

for $X,\ Y,\ Z$, and W tangent to M.

For the theory of submersion $\pi : M \to B$ of a Riemannian manifold M onto a Riemannian manifold B we follow B. O'Neill [7] and for the submersion of CR-submanifolds we follow Kobayashi [6].

Let (M, g) and (B, g_B) be two Riemannian manifolds with $\dim(M) = m$, $\dim(B) = n$ and $m > n$. A Riemannian submersion $\pi : M \to B$ is a map of M onto B satisfying the following axioms;

$(S_1)\pi$ has maximal rank,

that is, each derivative map π_* of π is onto and hence, for each $b \in B,\ \pi^{-1}(b)$ is a submanifold of M of dimension $= \dim M - \dim B$. The submanifolds $\pi^{-1}(b)$ are called fibers. A vector field on M is called vertical vector field if it is always tangent to fibres and it is called horizontal if it is always orthogonal to fibres.

The second axiom may now be stated in the following form;

(S_2) The differential π_* preserves the length of the horizontal vectors.

We recall that a vector field X on M is said to be basic, if $X \in \mathcal{D}$ and X is π-related to a vector field on B, *i.e.*, there exists a vector field X_* on B such that $(\pi_* X)_p = X_* \pi(p)$ for every $p \in M$.

We have the following lemma for basic vector fields [7],

Lemma 2.1. *Let X and Y be any basic vector fields on M. Then*

(i) $g(X, Y) = g_B(X_*, Y_*) o\pi$.
(ii) *The horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, that is $\pi_* \mathcal{H}[X, Y] = [X_*, Y_*] o\pi$.*
(iii) $[V, X] \in \mathcal{D}^{\perp}$, *for any $V \in \mathcal{D}^{\perp}$.*
(iv) *$\mathcal{H}(\nabla_X Y)$ is a basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where ∇^* is the Riemannian connection on B.*

Let $\bar{\nabla}$, ∇ and ∇^* denote Riemannian connections on $\overline{M}$, M and B respectively. For the connection ∇^* we define corresponding connection $\tilde{\nabla}^*$ for basic vector fields on M by

$$\tilde{\nabla}^*_X Y = \mathcal{H}(\nabla_X Y). \tag{2.12}$$

Then $\tilde{\nabla}^*_X Y$ is a basic vector field, and by Lemma 2.1, we have

$$\pi_* \left(\tilde{\nabla}^*_X Y \right) = \nabla^*_{X_*} Y_*. \tag{2.13}$$

We define a tensor field C on M by

$$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X, Y) \tag{2.14}$$

for any $X,\ Y \in \mathcal{D}$, where $C(X, Y)$ is the vertical part of $\nabla_X Y$, *i.e.*, $\mathcal{V}(\nabla_X Y) = C(X, Y)$. It has been observed that C is skew symmetric and satisfies

$$C(X, Y) = \frac{1}{2} \mathcal{V}[X, Y]$$

for any $X,\ Y \in \mathcal{D}$. Also for $X \in \mathcal{D}$ and $V \in \mathcal{D}^{\perp}$, we define an operator $\mathcal{A}$ on M by

$$\nabla_X V = \mathcal{V}(\nabla_X V) + \mathcal{A}_X V, \tag{2.15}$$

where $\mathcal{A}_X V$ is the horizontal part of $\nabla_X V$. Since $[V, X] \in \mathcal{D}^{\perp}$ for any basic vector field X and $V \in \mathcal{D}^{\perp}$, we have

$$\mathcal{H}(\nabla_X V) = \mathcal{H}(\nabla_V X) = \mathcal{A}_X V. \tag{2.16}$$

The operator C and $\mathcal{A}$ are related by

$$g(\mathcal{A}_X V, Y) = -g(V, C(X, Y)), X, Y \in \mathcal{D} \quad \text{and} \quad V \in \mathcal{D}^{\perp}. \tag{2.17}$$

The operator C in (2.14) was introduced by Kobayashi [6]. For vertical vector fields we introduced an operator L defined in the following manner;

For U, $V \in \mathcal{D}^{\perp}$, we define L by

$$\nabla_U V = \hat{\nabla}_U V + L(U, V), \tag{2.18}$$

where $\hat{\nabla}_U V = \mathcal{V}(\nabla_U V)$ and $L(U, V) = \mathcal{H}(\nabla_U V)$. For horizontal vector field X and vertical vector field V we set

$$\nabla_V X = \mathcal{H}(\nabla_V X) + \mathcal{T}_V X, \tag{2.19}$$

where $\mathcal{T}_V X = \mathcal{V}(\nabla_V X)$.

Moreover, if X is basic, $[V, X] \in \mathcal{D}^{\perp}$ for $V \in D^{\perp}$ and we obtain

$$\mathcal{H}(\nabla_V X) = \mathcal{H}(\nabla_X V) = \mathcal{A}_X V.$$

Hence for a basic vector field X and $V \in D^{\perp}$ we have

$$\nabla_V X = \mathcal{A}_X V + \mathcal{T}_V X. \tag{2.20}$$

The operators $\mathcal{T}$ and L are related by

$$g(\mathcal{T}_V X, W) = -g(L(V, W), X). \tag{2.21}$$

Let R^* be the curvature tensor corresponding to the connection ∇^* of the base manifold B then R^* and R are related by

$$\begin{aligned} R(X, Y, Z, H) = {} & R^*(X_*, Y_*, Z_*, H_*) + g(C(X, Z), C(Y, H)) \\ & - g(C(Y, Z), C(X, H) + 2g(C(X, Y), C(Z, H)) \end{aligned} \tag{2.22}$$

for the horizontal vector fields X, Y, Z and H on M.

3. Submersion of generic submanifolds

In this section, we define the submersion of the generic submanifold of a Kaehler manifold onto an almost Hermitian manifold and discuss the impact such submersions $\pi : M \to B$ on the geometry of generic submanifold M.

Let M ba a generic submanifold of an almost Hermitian manifold $\overline{M}$ with distribution $\mathcal{D}$ and $\mathcal{D}^{\perp}$ and the normal bundle $T^{\perp}M$. We assume that

(i) $\mathcal{D}^{\perp}$ is the kernel of π_* that is $\pi_*(\mathcal{D}^{\perp}) = \{0\}$.
(ii) $\pi_*(\mathcal{D}_p) = T_{\pi(p)}B$ is a complex isometry, where $p \in M$ and $T_{\pi(p)}B$ is the tangent space of B at $\pi(p)$.

Now we have the following lemma;

Lemma 3.1. *Let $\pi : M \to B$ be a submersion of generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B. Then for horizontal vector fields X, Y we have*

$$C(X, JY) = PC(X, Y) + th(X, Y),$$
$$h(X, JY) = FC(X, Y) + fh(X, Y).$$

Proof. Since $\overline{M}$ is a Kaehler manifold, we have

$$\bar{\nabla}_X JY = J\,\bar{\nabla}_X Y$$

for all $X, Y \in \mathcal{D}$

On using (2.4), (2.7), (2.12) and (2.14), we have

$$\begin{aligned}\tilde{\nabla}^*_X JY + C(X, JY) + h(X, JY) &= J\big(\tilde{\nabla}^*_X Y + C(X, Y)\big) + Jh(X, Y)\\ &= J\tilde{\nabla}^*_X Y + PC(X, Y) + FC(X, Y)\\ &\quad + th(X, Y) + fh(X, Y).\end{aligned} \tag{3.1}$$

Comparing the horizontal, vertical and normal parts, we get

$$\tilde{\nabla}^*_X JY = J\tilde{\nabla}^*_X Y, \tag{3.2}$$
$$C(X, JY) = PC(X, Y) + th(X, Y), \tag{3.3}$$
$$h(X, JY) = FC(X, Y) + fh(X, Y) \tag{3.4}$$

and hence the result. □

From above lemma, we have the following corollary;

Corollary 3.1. *Let $\pi : M \to B$ be a submersion of generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B. Then*

$$C(X, JY) + h(X, JY) = JC(X, Y) + Jh(X, Y)$$

for all $X, Y \in \mathcal{D}$.

Lemma 3.2. *Let M be a generic submanifold of a Kaehler manifold $\overline{M}$ and $\pi : M \to B$ be a submersion from generic submanifold M onto an almost Hermitian manifold B. Then for any vertical vector fields V, W we have the following:*

$$L(V, PW) - \mathcal{H}(\tilde{\mathcal{A}}_{FW} V) = JL(V, W) \tag{3.5}$$
$$\hat{\nabla}_V PW - \mathcal{V}(\tilde{\mathcal{A}}_{FW} V) = P\,\hat{\nabla}_V W + th(V, W) \tag{3.6}$$
$$h(V, PW) + \nabla^{\perp}_V FW = F\,\hat{\nabla}_V W + fh(V, W). \tag{3.7}$$

Proof. Since $\overline{M}$ is a Kaehler manifold, we have $\bar{\nabla}_V JW = J\,\bar{\nabla}_V W$, for any $V, W \in D^{\perp}$. By using (2.7), we get

$$\bar{\nabla}_V PW + \bar{\nabla}_V FW = J\,\bar{\nabla}_V W.$$

On using Gauss and Weingarten formulae, we have

$$\nabla_V PW + h(V, PW) + (-\tilde{\mathcal{A}}_{FW} V) + \nabla^{\perp}_V FW = J(\nabla_V W + h(V, W)). \tag{3.8}$$

Further on using Eqs. (2.7), (2.8), and (2.18) in (3.8), we get

$$
\begin{aligned}
&L(V,PW) + \hat{\nabla}_V PW + h(V,PW) - \mathcal{H}(\tilde{\mathcal{A}}_{FW}V) - \mathcal{V}(\tilde{\mathcal{A}}_{FW}V) + \nabla^{\perp}_V FW \\
&\quad = J(\hat{\nabla}_V W + L(V,W) + h(V,W)) \\
&\quad = P\hat{\nabla}_V W + F\hat{\nabla}_V W + JL(V,W) + th(V,W) + fh(V,W). \qquad (3.9)
\end{aligned}
$$

Comparing the horizontal, vertical and normal parts in (3.9), we have

$$
\begin{aligned}
&L(V,PW) - \mathcal{H}(\tilde{\mathcal{A}}_{FW}V) = JL(V,W), \\
&\hat{\nabla}_V PW - \mathcal{V}(\tilde{\mathcal{A}}_{FW}V) = P\hat{\nabla}_V W + th(V,W), \\
&h(V,PW) + \nabla^{\perp}_V FW = F\hat{\nabla}_V W + fh(V,W),
\end{aligned}
$$

which completes the proof. □

Proposition 3.1. *Let M be a generic submanifold of a Kaehler manifold $\overline{M}$ and $\pi : M \to B$ be a submersion from generic submanifold M onto an almost Hermitian manifold B. Then*

$$\mathcal{A}_{JX}V = J\mathcal{A}_X V, \quad \text{for any} \quad X \in \mathcal{D} \text{ and } V \in \mathcal{D}^{\perp}.$$

Proof. Let X be a basic vector field, Y in $\mathcal{D}$ and $V \in \mathcal{D}^{\perp}$, we have

$$
\begin{aligned}
g(\mathcal{A}_{JX}V, Y) &= g(\mathcal{H}\nabla_{JX}V, Y) = g(\nabla_{JX}V, Y) = g([JX, V], Y) + g(\nabla_V JX, Y) \\
&= g(\nabla_V JX, Y) = g(\bar{\nabla}_V JX, Y) = g(J\bar{\nabla}_V X, Y) = -g(\bar{\nabla}_V X, JY) \\
&= -g(\nabla_V X, JY) = -g(\mathcal{A}_X V, JY) = g(J\mathcal{A}_X V, Y)
\end{aligned}
$$

Non-degeneracy of g gives the desired result. □

From above proposition and (2.17) it follows that

Proposition 3.2. *Let M be a generic submanifold of a Kaehler manifold $\overline{M}$ and $\pi : M \to B$ be a submersion from generic submanifold M onto an almost Hermitian manifold B. Then*

$$C(JX, JY) = C(X, Y), \text{ for all } X, Y \in \mathcal{D}.$$

As a consequence of the above result, we have the following:

Corollary 3.2. *For horizontal vector fields X and Y, we have*

$$C(X, JY) = -C(JX, Y).$$

Definition 3.1. A generic submanifold M is said to be mixed totally geodesic, if $h(X, V) = 0$, for any $X \in \mathcal{D}$ and $V \in \mathcal{D}^{\perp}$.

Proposition 3.3. *A generic submanifold M of a Kaehler manifold $\overline{M}$ is mixed totally geodesic if and only if $\tilde{\mathcal{A}}_N V \in \mathcal{D}^{\perp}$ (respectively $\tilde{\mathcal{A}}_N X \in \mathcal{D}$), for $V \in \mathcal{D}^{\perp}$ (respectively $X \in \mathcal{D}$) and $N \in \Gamma(T^{\perp}M)$.*

Proof. Let M be a mixed totally geodesic, then by (2.6), we have

$$g(\tilde{\mathcal{A}}_N V, X) = g(h(X, V), N) = 0$$

for any $X \in \mathcal{D}$ and $V \in \mathcal{D}^{\perp}$, which implies that

$$\tilde{\mathcal{A}}_N V \in \mathcal{D}^{\perp} \text{ for } V \in \mathcal{D}^{\perp}.$$

Similarly, we can prove that $\tilde{\mathcal{A}}_N X \in \mathcal{D}$ for $X \in \mathcal{D}$.

Conversely, suppose that $\tilde{\mathcal{A}}_N V \in \mathcal{D}^{\perp}$ for any $V \in \mathcal{D}^{\perp}$ and $N \in (T^{\perp}M)$. Then for any $X \in \mathcal{D}$,

$$g(\tilde{\mathcal{A}}_N V, X) = 0.$$

Again, by using (2.6), we have

$$g(h(X, Y), N) = 0 \text{ for } N \in \Gamma(T^{\perp}M). \tag{3.10}$$

Since $h(X, V) \in \Gamma(T^{\perp}M)$, from (3.10) it follows that $h(X, V) = 0$, i.e., M is mixed totally geodesic. Which completes the proof. □

Now, we have

Theorem 3.1. *Let M be a generic submanifold of a Kaehler manifold $\overline{M}$ and $\pi : M \to B$ be a submersion from generic submanifold M onto an almost Hermitian manifold B. Then B is a Kaehler manifold.*

Proof. From (3.2), for any basic vector fields X and Y, we have

$$\tilde{\nabla}^*_X JY = J\tilde{\nabla}^*_X Y.$$

Operating π_* on the above equation to project it down on B and using Lemma 2.1, we get

$$\nabla^*_{X_*} J' Y_* = J' \nabla^*_{X_*} Y_*,$$
$$\left(\nabla^*_{X_*} J'\right) Y_* = 0$$

for any vector fields $X_*,\ Y_* \in TB$, where $\pi_* X = X_*$ and $\pi_* Y = Y_*$ and J' is the almost complex structure on B. This proves that B is a Kaehler manifold. □

Now, we recall that on a Riemannian manifold M, a distribution S is said to be parallel if $\nabla_X Y \in S$, where ∇ is the Riemannian connection on M. From the definition of Riemannian submersion $\pi : M \to B$ of a Riemannian manifold M onto a Riemannian manifold B, it follows that the vertical distribution is always integrable and its integral manifold are the fibers [7], which are closed submanifolds of M. If in addition, $\mathcal{D}^{\perp}$ is parallel, then we have

Theorem 3.2. *Let M be a generic submanifold of a Kaehler manifold $\overline{M}$ and $\pi : M \to B$ be a submersion from M onto an almost Hermitian manifold B. If $\mathcal{D}$ is integrable and $\mathcal{D}^{\perp}$ is parallel, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is an invariant submanifold and M_2 is a purely real submanifold of $\overline{M}$.*

Let $\bar{\mathcal{H}}$ and $\mathcal{H}^*$ denote the holomorphic sectional curvatures of $\overline{M}$ and B respectively. In order to compare the holomorphic sectional curvatures of $\overline{M}$ with that of B, we calculate the bisectional curvature. For this, we set $Z = JW$, $Y = JX$ in (2.11) and (2.22) and get

$$\begin{aligned}\bar{R}(W, JW, X, JX) &= R(W, JW, X, JX) + g(h(X, JW), h(JX, W)) \\ &\quad - g(h(JX, JW), h(X, W)) \qquad (3.11)\\ R(W, JW, X, JX) &= R^*(W_*, J'W_*, X_*, J'X_*) - g(C(JX, JW), C(X, W)) \\ &\quad + g(C(X, JW), C(JX, W)) + 2g(C(X, JX), C(JW, W)) \qquad (3.12)\end{aligned}$$

for any basic vector fields X, Y, Z, and W on M.

From (3.11) and (3.12), we have

$$\begin{aligned}\bar{R}(W, JW, X, JX) &= R^*(W_*, J'W_*, X_*, J'X_*) + g(h(X, JW), h(JX, W)) \\ &\quad - g(h(JX, JW), h(X, W)) - g(C(JX, JW), C(X, W)) \\ &\quad + g(C(X, JW), C(JX, W)) \\ &\quad + 2g(C(X, JX), C(W, JW)). \end{aligned} \tag{3.13}$$

From the above equation, we have the following theorem.

Theorem 3.3. *Let $\overline{M}$ be a Kaehler manifold and M be a generic submanifold of $\overline{M}$ with $\mathcal{D}$ integrable. Let B be an almost Hermitian manifold and $\pi : M \to B$ be a submersion then the bisectional curvatures K and K^* of $\overline{M}$ and B respectively satisfy*

$$\bar{K}(W, X) = K^*(W_*, X_*) + \|h(X, JW)\|^2 + \|h(X, W)\|^2$$

for any X, W in $\mathcal{D}$.

Proof. Since $\mathcal{D}$ is integrable, so we have

$$h(X, JW) = h(JX, W), \text{ for } X, W \in \mathcal{D}. \tag{3.14}$$

Also, we have

$$C(X, Y) = 0, \text{ for any } X, Y \in \mathcal{D}. \tag{3.15}$$

Using the relation (3.14) and (3.15) in (3.13), we get the result. □

In order to compare the holomorphic sectional curvatures of $\overline{M}$ and B, we have the following theorem whose proof follows from Theorem 3.3.

Theorem 3.4. *Let π be a submersion from a generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B. If the horizontal distribution $\mathcal{D}$ is integrable, then the holomorphic sectional curvatures of $\overline{M}$ and B satisfy*

$$\bar{\mathcal{H}}(X) = \mathcal{H}^*(X_*) + \|h(X, JX)\|^2 + \|h(X, X)\|^2 \tag{3.16}$$

for any $X \in \mathcal{D}$.

From above result, we have

Corollary 3.3. *Let π be a submersion from generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B. If the horizontal distribution $\mathcal{D}$ is integrable, then the holomorphic sectional curvature $\bar{\mathcal{H}}$ and $\mathcal{H}^*$ of $\overline{M}$ and B respectively, are equal if and only if M is $\mathcal{D}$-totally geodesic.*

4. Submersions with totally geodesic fibres

In this section we discuss the submersion of generic submanifold of a Kaehler manifold onto an almost Hermitian manifold with totally geodesic fibres and we assume that $v = 0$.

Proposition 4.1. *Let π be a submersion from a generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B and assume that $v = 0$. If the fibers are totally geodesic submanifolds of M then M is mixed totally geodesic, i.e., $h(X, V) = 0$, for any $X \in \mathcal{D}$ and $V \in \mathcal{D}^{\perp}$.*

Proof. Since fibers are totally geodesic, *i.e.*, $L(V, W) = 0$ for any $V, W \in \mathcal{D}^{\perp}$, then by (3.5), we have

$$\mathcal{H}(\tilde{\mathcal{A}}_{FW}V) = 0,$$

which implies that

$$\tilde{\mathcal{A}}_{FW}V \in \mathcal{D}^{\perp}.$$

Now for any $X \in \mathcal{D}$, using (2.6), we get

$$\begin{aligned} 0 &= g(\tilde{\mathcal{A}}_{FW}V, X), \\ &= g(h(V, X), FW). \end{aligned}$$

Therefore by the non-degeneracy of g we get the result. $\square$

Remark 4.1. The converse of the above result is also true for the submersion of CR-submanifold of a Kaehler manifold [4].

For the endomorphism $P : TM \to TM$, we have

$$(\bar{\nabla}_E P)F = \nabla_E PF - P(\nabla_E F) \tag{4.1}$$

for any vector fields E and F tangent to M. The endomorphism P is said to be parallel, if $\bar{\nabla} P = 0$ or $(\bar{\nabla}_E P)F = 0$, for any vector E, F tangent to M.

Using (2.18), for any $V, W \in D^{\perp}$, we have

$$(\bar{\nabla}_V P)W = \hat{\nabla}_V PW + L(V, PW) - P(\hat{\nabla}_V W) - PL(V, W). \tag{4.2}$$

Now, if we suppose that the fibers are totally geodesic then above equation gets the form

$$(\bar{\nabla}_V P)W = \hat{\nabla}_V PW - P\,\hat{\nabla}_V W. \tag{4.3}$$

Further, if P is parallel, then (4.3) yields

$$\hat{\nabla}_V PW = P\,\hat{\nabla}_V W. \tag{4.4}$$

If we now consider that the fibers are totally geodesic and P is parallel, then by using (3.5) and (3.6) of Lemma 3.2, we have

$$\tilde{\mathcal{A}}_{FW}V = -th(V, W).$$

Since $v = 0$, therefore $fh(V, W) = 0$ and it shows that

$$\tilde{\mathcal{A}}_{FW}V = -Jh(V, W). \tag{4.5}$$

Proposition 4.2. *Let π be a submersion from a generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B with totally geodesic fibers. Then*

$$\bar{R}(X, V, Y, W) = -g((\nabla_V C)(X, Y), W) + g(\mathcal{A}_X V, \mathcal{A}_Y W) - g(h(X, Y), h(V, W))$$

for any $X, Y \in \mathcal{D}$ and $V, W \in \mathcal{D}^\perp$.

Proof. From (1.29) [5] we have

$$\begin{aligned} R(V, X, Y, W) = {} & g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W) \\ & + g(\mathcal{A}_X V, \mathcal{A}_Y W) - g(\mathcal{T}_V X, \mathcal{T}_W Y). \end{aligned} \tag{4.6}$$

Now using (4.6) in (2.11), we get

$$\begin{aligned} \bar{R}(X, V, Y, W) = {} & -g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W) - g(\mathcal{A}_X V, \mathcal{A}_Y W) \\ & - g(\mathcal{T}_V X, \mathcal{T}_W Y) - g(h(X, W), h(V, Y)) + g(h(X, Y), h(V, W)). \end{aligned}$$

Since the fibers are totally geodesic, then

$$g(\mathcal{T}_V X, \mathcal{T}_W Y) = -g(X, L(V, \mathcal{V}\nabla_W Y)) = 0.$$

Also, by Proposition 4.1, M is mixed totally geodesic *i.e.*, $h(X, V) = 0$ for any $X \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$. Hence we get the result. □

As an immediate consequence of Proposition 4.3 we have

Corollary 4.1. *Let π be a submersion from generic submanifold M of a Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold B. If the fibers are totally geodesic submanifolds of M then for unit vectors $X \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$*

$$\bar{K}(X \wedge V) = g(h(X, X), h(V, V)) - \|\mathcal{A}_X V\|^2.$$

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