

Original article

Statistical convergence of order α in probability

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Abstract. In this paper the ideas of different types of convergence of a sequence of random variables in probability, namely, statistical convergence of order α in probability, strong p -Cesàro summability of order α in probability, lacunary statistical convergence or S_θ -convergence of order α in probability, and N_θ -convergence of order α in probability have been introduced and their certain basic properties have been studied.

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1. INTRODUCTION AND BACKGROUND

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [14] and Steinhaus [31] as follows: If \mathbb{N} denotes the set of natural numbers and $K \subset \mathbb{N}$, then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K is defined by

$$\bar{d}(K) = \limsup_{n \rightarrow \infty} \frac{K(1, n)}{n} \quad \text{and} \quad \underline{d}(K) = \liminf_{n \rightarrow \infty} \frac{K(1, n)}{n}.$$

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If $\bar{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists, and it is denoted simply by

$$d(K) = \lim_{n \rightarrow \infty} \frac{K(1, n)}{n}.$$

A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be statistically convergent to a real number x if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ has natural density zero and we write $x_n \xrightarrow{S} x$. Statistical convergence has turned out to be one of the most active areas of research in summability theory after the work of Fridy [16] and Šalát [28]. Over the years a lot of work have been done to generalize this notion of statistical convergence and to introduce new summability methods related to it. Some of the most important concepts introduced are : lacunary statistical convergence by Fridy & Orhan [18] (for more results on this convergence see the paper of Li [24]), \mathcal{I} -convergence by Kostyrko et al. [23], (see [1, 8, 9, 11, 29] for recent advances and more references on this convergence), statistical convergence of order α by Bhunia et al. [2] (statistical convergence of order α was also independently introduced by Colak [4], more investigations in this direction and more applications can be found in [5]), lacunary statistical convergence of order α by Sengöl & Et. M [30], pointwise and uniform statistical convergence of order α for sequences of functions by Cinar et al. [3], λ -statistical convergence of order α of sequences of function by Et. M et al. [13], \mathcal{I} -statistical and \mathcal{I} -lacunary statistical convergence of order α by Savas & Das [10], open covers and selection principles by Das [6, 7]. The notion of statistical convergence has applications in different fields of mathematics: in number theory by Erdős & Tenenbaum [12], in statistics and probability theory by Fridy & Khan [17] and Ghosal [20–22], in approximation theory by Gadjiev & Orhan [19], in Hopfield neural network by Martinez et al. [25], in optimization by Pehlivan & Mamedov [26].

In particular in probability theory, if for each positive integer n , a random variable X_n is defined on a given event space S (same for each n) with respect to a given class of events Δ and a probability function $P : \Delta \rightarrow \mathbb{R}$ (where \mathbb{R} denotes the set of real numbers) then we say that $X_1, X_2, X_3, \dots, X_n, \dots$ is a sequence of random variables and as in analysis we denote this sequence by $\{X_n\}_{n \in \mathbb{N}}$.

From the practical point of view the discussion of a random variable X is highly significant if it is known that there exists a real constant c for which $P(|X - c| < \epsilon) \simeq 1$, where $\epsilon > 0$ is sufficiently small, that is, it is nearly certain that values of X lie in a very small neighbourhood of c .

For a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$, each X_n may not have the above property but it may happen that the aforementioned property (with respect to a real constant c) becomes more and more distinguishable as n gradually increases and the question of existence of such a real constant c can be answered by a concept of convergence in probability of the sequence $\{X_n\}_{n \in \mathbb{N}}$.

In this short paper we shall limit our discussion to four types of convergence of a sequence of random variables, namely,

- (i) statistical convergence of order α in probability,
- (ii) strong p -Cesàro summability of order α in probability,
- (iii) lacunary statistical convergence or S_θ -convergence of order α in probability,
- (iv) N_θ -convergence of order α in probability.

Our main aim in this paper is to establish some important theorems related to the modes of convergence (i)–(iv), which effectively extend and improve all the existing results in

this direction [2,4,18,20,24,27]. We also intend to establish the relation between these four summability notions. It is important to note that the methods of proofs and in particular the examples are not analogous to the real case.

2. STATISTICAL CONVERGENCE OF ORDER α IN PROBABILITY

We first recall the definition of statistical convergence of order α of a sequence of real numbers from [2,4] as follows:

Definition 2.1. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be statistically convergent of order α (where $0 < \alpha \leq 1$) to a real number x if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ has α -natural density zero, i.e

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0$$

and we write $x_n \xrightarrow{S^\alpha} x$.

Now we would like to introduce the definition of statistical convergence of order α in probability for a sequence of random variables as follows.

Definition 2.2. Let (S, Δ, P) be a probability space and $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function $P : \Delta \rightarrow \mathbb{R}$. Then the sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to be statistically convergent of order α (where $0 < \alpha \leq 1$) in probability to a random variable X (where $X : S \rightarrow \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0$$

or equivalently

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{k \leq n : 1 - P(|X_k - X| < \varepsilon) \geq \delta\}| = 0.$$

In this case we write $PS^\alpha - \lim P(|X_k - X| \geq \varepsilon) = 0$ or $PS^\alpha - \lim P(|X_k - X| < \varepsilon) = 1$ or by $X_n \xrightarrow{PS^\alpha} X$. The class of all sequences of random variables which are statistically convergent of order α in probability is denoted simply by PS^α .

Theorem 2.1. *If a sequence of constants $x_n \xrightarrow{S^\alpha} x$ then, regarding a constant as a random variable having one point distribution at that point, we may also write $x_n \xrightarrow{PS^\alpha} x$.*

Proof. Let $\varepsilon > 0$ be any arbitrarily small positive real number. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0.$$

Now let $\delta > 0$. So the set $\{k \in \mathbb{N} : P(|x_k - x| \geq \varepsilon) \geq \delta\} \subseteq K$ where $K = \{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}$. This shows that $x_n \xrightarrow{PS^\alpha} x$.

The following example shows that in general the converse of [Theorem 2.1](#) is not true. It also shows that there is a sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables which is statistically convergent in probability to a random variable X but it is not statistically convergent of order α in probability for $0 < \alpha < 1$.

Example 2.1. Let $\frac{r}{s}$ be a rational number between α and β . Let the probability density function of X_n be given by,

If $n = [m^{\frac{s}{r}}]$ let,

$$f_n(x) = \begin{cases} 1 & \text{where } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $n \neq [m^{\frac{s}{r}}]$ define,

$$f_n(x) = \begin{cases} \frac{nx^{n-1}}{2^n} & \text{where } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $0 < \varepsilon, \delta < 1$.

Then

$$P(|X_n - 2| \geq \varepsilon) = \begin{cases} 1 & \text{if } n = [m^{\frac{s}{r}}] \text{ for some } m \in \mathbb{N} \\ \left(1 - \frac{\varepsilon}{2}\right)^n & \text{if } n \neq [m^{\frac{s}{r}}] \text{ for any } m \in \mathbb{N}. \end{cases}$$

Consequently we have the inequality,

$$\lim_{n \rightarrow \infty} \left\{ \frac{n^{\frac{r}{s}} - 1}{n^\alpha} \right\} \leq \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{k \leq n : P(|X_k - 2| \geq \varepsilon) \geq \delta\}|$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n^\beta} |\{k \leq n : P(|X_k - 2| \geq \varepsilon) \geq \delta\}| \leq \lim_{n \rightarrow \infty} \frac{1}{n^\beta} \{n^{\frac{r}{s}} + 1\} + \lim_{n \rightarrow \infty} \frac{c}{n^\beta}$$

(where c is a fixed finite positive integer). This shows that X_n is statistically convergent of order β in probability to 2 but is not statistically convergent of order α in probability to 2 for any $\alpha < \beta$ and this is not the usual statistical convergence of order α of real numbers. So the converse of [Theorem 2.1](#) is not true. Also by taking $\beta = 1$, we see that $X_n \xrightarrow{PS} 2$ but $\{X_n\}_{n \in \mathbb{N}}$ is not statistically convergent of order α in probability to 2 for $0 < \alpha < 1$.

Theorem 2.2 (Elementary Properties). (i) If $X_n \xrightarrow{PS^\alpha} X$ and $X_n \xrightarrow{PS^\beta} Y$ then $P\{X = Y\} = 1$ for any α, β where $0 < \alpha, \beta \leq 1$

(ii) If $X_n \xrightarrow{PS^\alpha} X$ and $Y_n \xrightarrow{PS^\beta} Y$ then $X_n + Y_n \xrightarrow{PS^{\max\{\alpha, \beta\}}} X + Y$ for any α, β where $0 < \alpha, \beta \leq 1$

(iii) If $X_n \xrightarrow{PS^\alpha} X$ then $cX_n \xrightarrow{PS^\beta} cX$ where $0 < \alpha < \beta \leq 1$.

(iv) Let $0 < \alpha \leq \beta \leq 1$. Then $PS^\alpha \subset PS^\beta$ and this inclusion is strict whenever $\alpha < \beta$.

Proof. (i) Without loss of generality we assume $\beta \leq \alpha$. If possible let $P\{X = Y\} \neq 1$. Then there exists two positive real numbers ε and δ such that $P(|X - Y| \geq \varepsilon) = \delta > 0$.

Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n^\alpha} &\leq \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : P \left(|X_k - X| \geq \frac{\varepsilon}{2} \right) \geq \frac{\delta}{2} \right\} \right| \\ &+ \lim_{n \rightarrow \infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : P \left(|X_k - Y| \geq \frac{\varepsilon}{2} \right) \geq \frac{\delta}{2} \right\} \right| \end{aligned}$$

which is impossible because the left hand limit is not 0 whereas the right hand limit is 0. So $P\{X = Y\} = 1$.

The proofs of (ii), (iii) are straightforward and are therefore omitted.

(iv) The first part is obvious. The inclusion is proper as can be seen from [Example 2.1](#).

Remark 2.1. In Theorem 2 [2], we get, if $\alpha < \beta$ then $m_0^\alpha \subset m_0^\beta$ (i.e., statistical convergence of order $\alpha \Rightarrow$ statistical convergence of order β) and this inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$. But the above [Theorem 2.2\(iv\)](#) and [Example 2.1](#) shows that the inclusion is strict for any $\alpha < \beta$ (i.e., α, β may satisfy the inequality $\frac{1}{k+1} < \alpha < \beta < \frac{1}{k}$ for any $k \in \mathbb{N}$).

Definition 2.3. Let (S, Δ, P) be a probability space and $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function $P : \Delta \rightarrow \mathbb{R}$. A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be strong p -Cesàro summable of order α (where $0 < \alpha \leq 1$ and $p > 0$ is any fixed positive real number) in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \sum_{k=1}^n \{P(|X_k - X| \geq \varepsilon)\}^p = 0.$$

In this case we write $X_n \xrightarrow{PW_p^\alpha} X$. The class of all sequences of random variables which are strong p -Cesàro summable of order α in probability is denoted simply by PW_p^α .

Theorem 2.3. (i) Let $0 < \alpha \leq \beta \leq 1$. Then $PW_p^\alpha \subset PW_p^\beta$. This inclusion is strict whenever $\alpha < \beta$.

(ii) Let $0 < \alpha \leq 1$ and $0 < p < q < \infty$. Then $PW_q^\alpha \subset PW_p^\alpha$.

Proof. (i) The first part of this theorem is straightforward and hence omitted. For the second part we will give an example to show that there is a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ which is strong p -Cesàro summable of order β in probability to a random variable X but is not strong p -Cesàro summable of order α in probability for any $\alpha < \beta$.

Let $\frac{r}{s}$ be a rational number between α and β . We consider a sequence of random variables:

$$X_n \in \begin{cases} \{-1, 1\} & \text{with probability } \frac{1}{2}, \text{ if } n = [m^{\frac{s}{r}}] \text{ for some } m \in \mathbb{N} \\ \{0, 1\} & \text{with probability } P(X_n = 0) = 1 - \frac{1}{\sqrt[r]{n^2}} \\ & \text{and } P(X_n = 1) = \frac{1}{\sqrt[r]{n^2}}, \text{ if } n \neq [m^{\frac{s}{r}}] \text{ for any } m \in \mathbb{N}. \end{cases}$$

For $0 < \varepsilon < 1$ we get the inequality,

$$\lim_{n \rightarrow \infty} \frac{n^{\frac{r}{s}} - 1}{n^\alpha} \leq \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \sum_{k=1}^n \{P(|X_k - 0| \geq \varepsilon)\}^p$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n^\beta} \sum_{k=1}^n \{P(|X_k - 0| \geq \varepsilon)\}^p \leq \lim_{n \rightarrow \infty} \left[\frac{n^{\frac{r}{s}} + 1}{n^\beta} + \frac{1}{n^\beta} \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \right) \right].$$

This shows that $X_n \xrightarrow{PW_p^\beta} 0$ but $\{X_n\}_{n \in \mathbb{N}}$ is not strong p -Cesàro summable of order α in probability to 0.

(ii) The proof is straightforward.

Theorem 2.4. Let $0 < \alpha \leq \beta \leq 1$. Then $PW_p^\alpha \subset PS^\beta$.

Proof. For any $\varepsilon, \delta > 0$,

$$\begin{aligned} & \sum_{k=1}^n \{P(|X_k - X| \geq \varepsilon)\}^p \\ & \geq \sum_{\substack{k=1 \\ P(|X_k - X| \geq \varepsilon) \geq \delta}}^n \{P(|X_k - X| \geq \varepsilon)\}^p \geq \delta^p |\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}|. \\ & \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \sum_{k=1}^n \{P(|X_k - X| \geq \varepsilon)\}^p \\ & \geq \lim_{n \rightarrow \infty} \frac{1}{n^\beta} |\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \cdot \delta^p. \end{aligned}$$

This proves the theorem.

Note 2.1. If a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is strong p -Cesàro summable of order α in probability to X then it is statistically convergent of order α in probability to X i.e $PW_p^\alpha \subset PS^\alpha$.

But the converse of [Theorem 2.4](#) (or [Note 2.1](#)) is not generally true as can be seen from the following example.

Example 2.2. Let a sequence of random variable $\{X_n\}_{n \in \mathbb{N}}$ be defined by,

$$X_n \in \begin{cases} \{-1, 1\} & \text{with probability } \frac{1}{2}, \quad \text{if } n = m^m \text{ for some } m \in \mathbb{N} \\ \{0, 1\} & \text{with probability } P(X_n = 0) = 1 - \frac{1}{\sqrt[2^p]{n}}, P(X_n = 1) = \frac{1}{\sqrt[2^p]{n}}, \\ & \text{if } n \neq m^m \text{ for any } m \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon < 1$ be given.

Then

$$P(|X_n - 0| \geq \varepsilon) = \begin{cases} 1 & \text{if } n = m^m \text{ for some } m \in \mathbb{N} \\ \frac{1}{2\sqrt[n]{n}} & \text{if } n \neq m^m \text{ for any } m \in \mathbb{N}. \end{cases}$$

This implies $X_n \xrightarrow{PS^\alpha} 0$ for each $0 < \alpha \leq 1$. Now let $H = \{n \in \mathbb{N} : n \neq m^m \text{ for any } m \in \mathbb{N}\}$.

$$\begin{aligned} \text{Then } \frac{1}{n^\alpha} \sum_{k=1}^n \{P(|X_k - 0| \geq \varepsilon)\}^p &= \frac{1}{n^\alpha} \sum_{\substack{k=1 \\ k \in H}}^n \{P(|X_k - 0| \geq \varepsilon)\}^p \\ &\quad + \frac{1}{n^\alpha} \sum_{\substack{k=1 \\ k \notin H}}^n \{P(|X_k - 0| \geq \varepsilon)\}^p \\ &= \frac{1}{n^\alpha} \sum_{\substack{k=1 \\ k \in H}}^n \frac{1}{\sqrt{k}} + \frac{1}{n^\alpha} \sum_{\substack{k=1 \\ k \notin H}}^n 1 > \frac{1}{n^\alpha} \sum_{k=1}^n \frac{1}{\sqrt{k}} \\ &> \frac{1}{n^{\alpha-\frac{1}{2}}} \quad \left(\text{since } \sum_{k=1}^n \frac{1}{\sqrt{k}} > \sqrt{n} \text{ for } n \geq 2 \right). \end{aligned}$$

So X_n is not strong p -Cesàro summable of order α in probability to 0 for $0 < \alpha \leq \frac{1}{2}$.

Theorem 2.5. For $\alpha = 1$, $PW_p^\alpha = PS^\alpha$, in other words $PW_p = PS$.

Proof. That $PW_p \subset PS$ readily follows from [Theorem 2.4](#). For the converse part take $\varepsilon, \delta > 0$. Now

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \{P(|X_k - X| \geq \varepsilon)\}^p &= \frac{1}{n} \sum_{\substack{k=1 \\ P(|X_k - X| \geq \varepsilon) \geq \delta}}^n \{P(|X_k - X| \geq \varepsilon)\}^p \\ &\quad + \frac{1}{n} \sum_{\substack{k=1 \\ P(|X_k - X| \geq \varepsilon) < \delta}}^n \{P(|X_k - X| \geq \varepsilon)\}^p \\ &\leq \frac{1}{n} |\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}| + \frac{1}{n} .n .\delta^p. \end{aligned}$$

This shows that $PS \subset PW_p$. Hence the result follows.

3. LACUNARY STATISTICAL CONVERGENCE OF ORDER α IN PROBABILITY

First of all we would like to recall the definition of lacunary statistical convergence of order α for a sequence of real numbers from [\[30\]](#) as follows:

Definition 3.1. A real sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be lacunary statistically convergent of order α to a real number L if for any $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

where $\theta = \{k_r\}_{r \in \mathbb{N} \cup \{0\}}$ be the lacunary sequence (for definition of lacunary sequence see [18]), $h_r = (k_r - k_{r-1})$, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. In this case we write $x_n \xrightarrow{S_\theta^\alpha} L$.

Now we would like to introduce the definition of lacunary statistical convergence of order α in probability for a sequence of random variables as follows.

Definition 3.2. Let $\theta = \{k_r\}_{r \in \mathbb{N} \cup \{0\}}$ be a lacunary sequence and $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function $P : \Delta \rightarrow \mathbb{R}$. Then the sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to be lacunary statistically convergent or S_θ -convergent of order α (where $0 < \alpha \leq 1$) in probability to a random variable X if for any $\varepsilon, \delta > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0.$$

In this case we write $X_n \xrightarrow{PS_\theta^\alpha} X$. The class of all sequences of random variables which are S_θ -convergent of order α in probability is simply denoted by PS_θ^α .

Definition 3.3. Let $\theta = \{k_r\}_{r \in \mathbb{N} \cup \{0\}}$ be a lacunary sequence and $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function $P : \Delta \rightarrow \mathbb{R}$. The sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to be N_θ -convergent of order α (where $0 < \alpha \leq 1$) in probability to a random variable X if for any $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} P(|X_k - X| \geq \varepsilon) = 0.$$

In this case we write $X_n \xrightarrow{PN_\theta^\alpha} X$. The class of all sequences of random variables which are N_θ -convergent of order α in probability is simply denoted by PN_θ^α .

Theorem 3.1. For a fixed θ , the PS_θ^α -limit and PN_θ^α -limit of a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ are unique.

The proof is omitted.

However the PS_θ -limit of a sequence of random variables for two different lacunary sequences may not be equal as can be seen from the following example.

Example 3.1. Let $G : \mathbb{R}^+ \rightarrow \mathbb{N}$ be defined by,

$$G(x) = n \quad \text{if } n! < x \leq (n + 1)!$$

Let us take two lacunary sequences $\theta_1 = \{(2r)!\}$ and $\theta_2 = \{(2r + 1)!\}$. Let us define a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ by,

$$X_n \in \begin{cases} \{-1, 1\} & \text{with probability } P(X_n = -1) = \frac{1}{n}, P(X_n = 1) = 1 - \frac{1}{n}, \\ & \text{if } G(n) \text{ is even,} \\ \{0, 1\} & \text{with probability } P(X_n = 0) = 1 - \frac{1}{n}, P(X_n = 1) = \frac{1}{n}, \\ & \text{if } G(n) \text{ is odd,} \end{cases}$$

Let $0 < \varepsilon, \delta < 1$. For the lacunary sequence θ_1 ,

$$\begin{aligned} & \frac{1}{h_{r+1}} |\{k \in I_{r+1} : P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \\ &= \frac{1}{h_{r+1}} [\{(2r + 1)! - (2r)!\} + c_1] \rightarrow 0 \quad \text{as } r \rightarrow \infty \end{aligned}$$

where c_1 is a finite positive integer. This shows that $X_n \xrightarrow{PS_{\theta_1}} 0$.

For the lacunary sequence θ_2 ,

$$\begin{aligned} & \frac{1}{h_{r+1}} |\{k \in I_{r+1} : P(|X_k - 1| \geq \varepsilon) \geq \delta\}| \\ & \leq \frac{\{(2r + 2)! - (2r + 1)!\} + c_2}{(2r + 3)! - (2r + 1)!} \rightarrow 0 \quad \text{as } r \rightarrow \infty \end{aligned}$$

where c_2 is a finite positive integer. This shows that $X_n \xrightarrow{PS_{\theta_2}} 1$.

Remark 3.1. As in [Theorem 2.5](#), we can easily establish that $PN_{\theta} = PS_{\theta}$.

Theorem 3.2. Let $0 < \alpha \leq \beta \leq 1$. Then $PN_{\theta}^{\alpha} \subset PS_{\theta}^{\beta}$. This inclusion is strict whenever $\alpha < \beta$.

Proof. Let $\varepsilon, \delta > 0$. Now

$$\begin{aligned} & \sum_{k \in I_r} P(|X_k - X| \geq \varepsilon) \\ & \geq \sum_{\substack{k \in I_r \\ P(|X_k - X| \geq \varepsilon) \geq \delta}} P(|X_k - X| \geq \varepsilon) \geq \delta |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \end{aligned}$$

which implies that,

$$\begin{aligned} \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} P(|X_k - X| \geq \varepsilon) & \geq \frac{\delta}{h_r^{\alpha}} |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \\ & \geq \frac{\delta}{h_r^{\beta}} |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}|. \end{aligned}$$

As $r \rightarrow \infty$ we have, $\frac{1}{h_r^{\beta}} |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \rightarrow 0$. Hence the result follows.

For the second part of this theorem we will construct an example.

Let c be a rational number between α and β . Let a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ be defined by

$$X_n \in \begin{cases} \{-1, 1\} & \text{with probability } \frac{1}{2}, \text{ if } n \text{ is the first } [h_r^c] \text{ integers in the interval } I_r, \\ \{0, 1\} & \text{with probability } P(X_n = 0) = \left(1 - \frac{1}{n}\right) \text{ and } P(X_n = 1) = \frac{1}{n}, \\ & \text{if } n \text{ is other than the first} \\ & [h_r^c] \text{ integers in the interval } I_r. \end{cases}$$

Now let $0 < \varepsilon < 1$ and $\delta < 1$. Then we have,

$$P(|X_n - 0| \geq \varepsilon) = \begin{cases} 1 & \text{if } n \text{ is the first } [h_r^c] \text{ integers in the interval } I_r, \\ \frac{1}{n} & \text{if } n \text{ is other than the first } [h_r^c] \text{ integers in the interval } I_r. \end{cases}$$

Now $\frac{1}{h_r^\beta} |\{k \in I_r : P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{h_r^\beta} \{h_r^c + d\} = \left(\frac{1}{h_r^{\beta-c}} + \frac{d}{h_r^\beta}\right) \rightarrow 0$ as $r \rightarrow \infty$

since $c < \beta \leq 1$ where d is a finite positive integer. So $X_n \xrightarrow{PS_\theta^\beta} 0$.

We also have the inequality

$$\frac{h_r^c - 1}{h_r^\alpha} \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} P(|X_k - 0| \geq \varepsilon)$$

which shows that X_n is not N_θ -convergent of order α in probability to 0 since $0 < \alpha < c$. So the inequality is strict whenever $\alpha < \beta$.

Note 3.1. We show that whenever for bounded sequences, the equality between the two sets PN_θ^α and PS_θ^β may not be true in general when $\alpha < \beta$.

Theorem 3.3. *If $0 < \alpha \leq \beta \leq 1$ then $PS^\alpha \subset PS_\theta^\beta$ if and only if $\liminf q_r > 1$. If $\liminf q_r = 1$ then there exists a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ which is not S_θ -convergent of order β in probability but is statistically convergent of order α in probability for some α, β .*

Proof. Let $\liminf q_r > 1$ and $X_n \xrightarrow{PS^\alpha} X$. Then for each $\delta > 0$ we can find q_r for sufficiently large r such that $q_r \geq (1 + \delta) \Rightarrow \frac{h_r}{k_r} \geq \frac{\delta}{(1+\delta)} \Rightarrow \frac{1}{h_r^\alpha} \leq \frac{1}{k_r^\alpha} \frac{(1+\delta)^\alpha}{\delta^\alpha}$. Let $\varepsilon > 0$.

$$\begin{aligned} \frac{1}{h_r^\beta} |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| &\leq \frac{1}{h_r^\alpha} |\{k \leq k_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \\ &\leq \frac{(1 + \delta)^\alpha}{\delta^\alpha} \frac{1}{k_r^\alpha} |\{k \leq k_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}|. \end{aligned}$$

This shows that $X_n \xrightarrow{PS_\theta^\beta} X$.

Now let $PS^\alpha \subset PS_\theta^\beta$ (where $0 < \alpha \leq \beta \leq 1$). If possible let $\liminf q_r = 1$. From [15] we can find a subsequence $k_{r(j)}$ satisfying $\frac{k_{r(j)}}{k_{r(j)-1}} < (1 + \frac{1}{j})$ and $\frac{k_{r(j)-1}}{k_{r(j-1)}} > j$ where $j \in \mathbb{N}$.

We define a sequence of random variables by

$$X_n \in \begin{cases} \{-1, 1\} & \text{with probability } \frac{1}{2} \text{ if } n \in I_{r(j)} \quad \text{where } j \in \mathbb{N} \\ \{0, 1\} & \text{with probability } P(X_n = 0) = 1 - \frac{1}{n^2} \text{ and } P(X_n = 1) = \frac{1}{n^2} \\ & \text{if } n \notin I_{r(j)} \text{ for any } j \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon < 1$ and $\delta < 1$. Now

$$P(|X_n - 0| \geq \varepsilon) = 1 \quad \text{if } n \in I_{r(j)} \text{ where } j \in \mathbb{N},$$

and

$$P(|X_n - 0| \geq \varepsilon) = \frac{1}{n^2} \quad \text{if } n \notin I_{r(j)} \text{ for any } j \in \mathbb{N}.$$

Now $\frac{1}{h_{r(j)}^\beta} |\{k \in I_{r(j)} : P(|X_n - 0| \geq \varepsilon) \geq \delta\}| = \frac{1}{h_{r(j)}^\beta} h_{r(j)} \rightarrow \infty$ as $r \rightarrow \infty$. But as $\frac{1}{h_{r(j)}^\beta} |\{k \in I_{r(j)} : P(|X_n - 0| \geq \varepsilon) \geq \delta\}|$ is a subsequence of the sequence $\frac{1}{h_r^\beta} |\{k \in I_r : P(|X_n - 0| \geq \varepsilon) \geq \delta\}|$, this shows that X_n is not S_θ -convergent of order β (where $0 < \beta \leq 1$) in probability to 0.

Finally let $\alpha = 1$. If we take t sufficiently large such that $k_{r(j)-1} < t \leq k_{r(j+1)-1}$ then we observe that

$$\begin{aligned} \frac{1}{t} \sum_{k=1}^t P(|X_k - 0| \geq \varepsilon) &\leq \frac{k_{r(j-1)} + h_{r(j)}}{k_{r(j)-1}} + \frac{1}{t} \left\{ 1 + \frac{1}{2^2} + \dots + \frac{1}{t^2} \right\} \\ &\leq \frac{2}{j} + \frac{1}{t} \left\{ 1 + \frac{1}{2^2} + \dots + \frac{1}{t^2} \right\} \rightarrow 0 \text{ if } j, t \rightarrow \infty. \end{aligned}$$

This shows that $X_n \xrightarrow{PS} 0$ and we conclude that $\liminf q_r$ must be > 1 .

Theorem 3.4. Let $X_n \xrightarrow{PS^\alpha} X$ and $X_n \xrightarrow{PS_\theta^\beta} Y$. If $\liminf q_r > 1$ and $0 < \beta \leq \alpha \leq 1$ then $P\{X = Y\} = 1$.

Proof. Let $\varepsilon > 0$ be any small positive real number and if possible let $P(|X - Y| \geq \varepsilon) = \delta > 0$. Now we have the inequality $\{P(|X - Y| \geq \varepsilon) \geq \delta\} \subseteq \{P(|X_n - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\} + \{P(|X_n - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}$.

So $\{k \in I_r : P(|X - Y| \geq \varepsilon) \geq \delta\} \subseteq \{k \in I_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\} \cup \{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}$

$\Rightarrow |\{k \in I_r : P(|X - Y| \geq \varepsilon) \geq \delta\}| \leq |\{k \in I_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| + |\{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}|$

$\Rightarrow |\{k \in I_r : P(|X - Y| \geq \varepsilon) \geq \delta\}| \leq |\{k \leq k_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| + |\{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}|$

$\Rightarrow h_r \leq |\{k \leq k_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| + |\{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}|$

$\Rightarrow \left\{ \frac{h_r}{k_r} \right\}^\alpha \leq \frac{1}{k_r^\alpha} |\{k \leq k_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| + \frac{1}{k_r^\beta} |\{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}|$

$$\begin{aligned} &\Rightarrow \left\{ \frac{h_r}{k_r} \right\}^\alpha \leq \frac{1}{k_r^\alpha} |\{k \leq k_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| + \left\{ \frac{h_r}{k_r} \right\}^\beta \frac{1}{h_r^\beta} |\{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| \\ &\Rightarrow \left\{ 1 - \frac{1}{q_r} \right\}^\alpha \leq \frac{1}{k_r^\alpha} |\{k \leq k_r : P(|X_k - X| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}| + \left\{ 1 - \frac{1}{q_r} \right\}^\beta \frac{1}{h_r^\beta} |\{k \in I_r : P(|X_k - Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\}|. \end{aligned}$$

Taking $r \rightarrow \infty$ on both sides we see that the left hand side does not tend to zero since $\liminf q_r > 1$ but the right hand side tends to zero. This is a contradiction. So we must have $P\{X = Y\} = 1$.

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REFERENCES

- [1] M. Balcerzak, K. Dems, A. KomisarSKI, Statistical convergence and ideal convergence for sequence of functions, *J. Math. Anal. Appl.* 328 (2007) 715–729.
- [2] S. Bhunia, P. Das, S. Pal, Restricting statistical convergence, *Acta Math. Hungar.* 134 (1–2) (2012) 153–161.
- [3] M. Cinar, M. Karakas, M. Et, On pointwise and uniform statistical convergence of order α for sequences of functions, *Fixed Point Theory Appl.* 33 (2013) 11.
- [4] R. Colak, Statistical convergence of order α , in: *Modern Methods in Analysis and Its Applications*, Anamaya Pub, New Delhi, India, 2010, pp. 121–129.
- [5] R. Çolak, A.C. Bektaş, λ -statistical convergence of order α , *Acta Math. Scientia* 31B (3) (2011) 953–959.
- [6] P. Das, Some further results on ideal convergence in topological space, *Topology Appl.* 159 (2012) 2621–2626.
- [7] P. Das, Certain types of open covers and selection principles using ideals, *Houston J. Math.* 139 (2) (2013) 637–650.
- [8] P. Das, S. Ghosal, On \mathcal{I} -Cauchy nets and completeness, *Topology Appl.* 157 (2010) 1152–1156.
- [9] P. Das, S. Ghosal, When \mathcal{I} -Cauchy nets in complete uniform spaces are \mathcal{I} -convergent, *Topology Appl.* 158 (2011) 1529–1533.
- [10] P. Das, E. Savas, On \mathcal{I} -statistical and \mathcal{I} -lacunary statistical convergence of order α , *J. Bull. Irani. Mathcal. Society* (2013) in press.
- [11] P. Das, E. Savas, S. Ghosal, On generalization of certain summability methods using ideals, *Appl. Math. Lett.* 24 (2011) 1509–1514.
- [12] P. Erdős, G. Tenenbaum, Sur les densités de certaines suites d’entiers, *Proc. London. Math. Soc.* 3 (59) (1989) 417–438.
- [13] M. Et, M. Cinar, M. Karakas, On λ -statistical convergence of order α of sequences of function, *J. Inequal. Appl.* 204 (2013) 8.
- [14] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2 (1951) 241–244.
- [15] R.A. Freedman, J.J. Sember, M. Raphael, Some Cesaro-type summability space, *Proc. Lond. Math. Soc.* 37 (3) (1978) 508–520.
- [16] A.J. Fridy, On statistical convergence, *Analysis* 5 (1985) 301–313.
- [17] A.J. Fridy, K.M. Khan, Tauberian theorems via statistical convergence, *J. Math. Anal. Appl.* 228 (1998) 73–95.
- [18] A.J. Fridy, C. Orhan, Lacunary statistical convergence, *Pacific. J. Math.* 160 (1993) 43–51.
- [19] D.A. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* 32 (2002) 129–138.
- [20] S. Ghosal, Statistical convergence of a sequence of random variables and limit theorems, *Appl. Math.* 4 (58) (2013) 423–437.
- [21] S. Ghosal, \mathcal{I} -statistical convergence of a sequence of random variables in probability, *Afrika Matematika*. <http://dx.doi.org/10.1007/s13370-013-0142-x>.
- [22] S. Ghosal, S_λ -statistical convergence of a sequence of random variables, *J. Egypt. Math. Soc.* <http://dx.doi.org/10.1016/j.joems.2014.03.007>.
- [23] P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak, \mathcal{I} -convergence and extremal \mathcal{I} -limit points, *Math. Slovaca* 55 (2005) 443–464.

- [24] J. Li, Lacunary statistical convergence and inclusion properties between lacunary methods, *Int. J. Math. Sci.* 23 (3) (2000) 175–180.
- [25] G.V Martinez, S.G Torrubia, T.C Blanc, A statistical convergence application for the Hopfield networks, *Info. Theo. and Appl.* 15 (1) (2008) 84–88.
- [26] S. Pehlivan, A.M. Mamedov, Statistical cluster points and turnpike, *Optimization* 48 (2000) 93–106.
- [27] K.V. Rohatgi, *An Introduction to Probability Theory and Mathematical Statistics*, Second ed., Wiley Eastern Limited, 2006.
- [28] T. Šalát, On Statistically convergent sequences of real numbers, *Math. Slovaca* 30 (1980) 139–150.
- [29] E. Savas, P. Das, A generalized statistical convergence via ideals, *Appl. Math. Lett.* 24 (2011) 826–830.
- [30] H. Sengöl, M. Et, On lacunary statistical convergence of order α , *Acta Math. Scientia* 34B (2) (2014) 473–482.
- [31] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951) 73–74.