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Some results on Whitney numbers of Dowling lattices

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Abstract. In this paper, we study some properties of Whitney numbers of Dowling lattices and related polynomials. We answer the following question: there is a relation between Stirling and Eulerian polynomials. Can we find a new relation between Dowling polynomials and other polynomials generalizing Eulerian polynomials? In addition, some congruences for the Dowling numbers are given.

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1. Introduction

In 1973, Dowling [10] introduced a class of geometric lattices called the Dowling lattices. These lattices denoted $Q_n(G)$ are indexed by a positive integer n (rank) and a finite group G of order $m \geqslant 1$. The most important example of Dowling lattices is obtained by letting G be the trivial group (e), then $Q_n(e)$ is the geometric lattice of partitions Π_{n+1} of the set $\{0, 1, \ldots, n\}$.

Using Möbius function of a finite partially order set, Dowling gave the characteristic polynomial of $Q_n(G)$

$$P_n(\upsilon; m) = m^n \left(\frac{\upsilon - 1}{m}\right)_n,$$

where $(x)_n$ is the falling factorial defined by $(x)_n = x(x-1)\cdots(x-n+1), (x)_0 = 1$.

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It is well known that the Whitney numbers of the first kind $w_m(n,k)$ are the coefficient of υ^k of the characteristic polynomial $P_n(\upsilon;m)$ of $Q_n(G)$ and Whitney numbers of the second kind $W_m(n,k)$ are the number of elements of corank k of $Q_n(G)$. Dowling proved that the Whitney numbers of Dowling lattices of both kinds satisfy the orthogonality relations and also satisfy the following recursions

$$w_m(n,k) = (1 + m(n-1))w_m(n-1,k) + w_m(n-1,k-1)$$

and

$$W_m(n,k) = (1 + mk)W_m(n-1,k) + W_m(n-1,k-1).$$

In 1996, Benoumhani [2,4] established most properties (generating functions, explicit formulas, recurrence relations, congruences, concavity) of Whitney numbers of Dowling lattices. He also introduced two kinds of polynomials [2,3] related to Whitney numbers of Dowling lattices: the Dowling polynomials $D_m(n,x)$ and Tanny–Dowling polynomials $\mathcal{F}_m(n,x)$. The results reported in the present paper are complementary to those obtained by Benoumhani and make points, especially in Eulerian–Dowling polynomials. More precisely, the question which was asked by Benoumhani in [2,3] is: there is a relation between Stirling and Eulerian polynomials. Can we find a new relation between $\sum_k W_m(n,k)x^k$ and other polynomials generalizing Eulerian polynomials? The answer to the previous question is yes.

The present paper is organized as follows. We first introduce in Section 2, our notations and definitions. Then we present in Section 3 some properties and some combinatorial identities related to the Dowling polynomials and the Tanny–Dowling polynomials. The answer to the previous question is in Section 4. Some congruences for Dowling numbers are presented in Section 5. Finally, the r-Dowling numbers are also considered in Section 6.

2. Definitions and notations

In this section, we introduce some definitions and notations which are useful in the rest of the paper. The (signed) Stirling numbers of the first kind $s(n,k)$ are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k.$$

Thus $s(0,0) = 1$, but $s(n,0) = 0$ for $n \geqslant 1$, and it is also convenient to define $s(n,k) = 0$ if $k < 0$ or $k > n$. The recurrence

$$s(n+1,k) = s(n,k-1) - n\ s(n,k), \tag{1}$$

is well known and easy to see, and we also have the generating function

$$\frac{1}{k!}(\ln(1+x))^k = \sum_{n \geqslant k} s(n,k)\frac{x^n}{n!}. \tag{2}$$

The Stirling numbers of the second kind, denoted $S(n,k)$, appear as coefficients when converting powers to binomial coefficients

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k}.$$

They have a combinatorial interpretation involving set partitions. Specifically, $S(n,k)$, is the number of ways to partition a set of n elements into exactly k nonempty subsets $(0 \leqslant k \leqslant n)$. The Stirling numbers of the second kind can be enumerated by the following recurrence relation

$$S(n+1,k) = kS(n,k) + S(n,k-1),$$

or explicitly

$$S(n,k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

The number of all partitions is the Bell number ϕ_n, thus

$$\phi_n = \sum_{k=0}^{n} S(n,k).$$

The polynomials

$$\phi_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

are called Bell polynomials or exponential polynomials. The first few Bell polynomials are

$$\begin{aligned}
\phi_0(x) &= 1,\\
\phi_1(x) &= x,\\
\phi_2(x) &= x^2 + x,\\
\phi_3(x) &= x^3 + 3x^2 + x,\\
\phi_4(x) &= x^4 + 6x^3 + 7x^2 + x.
\end{aligned}$$

The exponential generating function for the polynomials $\phi_n(x)$ is

$$\sum_{n\geqslant 0}\phi_n(x)\frac{z^n}{n!} = \exp(x(e^z - 1)).$$

Now, if $\omega_n(x)$ and $\phi_n(x)$ are ordinary and exponential generating functions of the sequence $k!S(n,k)$, then (cf. [19])

$$\omega_n(x) = \int_0^{+\infty} \phi_n(\lambda x)e^{-\lambda}d\lambda.$$

The polynomials

$$\omega_n(x) = \sum_{k=0}^{n} k!S(n,k)x^k,$$

are called geometric polynomials [21]. The first few geometric polynomials are

$$\omega_0(x) = 1,$$
$$\omega_1(x) = x,$$
$$\omega_2(x) = 2x^2 + x,$$
$$\omega_3(x) = 6x^3 + 6x^2 + x,$$
$$\omega_4(x) = 24x^4 + 36x^3 + 14x^2 + x.$$

The numbers $\omega_n(1)$ called ordered Bell numbers or Fubini numbers, count the number of ordered partitions of $\{1, 2, \ldots, n\}$.

As Comtet in [9, p. 244], we define the Eulerian polynomials $A_n(x)$ by

$$A_n(x) = \delta_{n,0} + \sum_{k=1}^{n} \left\langle {n \atop k-1} \right\rangle x^k, \tag{3}$$

where $\left\langle {n \atop k} \right\rangle$ are the Eulerian numbers. $\left\langle {n \atop k-1} \right\rangle$ is the number of permutations of length n with exactly k rises (i.e., the number of times it goes from a lower to a higher number, reading left to right).

The first few Eulerian polynomials are

$$A_0(x) = 1,$$
$$A_1(x) = x,$$
$$A_2(x) = x^2 + x,$$
$$A_3(x) = x^3 + 4x^2 + x,$$
$$A_4(x) = x^4 + 11x^3 + 11x^2 + x.$$

Using the Frobenius [11] result

$$A_n(x) = \delta_{n,0} + x\sum_{k=1}^{n} k!S(n,k)(x-1)^{n-k} \tag{4}$$

$$= \sum_{k=0}^{n} k!S(n+1,k+1)(x-1)^{n-k}, \tag{5}$$

we can easily establish the following connection between the Eulerian polynomials and the geometric polynomials

$$A_n(x) = \delta_{n,0} + x(x-1)^n \omega_n\left(\frac{1}{x-1}\right) - x(x-1)^n,$$

or

$$\omega_n(x) = \frac{x^{n+1}}{1+x}\left(A_n\left(\frac{1+x}{x}\right) - \delta_{n,0}\right) + 1. \tag{6}$$

Substituting (3) in (6) we get

$$\omega_n(x) = 1 + \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle (1+x)^k x^{n-k},$$

since $\left\langle {n \atop n} \right\rangle = \delta_{n,0}$, we obtain the relationship between geometric polynomials and Eulerian numbers (see for instance [21, p. 737])

$$\omega_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle (1+x)^k x^{n-k}. \tag{7}$$

It has been shown by Benoumhani that the first and second kind Whitney numbers of Dowling lattices are defined respectively by

$$\sum_{n \geqslant k} w_m(n,k) \frac{z^n}{n!} = \frac{(1+mz)^{-\frac{1}{m}} (\ln(1+mz))^k}{m^k k!}, \tag{8}$$

$$\sum_{n \geqslant 0} W_m(n,k) \frac{z^n}{n!} = \frac{e^z}{m^k k!} (e^{mz}-1)^k, \tag{9}$$

or explicitly by

$$w_m(n,k) = \sum_{i=0}^{n} (-1)^{i-k} \binom{i}{k} m^{n-i} s(n,i), \tag{10}$$

$$W_m(n,k) = \sum_{i=k}^{n} \binom{n}{i} m^{i-k} S(i,k) \tag{11}$$

$$= \frac{1}{m^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (mi+1)^n. \tag{12}$$

For more details on Whitney numbers of Dowling lattices see [2,10].

The Dowling polynomials and Tanny–Dowling polynomials were evidently first introduced by Benoumhani [2,3]. They are usually defined in the following way:

$$D_m(n,x) = \sum_{k=0}^{n} W_m(n,k) x^k \tag{13}$$

$$\mathcal{F}_m(n,x) = \sum_{k=0}^{n} k! W_m(n,k) x^k. \tag{14}$$

It is not difficult to see that

$$\mathcal{F}_m(n,x) = \int_0^{+\infty} D_m(n,\lambda x) e^{-\lambda} d\lambda.$$

3. Some properties of the Dowling polynomials

Theorem 1. *For $m \geqslant 1$, the Whitney numbers of the second kind $W_m(n,k)$ satisfy the recursion*

$$W_{m+1}(n,k) = \frac{1}{(m+1)^k m^{n-k}} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j W_m(j,k), \tag{15}$$

with $W_1(n,k) = S(n+1,k+1)$.

Proof. Expression (12) may be rewritten as

$$\begin{aligned} W_{m+1}(n,k) &= \frac{(m+1)^{n-k}}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \left(i + \frac{1}{m+1}\right)^n \\ &= \frac{(m+1)^n}{(m+1)^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \left(i + \frac{1}{m} - \frac{1}{m(m+1)}\right)^n \\ &= \frac{(m+1)^n}{(m+1)^k} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{k-i}}{k!} \sum_{j=0}^{n} \binom{n}{j} \left(i + \frac{1}{m}\right)^{n-j} \left(\frac{-1}{m(m+1)}\right)^j \\ &= (m+1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{m^k}{(m+1)^k m^{n-j}} \left(-\frac{1}{m(m+1)}\right)^j \frac{1}{m^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (mi+1)^{n-j} \\ &= \frac{1}{(m+1)^k m^{n-k}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (m+1)^{n-j} W_m(n-j,k), \end{aligned}$$

which is the required expression (15). □

Theorem 2. *For* $m \geqslant 1$, *the Dowling polynomials* $D_m(n,x)$ *satisfy the recursion*

$$D_{m+1}(n,x) = \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j D_m\left(j, \frac{mx}{m+1}\right), \tag{16}$$

with $D_1(n,x) = x^{-1}\phi_{n+1}(x)$.

Proof. By using (13) and (15), we obtain

$$D_{m+1}(n,x) = \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j \sum_{k=0}^{n} W_m(j,k) \left(\frac{m}{m+1} x\right)^k,$$

we arrive at the desired result. □

Theorem 3. *For* $m \geqslant 1$, *we have*

$$\mathcal{F}_{m+1}(n,x) = \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j \mathcal{F}_m\left(j, \frac{m}{m+1} x\right), \tag{17}$$

with $\mathcal{F}_1(n,x) = \left(1 + \frac{1}{x}\right)\omega_n(x) - \frac{\delta_{n,0}}{x}$.

Proof. Combining (14) and (15), we easily arrive at the desired result. □

Theorem 4. *For $m \geqslant 1$, the Dowling polynomials $D_m(n,x)$ satisfy*

$$D_m(n,x) = \sum_{i=0}^{n} \binom{n}{i} m^i \phi_i\left(\frac{x}{m}\right). \tag{18}$$

Proof. By using (13) and (11), we get

$$D_m(n,x) = \sum_{i=0}^{n} \binom{n}{i} m^i \sum_{k=0}^{n} S(i,k) \left(\frac{x}{m}\right)^k. \quad \square$$

We note that the identity (18) can be viewed as a binomial transform. Given a sequence α_k, its binomial transform β_k is the sequence defined by

$$\beta_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k, \text{ with inversion } \alpha_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \beta_k.$$

From this observation we obtain

Corollary 1

$$\phi_n\left(\frac{x}{m}\right) = \frac{1}{m^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} D_m(i,x). \tag{19}$$

By substituting $m = 1$ in (18) and (19), we obtain the well-known results.

$$x\phi_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \phi_{i+1}(x), \tag{20}$$

$$\phi_{n+1}(x) = x \sum_{i=0}^{n} \binom{n}{i} \phi_i(x). \tag{21}$$

As the result of Chen [7] for the binomial transform, we have

$$\sum_{k=0}^{l} \binom{l}{k} \binom{n+k}{s} \alpha_{n+k-s} = \sum_{k=0}^{n} \binom{n}{k} nk \binom{l+k}{s} (-1)^{n-k} \beta_{l+k-s}. \tag{22}$$

Substituting $\alpha_k := m^k \phi_k\left(\frac{x}{m}\right), \beta_k := D_m(k,x)$ and $l = s = n$ into (22), we get a curious identity of Simons type (see [5]) which has the interesting property that the binomial coefficients on both sides are the same

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} m^k \phi_k\left(\frac{x}{m}\right) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} D_m(k,x), \tag{23}$$

and for $m = 1$, we have a curious identity for Bell polynomials

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x\phi_k(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \phi_{k+1}(x). \tag{24}$$

Now, setting $m = 1, x := 2x$ in (16) and setting $m = 2, x := 2x$ in (18), we get another curious identity for Bell polynomials

Corollary 2. *The following formula holds true*

$$\sum_{k=0}^{n}\binom{n}{k}2^k x\phi_k(x) = \sum_{k=0}^{n}\binom{n}{k}2^k(-1)^{n-k}\phi_{k+1}(x).$$

Similarly, we obtain

Theorem 5. *For $m \geqslant 1$, we have*

$$\mathcal{F}_m(n,x) = \sum_{i=0}^{n}\binom{n}{i}m^i\omega_i\left(\frac{x}{m}\right), \tag{25}$$

$$\omega_n\left(\frac{x}{m}\right) = \frac{1}{m^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\mathcal{F}_m(i,x), \tag{26}$$

$$x\omega_n(x) = (-1)^{n+1} + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}(x+1)\omega_i(x), \tag{27}$$

$$(1+x)\omega_n(x) = \delta_{n,0} + \sum_{i=0}^{n}\binom{n}{i}x\omega_i(x), \tag{28}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{l+k}{s}(-1)^{n-k}\mathcal{F}_m(l+k-s,x) = \sum_{k=0}^{l}\binom{l}{k}\binom{n+k}{s}m^{n+k-s}\omega_{n+k-s}\left(\frac{x}{m}\right), \tag{29}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x\omega_k(x) = 1 + \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\omega_k(x), \tag{30}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^k x\omega_k(x) = (-1)^n + \sum_{k=0}^{n}\binom{n}{k}2^k(-1)^{n-k}(1+x)\omega_k(x). \tag{31}$$

Proof. The identity (25) can be found in [3], here we give a simple proof. Combining (14) and (11), we get

$$\sum_{k=0}^{n}k!W_m(n,k)x^k = \sum_{i=0}^{n}\binom{n}{i}m^i\sum_{k=0}^{n}k!S(i,k)\left(\frac{x}{m}\right)^k.$$

The relation (26) is the inverse binomial transform of identity (25). As a special case, we get (27) and (28) by using Theorem 3 and setting $m = 1$ in (25) and (26).

By substituting $\alpha_k := m^k\omega_k\left(\frac{x}{m}\right)$and $\beta_k := \mathcal{F}_m(k,x)$ in (22), we get (29). The relation (30) is a special case, by setting $m = 1$ and $n = s = l$ in (29).

Finally, setting $m = 1, x := 2x$ in (17) and setting $m = 2, x := 2x$ in (25), we get (31). This completes the proof. $\square$

The Hankel transform of a sequence α_n is the sequence of Hankel determinants $H_n(\alpha_n)$, where $H_n(\alpha_n) = \det(\alpha_{i+j})_{0\leqslant i,j\leqslant n}$. It is well-known that the Hankel transform of sequences α_n and β_n are equal (see [13]).

In 2000, Suter [20] proved that $H_n(D_m(n,1)) = m^{\binom{n+1}{2}}\prod_{k=1}^{n} k!$, we shall give the following generalization

Corollary 3.

$$H_n(D_m(n,x)) = (xm)^{\binom{n+1}{2}}\prod_{k=1}^{n} k!.$$

Proof. Using the fact that $H_n(D_m(n,x)) = H_n\left(m^n\phi_n\left(\frac{x}{m}\right)\right)$ and $H_n(\phi_n(x)) = x^{\binom{n+1}{2}}\prod_{k=1}^{n} k!$ (cf. [17]). $\square$

4. The Eulerian–Dowling polynomials

In this section, we define the Eulerian–Dowling polynomials and we derive some elementary properties. According to (5), the following definition provides a natural generalization of Eulerian polynomials.

Definition 1. The Eulerian–Dowling polynomials $\mathcal{A}_m(n,x)$ are defined by

$$\mathcal{A}_m(n,x) = \sum_{i=0}^{n} i!W_m(n,i)(x-1)^{n-i} \tag{32}$$

$$= (x-1)^n \mathcal{F}_m\left(n, \frac{1}{x-1}\right). \tag{33}$$

From the above definition, we can rewrite $\mathcal{A}_m(n,x)$ as

$$\mathcal{A}_m(n,x) = \sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n-i}{k} i!W_m(n,i)(-1)^{n-i-k}x^k = \sum_{k=0}^{n}\left(\sum_{i=0}^{n}(-1)^{n-i-k}\binom{n-i}{k} i!W_m(n,i)\right)x^k.$$

Now, we define the Eulerian–Dowling numbers $a_m(n,k)$ by

$$a_m(n,k) = \sum_{i=0}^{n}(-1)^{n-i-k} i!\binom{n-i}{k} W_m(n,i). \tag{34}$$

For $m = 1$, we have

$$a_1(n,k) = \delta_{n,0} + \left\langle {n \atop k-1} \right\rangle.$$

The following elementary properties of the Eulerian–Dowling polynomials are given

Theorem 6. *The exponential generating function for* $\mathcal{A}_m(n,x)$ *is*

$$\sum_{n\geqslant 0} \mathcal{A}_m(n,x)\frac{z^n}{n!} = \frac{m(x-1)e^{(x-1)z}}{m(x-1)+1-e^{m(x-1)z}}.$$

Proof. From (32) and (9)

$$\begin{aligned}\sum_{n\geqslant 0} \mathcal{A}_m(n,x)\frac{z^n}{n!} &= \sum_{i\geqslant 0} i!\frac{1}{(x-1)^i}\sum_{n\geqslant i} W_m(n,i)\frac{(z(x-1))^n}{n!} \\ &= \exp(z(x-1))\sum_{i\geqslant 0}\left(\frac{\exp(zm(x-1))-1}{m(x-1)}\right)^i,\end{aligned}$$

we arrive at the desired result. □

In [2,3], Benoumhani asked for the analogue of (7) for $\mathcal{F}_m(n,x)$. The answer to the previous question is given in the following theorem

Theorem 7

$$\mathcal{F}_m(n,x) = \sum_{k=0}^{n} a_m(n,k)(1+x)^k x^{n-k}. \tag{35}$$

Proof. From (33), we can write $\mathcal{F}_m(n,x)$ as

$$\mathcal{F}_m(n,x) = x^n \mathcal{A}_m\left(n, \frac{1+x}{x}\right) = x^n \sum_{k=0}^{n} a_m(n,k)\left(\frac{1+x}{x}\right)^k,$$

which completes the proof. □

As a special case, we have the well known result

Corollary 4

$$\omega_n(1) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle 2^k.$$

Proof. By setting $m = 1, x = 1$ in (35) and using Theorem 3, we get

$$2\omega_n(1) - \delta_{n,0} = \sum_{k=0}^{n}\left(\delta_{n,0} + \left\langle {n \atop k-1} \right\rangle\right)2^k,$$

from which it follows that

$$\omega_n(1) = \delta_{n,0} + \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle 2^k,$$

since $\left\langle {n \atop n} \right\rangle = \delta_{n,0}$, we obtain the result. $\square$

5. Congruences for Dowling numbers

By using the Gessel method [12], we shall give some congruences for the Dowling numbers. We consider the polynomials $R_{n,k}^{(m)}(t)$ for fixed m, defined by the exponential generating function

$$\sum_{n \geqslant k} R_{n,k}^{(m)}(t) \frac{z^n}{n!} = e^{-tz}(1+mz)^{-\frac{1}{m}} \frac{(\ln(1+mz))^k}{m^k k!}. \tag{36}$$

Theorem 8. *The following explicit representation formula holds true*

$$R_{n,k}^{(m)}(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} w_m(n-j,k) t^j. \tag{37}$$

Here $w_m(n,k)$ are the Whitney numbers of the first kind.

Proof. From the generating function (8) we have

$$\sum_{n \geqslant k} R_{n,k}^{(m)}(t) \frac{z^n}{n!} = \sum_{n \geqslant 0} (-1)^n t^n \frac{z^n}{n!} \sum_{n \geqslant k} w_m(n,k) \frac{z^n}{n!} = \sum_{n \geqslant 0} \frac{z^n}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} w(n-j,k) t^j.$$

Equating the coefficients of $\frac{z^n}{n!}$ we get the result. $\square$

Theorem 9. *The double generating function for $R_{n,k}^{(m)}(t)$ is*

$$\sum_{n,k \geqslant 0} R_{n,k}^{(m)}(t) \frac{z^n}{n!} u^k = e^{-tz}(1+mz)^{\frac{u-1}{m}}. \tag{38}$$

Proof. From (36) and using the fact that $R_{n,k}^{(m)}(t) = 0$ for $k > n$, we get

$$\begin{aligned}\sum_{n \geqslant k} R_{n,k}^{(m)}(t) \frac{z^n}{n!} \sum_k u^k &= (1+mz)^{-\frac{1}{m}} e^{-tz} \sum_k \left(\frac{\ln(1+mz)u}{m} \right)^k \frac{1}{k!} \\ &= (1+mz)^{-\frac{1}{m}} e^{-z} \exp(\ln(1+mz)^{\frac{u}{m}}). \quad \square\end{aligned}$$

Theorem 10. *The* $R_{n,k}^{(m)}(t)$ *satisfy the following recurrence relation*

$$R_{n+1,k}^{(m)}(t) = R_{n,k-1}^{(m)}(t) - ((1+t) + mn)R_{n,k}^{(m)}(t) - mntR_{n-1,k}^{(m)}(t), \tag{39}$$

with initial conditions $R_{0,0}^{(m)}(t) = 1$ *and* $R_{n,k}^{(m)}(t) = 0$ *if* $k > n$ *or* $k < 0$.

Proof. Let $R(u, z)$ be the double generating function (38). Then by differentiation with respect to z we obtain

$$(1 + mz)\frac{d}{dz}R(u, z) = (u - 1 - t(1 + mz))R(u, z),$$

or equivalently

$$\sum_{n,k}\left(R_{n+1,k}^{(m)}(t) + mnR_{n,k}^{(m)}(t)\right)\frac{z^n}{n!}u^k = \sum_{n,k}\left(R_{n,k-1}^{(m)}(t) - (1+t)R_{n,k}^{(m)}(t) - tmnR_{n-1,k}^{(m)}(t)\right)\frac{z^n}{n!}u^k.$$

Comparing the coefficients of $\frac{z^n}{n!}u^k$ on both sides of the above equation, we arrive at the desired result. □

Taking $t = 1$ in (39) and a little computation gives the following table of values : Table 1

Theorem 11.

$$\sum_{k=0}^{n} R_{n,k}^{(m)}(t)D_m(i+k, t) = t^n n! \sum_{j=0}^{i} m^{i-j}\binom{i}{j}S(i-j, n)D_m(j, t) = \begin{cases} n!m^n t^n, & i = n \\ 0, & 0 \leqslant i < n \end{cases}, \tag{40}$$

where $R_{n,k}^{(m)}(t)$ *is defined in* (37).

Proof. Let $f(x)$ be the generating function for the Dowling polynomials, so that

$$f(x) = \exp\left(x + \frac{t}{m}(e^{mx} - 1)\right).$$

Then $f(x)$ satisfies the functional equation

$$f(x + y) = f(x)\exp\left(y + \frac{t}{m}(e^{my} - 1)e^{mx}\right). \tag{41}$$

Table 1 $R_{n,k}^{(m)}(1)$.

$n \setminus k$	0	1	2	3	4
0	1				
1	-2	1			
2	$m+4$	$-m-4$	1		
3	$-2m^2-6m-8$	$2m^2+9m+12$	$-3m-6$	1	
4	$6m^3+19m^2+24m+16$	$-6m^3-30m^2-48m-32$	$11m^2+30m+24$	$-6m-8$	1

Using Taylor's theorem, we have

$$f(x+y)=\sum_{k\geqslant 0}f^{(k)}(x)\frac{y^k}{k!},$$

where $f^{(k)}(x)=\frac{d^k}{dx^k}f(x)$, it follows that

$$f^{(k)}(x)=\sum_{i\geqslant 0}D_m(i+k,t)\frac{x^i}{i!}. \tag{42}$$

Now, set $z=\frac{\exp(my)-1}{m}$ in (41), we get

$$\sum_{k\geqslant 0}f^{(k)}(x)\frac{[\ln(1+mz)]^k}{m^k k!}=f(x)(1+mz)^{\frac{1}{m}}\exp(tze^{mx}).$$

Multiplying both sides by $e^{-tz}(1+mz)^{-\frac{1}{m}}$ we obtain

$$f(x)\exp(tz(e^{mx}-1))=\sum_{k\geqslant 0}f^{(k)}(x)e^{-tz}(1+mz)^{-\frac{1}{m}}\frac{[\ln(1+mz)]^k}{m^k k!},$$

and

$$f(x)\exp(tz(e^{mx}-1))=\sum_{n\geqslant 0}f(x)\frac{t^n z^n}{n!}(e^{mx}-1)^n.$$

Since

$$e^{-tz}(1+mz)^{-\frac{1}{m}}\frac{[\ln(1+mz)]^k}{m^k k!}=\sum_{n\geqslant k}R_{n,k}^{(m)}(t)\frac{z^n}{n!},$$

and

$$\frac{1}{n!}(e^{mx}-1)^n=\sum_{i\geqslant n}m^i S(i,n)\frac{x^i}{i!}. \tag{43}$$

It follows from (42) that

$$\begin{aligned}\sum_{k\geqslant 0}f^{(k)}(x)(1+mz)^{-\frac{1}{m}}e^{-tz}\frac{[\ln(1+mz)]^k}{m^k k!}&=\sum_{k\geqslant 0}f^{(k)}(x)\sum_{n\geqslant 0}R_{n,k}^{(m)}(t)\frac{z^n}{n!}\\&=\sum_{n\geqslant 0}\sum_{i\geqslant 0}\frac{x^i}{i!}\frac{z^n}{n!}\sum_{k=0}^{n}R_{n,k}^{(m)}(t)D_m(i+k,t),\end{aligned}$$

and by (43), we get

$$\begin{aligned}\sum_{n\geqslant 0}t^n z^n f(x)\frac{1}{n!}(e^{mx}-1)^n&=\sum_{n\geqslant 0}t^n z^n\sum_{n\geqslant 0}D_m(n,t)\frac{x^n}{n!}\sum_{i\geqslant n}m^i S(i,n)\frac{x^i}{i!}\\&=\sum_{n\geqslant 0}\sum_{i\geqslant 0}\frac{x^i}{i!}\frac{z^n}{n!}t^n n!\sum_{j=0}^{i}m^{i-j}\binom{i}{j}S(i-j,n)D_m(j,t).\end{aligned}$$

Equating coefficients of $\frac{x^i}{i!}\frac{z^n}{n!}$, we get the results. □

It is clear from (40) that the right-hand side is divisible by $n!$.

Corollary 5. *Let n, i be non-negative integers with $i \leqslant n$, we have*

$$\sum_{k=0}^{n} R_{n,k}^{(m)}(t) D_m(i+k, t) \equiv 0 (\text{mod } n!). \tag{44}$$

Let us give a short list of these congruences by taking $t = 1$ in (44) and using the Table 1.

$$mD_m(i) + mD_m(i+1) + D_m(i+2) \equiv 0 \ (\text{mod } 2),$$

$$(4m^2 - 2)D_m(i) + (2m^2 + 3m)D_m(i+1) + 3mD_m(i+2) + D_m(i+3) \equiv 0 \ (\text{mod } 6).$$

6. *r*-Dowling polynomials

In 1984, Broder [6] generalized the Stirling numbers of the second kind to the so-called r-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r$ as follows: is the number of partitions of $\{1, 2, \ldots, n\}$ into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. They may be defined recursively as follows

$$\begin{aligned} \left\{ {n \atop k} \right\}_r &= 0, \quad n < r, \\ \left\{ {n \atop k} \right\}_r &= \delta_{k,r}, \quad n = r, \\ \left\{ {n \atop k} \right\}_r &= k\left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r, \quad n > r, \end{aligned} \tag{45}$$

where $\delta_{k,r}$ is the Kronecker symbol.

The r-Whitney numbers of both kinds have appeared in [15] as a common generalization of Whitney numbers and r-Stirling numbers. Recently, Choen and Jung [8] have used these numbers to extend earlier results of Benoumhani. They defined the r-Dowling polynomials by means of

$$D_{m,r}(n, x) = \sum_{k=0}^{n} W_{m,r}(n, k) x^k, \tag{46}$$

where $W_{m,r}(n, k)$ is the r-Whitney numbers of the second kind of the Dowling lattices $Q_n(G)$ defined by

$$W_{m,r}(n, k) = \sum_{j=k}^{n} \binom{n}{j} m^{j-k} (r - rm)^{n-j} \left\{ {j+r \atop k+r} \right\}_r, \tag{47}$$

or expressed in terms of the Stirling numbers of the second kind

$$W_{m,r}(n,k) = \sum_{j=k}^{n} \binom{n}{j} m^{j-k} r^{n-j} S(j,k). \tag{48}$$

Note that (46) reduces to the Dowling polynomials by setting $r = 1$ and to the r-Bell polynomials $B_r(n, x)$ by setting $m = 1$. In another recent paper the author [18] has shown the relationship of r-Bell numbers to the Bell numbers by

$$B_r(n, 1) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r = \sum_{k=0}^{r} s(r,k)\phi_{n+k}.$$

Hence we have

$$B_r(n, x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k = \sum_{k=0}^{r} s(r,k)x^{-r}\phi_{n+k}(x). \tag{49}$$

In this section, we show all the results of Section 3 concerning the Dowling polynomials can be extended to r-Dowling polynomials. In particular, the relationship of r-Dowling polynomials to the Bell polynomials.

Theorem 12. *The r-Dowling polynomials may be expressed in terms of the Bell polynomials*

$$D_{m,r}(n, x) = \sum_{j=0}^{n} \binom{n}{j} m^j r^{n-j} \phi_j\left(\frac{x}{m}\right). \tag{50}$$

Proof. By using (46) and (48), we get the result. □

Now we want to generalize (21); setting $m = 1$ in (50), we obtain the well-known result (see [16])

$$B_r(n, x) = \sum_{j=0}^{n} \binom{n}{j} r^{n-j} \phi_j(x).$$

It follow from (49) that

Corollary 6

$$\sum_{k=0}^{n} \binom{n}{k} r^{n-k} x^r \phi_k(x) = \sum_{k=0}^{r} s(r,k)\phi_{n+k}(x). \tag{51}$$

Example 1. In [14], Mansour and Shattuck defined a sequence $(C_n)_{n\geqslant 1}$ with four parameters by means of

$$C_n(a,b,c,d) = abC_{n-1}(a,b,c,d) + cC_{n-1}(a+d,b,c,d),$$

where $C_0(a,b,c,d) = 1$ and they derived some formulas involving C_i and Bell polynomials $\phi_i(x)$ defined by

$$C_n(a,b,c,d) = b^n \sum_{j=0}^{n} a^{n-j} d^j \binom{n}{j} \phi_j\left(\frac{c}{bd}\right).$$

Now, if we assume that d divides a, then we deduce the following explicit formula

$$C_n(a,b,c,d) = \frac{(bd)^{n+a/d}}{c^{a/d}} \sum_{k=0}^{a/d} s(a/d,k)\phi_{n+k}\left(\frac{c}{bd}\right),$$

by setting $x = c/bd$ and $r = a/d$ in (51).

In particular, for $l \geqslant 1$

$$l^n C_n\left(1,1,\frac{1}{l},\frac{1}{l}\right) = \frac{1}{l^n} \sum_{k=0}^{l} s(l,k)\phi_{n+k}.$$

Corollary 7

$$m^n \phi_n\left(\frac{x}{m}\right) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} r^{n-k} D_{m,r}(k,x). \tag{52}$$

To generalize (20), substituting $m = 1$ in (52) and using (49), we get

Corollary 8. *The Bell polynomials satisfy the relation*

$$x^r \phi_n(x) = \sum_{k=0}^{n} \sum_{j=0}^{r} (-1)^{n-k} \binom{n}{k} r^{n-k} s(r,j)\phi_{k+j}(x). \tag{53}$$

We note that the identity (53) can be viewed as the inverse Stirling transform of (see for instance [1])

$$\phi_{n+r}(x) = \sum_{k=0}^{n} \sum_{j=0}^{r} j^{n-k} S(r,j) \binom{n}{k} x^j \phi_k(x).$$

Analogous formulas to (23), (24) can be derived. We omit all proofs.

Theorem 13. *The following results holds true*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\frac{m}{r}\right)^k \phi_k\left(\frac{x}{m}\right) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{1}{r^k} D_{m,r}(k,x), \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^r \frac{\phi_k(x)}{r^k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{1}{r^k} \sum_{j=0}^{r} s(r,j)\phi_{k+j}(x).$$

Theorem 14. *The r-Dowling polynomials have the Hankel transform*

$$H_n\left(\frac{D_{m,r}(n,x)}{r^n}\right) = \left(\frac{mx}{r^2}\right)^{\binom{n+1}{2}} \prod_{k=1}^{n} k!.$$

In particular, $H_n(D_{m,r}(n,x)) = H_n(D_m(n,x)) = (mx)^{\binom{n+1}{2}} \prod_{k=1}^{n} k!$.

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