Arab Journal of Mathematical Sciences

Arab J Math Sci 24 (1) (2018) 70–81

Original article

Some results of the f-biharmonic maps and applications

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Received 16 January 2017; revised 28 September 2017; accepted 8 October 2017
Available online 19 October 2017

Abstract. In this paper, we give some properties of the f-biharmonic maps, in the particular case represented by conformal maps between equidimensional manifolds. We construct a new example of the f-biharmonic maps and we characterize the p-biharmonicity of some particular cases.

Keywords: f-biharmonic map; p-harmonic map; p-biharmonic map; Conformal map

Mathematics Subject Classification: 31B30; 53C25; 58E20; 58E30

1. Introduction

Harmonic maps $\phi: (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds are the critical points of the energy functional

$$E(\phi) = \frac{1}{2}\int_D |d\phi|^2 dv_g,$$

for every compact domain $D \subset M$. The Euler–Lagrange equation associated to $E(\phi)$ is

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ. One can refer to [7–10] for background on harmonic maps. As the generalization of harmonic maps, biharmonic maps are defined as follows. The

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Peer review under responsibility of King Saud University.

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https://doi.org/10.1016/j.ajmsc.2017.10.001

map ϕ: $(M^m, g) \longrightarrow (N^n, h)$ is biharmonic if it is a critical point of the bienergy functional defined by

$$E_2(\phi) = \frac{1}{2}\int_D |\tau(\phi)|^2 dv_g.$$

The first variation formula for the bienergy which is derived in [12] shows that the Euler–Lagrange equation for bienergy is

$$\tau_2(\phi) = -Tr_g\left(\nabla^\phi\right)^2\tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0.$$

We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. Clearly, any harmonic map is biharmonic; therefore, it is interesting to construct non-harmonic biharmonic maps (see [2,4], [5] and [15–17] for examples). In [13], the author studies the f-harmonicity of some special maps from or into a doubly warped product manifold. He gives the conditions for f-harmonicity of projection maps and some characterizations for non-trivial f-harmonicity of the special product maps. Furthermore, he investigated non-trivial f-harmonicity of the product of two harmonic maps. The authors in [18] gave a method to produce biharmonic maps and f-biharmonic maps from given biharmonic maps and they construct many examples of biharmonic and f-biharmonic maps from the standard sphere S^2 and between two such spheres. In [14], the author obtains the first variation formula of the f-bienergy functional; he introduces the f-biharmonic maps, which are the natural generalization of biharmonic maps, and he studies some properties of the f-biharmonic maps. The authors in [6] study a subclass of generalized f-harmonic maps called generalized f-harmonic morphisms which pull back local harmonic functions to local generalized f-harmonic functions. In this paper, we give other constructions of f-biharmonic maps. We first characterize the f-biharmonicity of the conformal maps between equidimensional manifolds (Theorem 2). With this setting we obtain new examples of f-biharmonic maps . As a second step, we will study the p-biharmonicity of the map $\phi : (M^m, g) \longrightarrow (N^n, h)$ (Theorem 3) and we construct an example of p-biharmonic maps.

2. Preliminaries

2.1. *Harmonic and biharmonic maps*

Let $\phi : (M^m, g)(N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler–Lagrange equations:

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ. The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional:

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

Equivalently, ϕ is biharmonic if it satisfies the associated Euler–Lagrange equations:

$$\tau_2(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where ∇^{ϕ} is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on M, then

$$Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi) = \left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^M_{e_i}e_i}\right)\tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ.

2.2. f-biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds and let $f \in C^{\infty}(M)$ be a positive function. The map ϕ is said to be f-harmonic if it is a critical point of the f-energy functional:

$$E_f(\phi) = \frac{1}{2}\int_M f|d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is f-harmonic if it satisfies the associated Euler–Lagrange equations:

$$\tau_f(\phi) = f\tau(\phi) + d\phi(gradf) = f(\tau(\phi) + d\phi(grad\ln f)) = 0,$$

$\tau_f(\phi)$ is called the f-tension field of ϕ. The map ϕ is said to be f-biharmonic if it is a critical point of the f-bienergy functional:

$$E_{2,f}(\phi) = \frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g.$$

Equivalently, ϕ is f-biharmonic if it satisfies the associated Euler–Lagrange equations:

$$\tau_{2,f}(\phi) = f\tau_2(\phi) - (\Delta f)\tau(\phi) - 2\nabla_{gradf}\tau(\phi) = 0, \tag{1}$$

$\tau_{2,f}(\phi)$ is called the f-bitension field of ϕ.

3. The main results of f-biharmonic maps

In the first place, we can simplify the formula given by (1) to get the following remark.

Remark 1. A simple computation gives $\Delta f = f\Delta\ln f + f|grad\ln f|^2$ and $gradf = fgrad\ln f$. Then

$$\tau_{2,f}(\phi) = f\left\{\tau_2(\phi) - \left(\Delta\ln f + |grad\ln f|^2\right)\tau(\phi) - 2\nabla_{grad\ln f}\tau(\phi)\right\}. \tag{2}$$

We deduce that ϕ is f-biharmonic if and only if

$$\tau_2(\phi) - \left(\Delta\ln f + |grad\ln f|^2\right)\tau(\phi) - 2\nabla_{grad\ln f}\tau(\phi) = 0. \tag{3}$$

It is clear that any harmonic map is f-biharmonic. If the map ϕ is biharmonic ($\tau_2(\phi) = 0$), then ϕ is f-biharmonic if and only if

$$\left(\Delta\ln f + |grad\ln f|^2\right)\tau(\phi) + 2\nabla_{grad\ln f}\tau(\phi) = 0.$$

3.1. The case of conformal maps

We study conformal maps between equidimensional manifolds of dimension $n \geq 3$. Note that by the result in [3], any such map can have no critical points and so is a local conformal diffeomorphism. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exists a C^∞ function $\lambda : M \to \mathbb{R}_+^*$ such that for any $X, Y \in \Gamma(TM)$:

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y).$$

The function λ is called the dilation for the map ϕ. The tension field for a conformal map is given by (see [3]):

$$\tau(\phi) = (2-n)d\phi(grad \ln \lambda).$$

Theorem 1 (*[11]*)**.** *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, then the bitension field of ϕ is given by*

$$\tau_2(\phi) = (n-2)\, d\phi\, (H)$$

where

$$H = grad \Delta \ln \lambda - \frac{(n-6)}{2} grad \left(|grad \ln \lambda|^2 \right) + 2Ricci^M \left(grad \ln \lambda \right) - \left(2\left(\Delta \ln \lambda \right) + (n-2) \left| grad \ln \lambda \right|^2 \right) grad \ln \lambda .$$

For any $X \in \Gamma\,(TM)$, the Ricci operator $Ricci^M$ is defined by

$$Ricci^M(X) = Tr_g R^M\,(X, \cdot) \cdot = R^M\,(X, e_i)\, e_i$$

where $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame field on M (we sum over repeated indices).

The f-biharmonicity of the conformal map is given by the following theorem.

Theorem 2. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, then ϕ is f-biharmonic if and only if*

$$\begin{aligned} & grad \Delta \ln \lambda - \frac{(n-6)}{2} grad \left(|grad \ln \lambda|^2 \right) + 2\nabla_{grad \ln f} grad \ln \lambda \\ & - \left(2\left(\Delta \ln \lambda \right) + (n-2) \left| grad \ln \lambda \right|^2 - \Delta \ln f - |grad \ln f|^2 \right) grad \ln \lambda \\ & + 2 |grad \ln \lambda|^2 grad \ln f + 2Ricci^M \left(grad \ln \lambda \right) = 0. \end{aligned} \tag{4}$$

Proof of Theorem 2. By (2), the f-bitension field of ϕ is

$$\tau_{2,f}\,(\phi) = f \left\{ \tau_2\,(\phi) - \left(\Delta \ln f + |grad \ln f|^2 \right) \tau\,(\phi) - 2\nabla_{grad \ln f} \tau\,(\phi) \right\}. \tag{5}$$

Since ϕ is a conformal map, then

$$\tau(\phi) = (2-n)d\phi(grad \ln \lambda)$$

and

$$\tau_2(\phi) = (n-2)\, d\phi\, (H)$$

where

$$H = grad \Delta \ln \lambda - \frac{(n-6)}{2} grad \left(|grad \ln \lambda|^2 \right) + 2Ricci^M \left(grad \ln \lambda \right) - \left(2\left(\Delta \ln \lambda \right) + (n-2) \left| grad \ln \lambda \right|^2 \right) grad \ln \lambda ,$$

which gives us

$$\tau_{2,f}(\phi) = (n-2) f \left\{ d\phi(H) + \left(\Delta \ln f + |grad \ln f|^2\right) d\phi(grad \ln \lambda) + 2\nabla_{grad \ln f} d\phi(grad \ln \lambda)\right\}.$$

For the term $\nabla_{grad \ln f} d\phi(grad \ln \lambda)$, we have (see [3])

$$\begin{aligned}\nabla_{grad \ln f} d\phi(grad \ln \lambda) &= \nabla d\phi(grad \ln \lambda, grad \ln f) + d\phi\left(\nabla_{grad \ln f} grad \ln \lambda\right)\\ &= |grad \ln \lambda|^2 d\phi(grad \ln f) + d\phi\left(\nabla_{grad \ln f} grad \ln \lambda\right).\end{aligned}$$

Then

$$\tau_{2,f}(\phi) = (n-2) f d\phi\left(H + \left(\Delta \ln f + |grad \ln f|^2\right) grad \ln \lambda + 2|grad \ln \lambda|^2 grad \ln f + 2\nabla_{grad \ln f} grad \ln \lambda\right).$$

Finally, we conclude that the f-bitension field of ϕ is given by

$$\tau_{2,f}(\phi) = (n-2) f d\phi(H(\lambda, f))$$

where

$$\begin{aligned}H(\lambda, f) = {} & grad \Delta \ln \lambda - \frac{(n-6)}{2} grad\left(|grad \ln \lambda|^2\right) + 2\nabla_{grad \ln f} grad \ln \lambda\\ & - \left(2(\Delta \ln \lambda) + (n-2)|grad \ln \lambda|^2 - \Delta \ln f - |grad \ln f|^2\right) grad \ln \lambda\\ & + 2|grad \ln \lambda|^2 grad \ln f + 2Ricci^M(grad \ln \lambda).\end{aligned}$$

By considering the fact that $n \geq 3$, we deduce that the conformal map ϕ is f-biharmonic if and only if

$$\begin{aligned}H(\lambda, f) = {} & grad \Delta \ln \lambda - \frac{(n-6)}{2} grad\left(|grad \ln \lambda|^2\right) + 2\nabla_{grad \ln f} grad \ln \lambda\\ & - \left(2(\Delta \ln \lambda) + (n-2)|grad \ln \lambda|^2 - \Delta \ln f - |grad \ln f|^2\right) grad \ln \lambda\\ & + 2|grad \ln \lambda|^2 grad \ln f + 2Ricci^M(grad \ln \lambda) = 0.\end{aligned}$$

This completes the proof of Theorem 2. In the following part, we shall present an example of f-biharmonic maps.

Example 1. Let $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ $(n \geq 3)$ be the inversion defined by $\phi(x) = \frac{x}{|x|^2}$. ϕ is a conformal map with dilation $\lambda = \frac{1}{r^2}$ $(r = |x|)$. We suppose that $\ln f$ is radial $(\ln f = \alpha(r))$. Then by Theorem 2, we deduce that the map $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is f-biharmonic if and only if the function α satisfies the following differential equation

$$\frac{1}{r}\alpha'' + \frac{(n-7)}{r^2}\alpha' + \frac{1}{r}\left(\alpha'\right)^2 - \frac{4(n-4)}{r^3} = 0.$$

Let $\beta = \alpha'$, this equation becomes

$$\frac{1}{r}\beta' + \frac{(n-7)}{r^2}\beta + \frac{1}{r}\beta^2 - \frac{4(n-4)}{r^3} = 0.$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, then $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is f-biharmonic if and only if

$$a^2 + (n-8)a - 4(n-4) = 0.$$

This equation has two solutions $a = 4$ and $a = 4 - n$.

1. For $a = 4$, we obtain $f(r) = Cr^4$ and in this case the inversion $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}$ is f-biharmonic.
2. For $a = 4 - n$, we obtain $f(r) = Cr^{4-n}$; it follows that the inversion $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is f-biharmonic.

As a consequence of Theorem 2, if we suppose that $f = \lambda$, we obtain the following corollary.

Corollary 1. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, then ϕ is λ-biharmonic if and only if*

$$\begin{aligned} &grad \Delta \ln \lambda - \left(\Delta \ln \lambda + (n-5) |grad \ln \lambda|^2\right) grad \ln \lambda \\ &- \frac{(n-8)}{2} grad \left(|grad \ln \lambda|^2\right) + 2Ricci^M (grad \ln \lambda) = 0. \end{aligned}$$

In particular, we prove that the λ-biharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ where the dilation λ is radial ($\ln \lambda = \alpha(r)$, $r = |x|$ and $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$) is equivalent to an ordinary differential equation of the second order. More precisely, we have

Remark 2. Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ when we suppose that $\ln \lambda$ is radial ($\ln \lambda = \alpha(r)$, $r = |x|$ and $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$). A direct calculation gives

$$grad \ln \lambda = \alpha' \frac{\partial}{\partial r},$$

$$|grad \ln \lambda|^2 = \left(\alpha'\right)^2,$$

$$grad \left(|grad \ln \lambda|^2\right) = 2\alpha'\alpha'' \frac{\partial}{\partial r},$$

$$\Delta \ln \lambda = \alpha'' + \frac{n-1}{r}\alpha'$$

and

$$grad (\Delta \ln \lambda) = \left(\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha'\right) \frac{\partial}{\partial r}.$$

Therefore ϕ is λ-biharmonic if and only if the function α satisfies the following differential equation

$$\alpha''' - (n-7)\alpha'\alpha'' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha' - \frac{n-1}{r}\left(\alpha'\right)^2 - (n-5)\left(\alpha'\right)^3 = 0.$$

If we denote $\beta = \alpha'$, the λ-biharmonicity of ϕ is equivalent to the differential equation

$$\beta'' - (n-7)\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{n-1}{r}\beta^2 - (n-5)\beta^3 = 0.$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, we deduce that $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ is λ-biharmonic if and only if a is a solution of the algebraic equation

$$(n-5)a^2 + 6a + 2n - 4 = 0.$$

This equation has real solutions if and only if $n \in \{3, 4, 5, 6\}$.

1. If $n=3$, we find $a=\frac{3+\sqrt{17}}{2}$ or $a=\frac{3-\sqrt{17}}{2}$, so $\lambda=Cr^{\left(\frac{3+\sqrt{13}}{2}\right)}$ or $\lambda=Cr^{\left(\frac{3-\sqrt{13}}{2}\right)}$ $(C\in\mathbb{R}_{+}^{*})$. It follows that any conformal map $\phi:(\mathbb{R}^{3},g)\rightarrow(N^{3},h)$ of dilation $\lambda=Cr^{\left(\frac{3+\sqrt{13}}{2}\right)}$ or $\lambda=Cr^{\left(\frac{3-\sqrt{13}}{2}\right)}$ is λ-biharmonic.
2. If $n=4$, we find $a=3+\sqrt{13}$ or $a=3-\sqrt{13}$, so $\lambda=Cr^{(3+\sqrt{13})}$ or $\lambda=Cr^{(3-\sqrt{13})}$ $(C\in\mathbb{R}_{+}^{*})$. It follows that any conformal map $\phi:(\mathbb{R}^{4},g)\rightarrow(N^{4},h)$ of dilation $\lambda=Cr^{(3+\sqrt{13})}$ or $\lambda=Cr^{(3-\sqrt{13})}$ is λ-biharmonic.
3. If $n=5$, we find $a=-1$, so $\lambda=\frac{C}{r}$ $(C\in\mathbb{R}_{+}^{*})$. It follows that any conformal map $\phi:(\mathbb{R}^{5},g)\rightarrow(N^{5},h)$ of dilation $\lambda=\frac{C}{r}$ is λ-biharmonic.
4. If $n=6$, we find $a=-2$ or $a=-4$, so $\lambda=\frac{C}{r^{2}}$ or $\lambda=\frac{C}{r^{4}}$ $(C\in\mathbb{R}_{+}^{*})$. Then, in this case any conformal map $\phi:(\mathbb{R}^{6},g)\rightarrow(N^{6},h)$ of dilation $\lambda=\frac{C}{r^{2}}$ or $\lambda=\frac{C}{r}$ is λ-biharmonic. For example, the inversion $\phi:(\mathbb{R}^{n}\setminus\{0\},g_{\mathbb{R}^{n}})\longrightarrow(\mathbb{R}^{n}\setminus\{0\},g_{\mathbb{R}^{n}})$ defined by $\phi(x)=\frac{x}{|x|^{2}}$ is a conformal map with dilation $\lambda=\frac{1}{r^{2}}$ and it is λ-biharmonic if and only if $n=6$.

A particular and important case of f-harmonic and f-biharmonic maps is given by the p-harmonic and p-biharmonic maps $(p\geq 2)$.

3.2. *The case of p-harmonic and p-biharmonic maps*

The p-harmonic and p-biharmonic maps correspond to the case of the f-harmonic and f-biharmonic maps where the function f is defined by $f=\frac{2}{p}|d\phi|^{p-2}$ $(p\geq 2)$ for the p-harmonic maps and $f=\frac{2}{p}|\tau(\phi)|^{p-2}$ for the p-biharmonic maps. In this context, the p-energy and the p-bienergy of the map $\phi:(M^{m},g)\rightarrow(N^{n},h)$ are respectively defined by

$$E_{p}(\phi)=\frac{1}{p}\int_{M}|d\phi|^{p}dv_{g}$$

and

$$E_{2,p}(\phi)=\frac{1}{p}\int_{M}|\tau(\phi)|^{p}dv_{g}.$$

The map ϕ is said to be p-harmonic if it is a critical point of the p-energy. Equivalently, if we replace f by $\frac{2}{p}|d\phi|^{p-2}$, ϕ is p-harmonic if it satisfies the associated Euler–Lagrange equations:

$$\tau_{p}(\phi)=\frac{2}{p}|d\phi|^{p-2}\left(\tau(\phi)+(p-2)\,d\phi\,(grad\ln|d\phi|)\right)=0,$$

$\tau_{p}(\phi)$ is called the p-tension field of ϕ. One can refer to [1], [11] and [16] for background on p-harmonic maps. Similarly, the map ϕ is said to be p-biharmonic if it is a critical point of the p-bi-energy. Equivalently, if we replace $f=\frac{2}{p}|\tau(\phi)|^{p-2}$ in (2), a simple calculation gives

$$grad|\tau(\phi)|^{p-2}=\frac{p-2}{2}|\tau(\phi)|^{p-4}grad\left(|\tau(\phi)|^{2}\right),$$

and

$$\Delta|\tau(\phi)|^{p-2}=\frac{p-2}{2}|\tau(\phi)|^{p-4}\Delta\left(|\tau(\phi)|^{2}\right)+\frac{(p-2)(p-4)}{4}|\tau(\phi)|^{p-6}\left|grad\left(|\tau(\phi)|^{2}\right)\right|^{2}.$$

Then ϕ is p-biharmonic if it satisfies the associated Euler–Lagrange equations:

$$\begin{aligned}\tau_{2,p}(\phi) = {} & \frac{2}{p}|\tau(\phi)|^{p-2}\tau_2(\phi) - \frac{(p-2)}{p}|\tau(\phi)|^{p-4}\Delta\left(|\tau(\phi)|^2\right)\tau(\phi) \\ & - \frac{(p-2)(p-4)}{2p}|\tau(\phi)|^{p-6}\left|grad\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi) \\ & - \frac{2(p-2)}{p}|\tau(\phi)|^{p-4}\nabla_{grad\left(|\tau(\phi)|^2\right)}\tau(\phi) = 0.\end{aligned}$$

$\tau_{2,p}(\phi)$ is called the p-bitension field of ϕ. Then ϕ is p-biharmonic if and only if

$$\begin{aligned}& |\tau(\phi)|^4\tau_2(\phi) - \frac{(p-2)}{2}|\tau(\phi)|^2\Delta\left(|\tau(\phi)|^2\right)\tau(\phi) \\ & - \frac{(p-2)(p-4)}{4}\left|grad\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi) \\ & - (p-2)|\tau(\phi)|^2\nabla_{grad\left(|\tau(\phi)|^2\right)}\tau(\phi) = 0.\end{aligned} \tag{6}$$

Remark 3. By the expression of the p-bitension field of ϕ, we deduce that if ϕ is biharmonic and $|\tau(\phi)|^2$ is constant, then ϕ is p-biharmonic.

Therefore for the conformal map, we obtain the following result.

Theorem 3. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, then ϕ is p-biharmonic if and only if*

$$\begin{aligned}& |grad \ln\lambda|^4 grad\,\Delta\ln\lambda \\ & + \frac{(p-2)}{2}|grad\ln\lambda|^2\Delta\left(|grad\ln\lambda|^2\right)grad\ln\lambda \\ & - \frac{(n-4p+2)}{2}|grad\ln\lambda|^4 grad\left(|grad\ln\lambda|^2\right) \\ & + (p-4)|grad\ln\lambda|^4(\Delta\ln\lambda)\,grad\ln\lambda \\ & - (n-p(p-2)-2)|grad\ln\lambda|^6 grad\ln\lambda \\ & + (p-2)^2|grad\ln\lambda|^2 d\ln\lambda\left(grad\left(|grad\ln\lambda|^2\right)\right)grad\ln\lambda \\ & + \frac{(p-2)(p-4)}{4}\left|grad\left(|grad\ln\lambda|^2\right)\right|^2 grad\ln\lambda \\ & + (p-2)|grad\ln\lambda|^2\nabla_{grad\left(|grad\ln\lambda|^2\right)}grad\ln\lambda \\ & + 2|grad\ln\lambda|^4 Ricci^M(grad\ln\lambda) = 0.\end{aligned} \tag{7}$$

Proof of Theorem 3. By (6), ϕ is p-biharmonic if and only if

$$\begin{aligned}& |\tau(\phi)|^4\tau_2(\phi) - \frac{(p-2)}{2}|\tau(\phi)|^2\Delta\left(|\tau(\phi)|^2\right)\tau(\phi) \\ & - \frac{(p-2)(p-4)}{4}\left|grad\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi) \\ & - (p-2)|\tau(\phi)|^2\nabla_{grad\left(|\tau(\phi)|^2\right)}\tau(\phi) = 0.\end{aligned} \tag{8}$$

Since ϕ is a conformal map, then

$$|\tau(\phi)|^2 = (n-2)^2\lambda^2|grad\ln\lambda|^2,$$

$$\tau(\phi) = (2-n)\, d\phi\, (grad \ln \lambda)$$

and

$$\tau_2(\phi) = (n-2)\, d\phi\, (H)$$

where

$$\begin{aligned} H &= grad \Delta \ln \lambda - \frac{(n-6)}{2} grad \left(|grad \ln \lambda|^2\right) + 2Ricci^M\, (grad \ln \lambda) \\ &\quad - \left(2\,(\Delta \ln \lambda) + (n-2)\, |grad \ln \lambda|^2\right) grad \ln \lambda. \end{aligned}$$

We deduce that ϕ is p-biharmonic if and only if

$$\begin{aligned} &(n-2)^5 \lambda^4 |grad \ln \lambda|^4 d\phi\, (H) \\ &+ \frac{(n-2)^3\,(p-2)}{2} \lambda^2 |grad \ln \lambda|^2 \Delta \left(|\tau(\phi)|^2\right) d\phi\, (grad \ln \lambda) \\ &+ \frac{(n-2)\,(p-2)\,(p-4)}{4} \left|grad \left(|\tau(\phi)|^2\right)\right|^2 d\phi\, (grad \ln \lambda) \\ &+ (n-2)^3\,(p-2)\, \lambda^2 |grad \ln \lambda|^2 \nabla_{grad\left(|\tau(\phi)|^2\right)} d\phi\, (grad \ln \lambda) = 0. \end{aligned} \tag{9}$$

For the term $grad\left(|\tau(\phi)|^2\right)$, we have

$$grad \left(|\tau(\phi)|^2\right) = (n-2)^2 grad \left(\lambda^2 |grad \ln \lambda|^2\right),$$

then

$$\begin{aligned} grad \left(|\tau(\phi)|^2\right) &= (n-2)^2 \lambda^2 grad \left(|grad \ln \lambda|^2\right) \\ &\quad + 2(n-2)^2 \lambda^2 |grad \ln \lambda|^2 grad \ln \lambda. \end{aligned} \tag{10}$$

By (10), we deduce that

$$\left|grad \left(|\tau(\phi)|^2\right)\right|^2 = (n-2)^4 \lambda^4 \left|grad \left(|grad \ln \lambda|^2\right) + 2|grad \ln \lambda|^2 grad \ln \lambda\right|^2,$$

which gives us

$$\begin{aligned} \left|grad \left(|\tau(\phi)|^2\right)\right|^2 &= (n-2)^4 \lambda^4 \left|grad \left(|grad \ln \lambda|^2\right)\right|^2 \\ &\quad + 4(n-2)^4 \lambda^4 |grad \ln \lambda|^6 \\ &\quad + 4(n-2)^4 \lambda^4 |grad \ln \lambda|^2 d \ln \lambda \left(grad \left(|grad \ln \lambda|^2\right)\right). \end{aligned} \tag{11}$$

For the term $\Delta\left(|\tau(\phi)|^2\right)$, we have

$$\begin{aligned} \Delta \left(|\tau(\phi)|^2\right) &= (n-2)^2 \Delta \left(\lambda^2 |grad \ln \lambda|^2\right) \\ &= (n-2)^2 \lambda^2 \Delta \left(|grad \ln \lambda|^2\right) + (n-2)^2 |grad \ln \lambda|^2 \Delta \left(\lambda^2\right) \\ &\quad + 2(n-2)^2 d\lambda^2 \left(grad \left(|grad \ln \lambda|^2\right)\right). \end{aligned}$$

A simple calculation gives

$$\Delta \left(\lambda^2\right) = 2\lambda^2 \Delta \ln \lambda + 4\lambda^2 |grad \ln \lambda|^2$$

and

$$d\lambda^2 \left(grad \left(|grad \ln \lambda|^2\right)\right) = 2\lambda^2 d \ln \lambda \left(grad \left(|grad \ln \lambda|^2\right)\right).$$

It follows that

$$\begin{aligned}\Delta\left(|\tau(\phi)|^2\right) &= (n-2)^2\lambda^2\Delta\left(|grad\ln\lambda|^2\right) + 2(n-2)^2\lambda^2|grad\ln\lambda|^2\Delta\ln\lambda \\ &+ 4(n-2)^2\lambda^2|grad\ln\lambda|^4 \\ &+ 4(n-2)^2\lambda^2 d\ln\lambda\left(grad\left(|grad\ln\lambda|^2\right)\right).\end{aligned} \tag{12}$$

To complete the proof, we will simplify the term $\nabla_{grad\left(|\tau(\phi)|^2\right)}d\phi\,(grad\ln\lambda)$. By (10), we have

$$\begin{aligned}\nabla_{grad\left(|\tau(\phi)|^2\right)}d\phi\,(grad\ln\lambda) &= (n-2)^2\lambda^2\nabla_{grad\left(|grad\ln\lambda|^2\right)}d\phi\,(grad\ln\lambda) \\ &+ 2(n-2)^2\lambda^2|grad\ln\lambda|^2\nabla_{grad\ln\lambda}d\phi\,(grad\ln\lambda).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\nabla_{grad\left(|grad\ln\lambda|^2\right)}d\phi\,(grad\ln\lambda) &= |grad\ln\lambda|^2 d\phi\left(grad\left(|grad\ln\lambda|^2\right)\right) \\ &+ d\phi\left(\nabla_{grad\left(|grad\ln\lambda|^2\right)}grad\ln\lambda\right)\end{aligned}$$

and

$$\nabla_{grad\ln\lambda}d\phi\,(grad\ln\lambda) = |grad\ln\lambda|^2 d\phi\,(grad\ln\lambda) + d\phi\left(\nabla_{grad\ln\lambda}grad\ln\lambda\right),$$

then

$$\begin{aligned}\nabla_{grad\left(|\tau(\phi)|^2\right)}d\phi\,(grad\ln\lambda) &= 2(n-2)^2\lambda^2|grad\ln\lambda|^2 d\phi\left(grad\left(|grad\ln\lambda|^2\right)\right) \\ &+ (n-2)^2\lambda^2 d\phi\left(\nabla_{grad\left(|grad\ln\lambda|^2\right)}grad\ln\lambda\right) \\ &+ 2(n-2)^2\lambda^2|grad\ln\lambda|^4 d\phi\,(grad\ln\lambda).\end{aligned} \tag{13}$$

If we replace (11)–(13) in (9), we conclude that ϕ is p-biharmonic if and only if

$$\begin{aligned}&(n-2)^5\lambda^4|grad\ln\lambda|^4 d\phi\,(H) + \frac{(n-2)^5(p-2)}{2} \\ &\times\ \lambda^4|grad\ln\lambda|^2\Delta\left(|grad\ln\lambda|^2\right)d\phi\,(grad\ln\lambda) \\ &+ (n-2)^5(p-2)\,\lambda^4|grad\ln\lambda|^4(\Delta\ln\lambda)\,d\phi\,(grad\ln\lambda) \\ &+ (n-2)^5 p\,(p-2)\,\lambda^4|grad\ln\lambda|^6 d\phi\,(grad\ln\lambda) \\ &+ (n-2)^5(p-2)^2\lambda^4|grad\ln\lambda|^2 d\ln\lambda\left(grad\left(|grad\ln\lambda|^2\right)\right)d\phi\,(grad\ln\lambda) \\ &+ \frac{(n-2)^5(p-2)(p-4)}{4}\lambda^4\left|grad\left(|grad\ln\lambda|^2\right)\right|^2 d\phi\,(grad\ln\lambda) \\ &+ 2(n-2)^5(p-2)\,\lambda^4|grad\ln\lambda|^4 d\phi\left(grad\left(|grad\ln\lambda|^2\right)\right) \\ &+ (n-2)^5(p-2)\,\lambda^4|grad\ln\lambda|^2 d\phi\left(\nabla_{grad\left(|grad\ln\lambda|^2\right)}grad\ln\lambda\right) = 0.\end{aligned}$$

Finally, since

$$\begin{aligned}H &= grad\,\Delta\ln\lambda - \frac{(n-6)}{2}grad\left(|grad\ln\lambda|^2\right) + 2Ricci^M(grad\ln\lambda) \\ &- \left(2(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right)grad\ln\lambda\end{aligned}$$

and $n \geq 3$, it follows that ϕ is p-biharmonic if and only if

$$\begin{aligned}
&|grad \ln \lambda|^4 grad \Delta \ln \lambda + \frac{(p-2)}{2} |grad \ln \lambda|^2 \Delta \left(|grad \ln \lambda|^2\right) grad \ln \lambda \\
&- \frac{(n-4p+2)}{2} |grad \ln \lambda|^4 grad \left(|grad \ln \lambda|^2\right) \\
&+ (p-4) |grad \ln \lambda|^4 (\Delta \ln \lambda) grad \ln \lambda \\
&- (n - p(p-2) - 2) |grad \ln \lambda|^6 grad \ln \lambda \\
&+ (p-2)^2 |grad \ln \lambda|^2 d \ln \lambda \left(grad \left(|grad \ln \lambda|^2\right)\right) grad \ln \lambda \\
&+ \frac{(p-2)(p-4)}{4} \left|grad \left(|grad \ln \lambda|^2\right)\right|^2 grad \ln \lambda \\
&+ (p-2) |grad \ln \lambda|^2 \nabla_{grad\left(|grad \ln \lambda|^2\right)} grad \ln \lambda \\
&+ 2 |grad \ln \lambda|^4 Ricci^M (grad \ln \lambda) = 0.
\end{aligned}$$

Hence the proof is complete. As an application of Theorem 3, we construct an example of a p-biharmonic map.

Example 2. We consider the inversion $\phi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\} \, (n \geq 3)$ defined by $\phi(x) = \frac{x}{|x|^2}$. ϕ is a conformal map with dilation $\lambda = \frac{1}{r^2} \, (r = |x|)$. Note that ϕ is p-harmonic if and only if $n = p$. By Theorem 3, we deduce that the map ϕ is p-biharmonic if and only if

$$3p^2 - 3(n-4)p + 2(n-10) = 0.$$

By using the fact that $p \geq 2$, we deduce that the inversion is p-biharmonic if and only if

$$p = \frac{n-4}{2} + \frac{1}{6}\sqrt{9n^2 - 96n + 384},$$

where $n \geq 4$. For example for $p = 2$ (see [2]), the inversion is biharmonic non-harmonic if and only if $n = 4$.

Acknowledgments

The authors would like to thank the referees for some useful comments and their helpful suggestions that have improved the quality of this paper.

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