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Some results and examples of the biharmonic maps with potential

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Abstract. In this paper, we will study the class of biharmonic maps with potential, in the particular case represented by conformal maps between equidimensional manifolds. Some examples are constructed in particular cases (Euclidean space and sphere).

Keywords: Harmonic map with potential; Biharmonic map with potential; Conformal maps

Mathematics Subject Classification: 31B30; 53C25; 58E20; 58E30

1. Introduction

The notion of harmonic maps with potential was first suggested by A. Ratto and A. Fardoun (see [5] and [9]). Let (M^m, g) and (N^n, h) be Riemannian manifolds, H a smooth function on N, and let $\phi : M \to N$ be a smooth map. We consider the following energy functional

$$E_H(\phi) = \int_K (e(\phi) - H(\phi))\, dv_g \tag{1}$$

for any compact subset $K \subset M$. The Euler–Lagrange equation of $E_H(\phi)$ is

$$\tau_H(\phi) = \tau(\phi) + \left(grad^N H\right) \circ \phi = 0, \tag{2}$$

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where $\tau(\phi) = Tr_g \nabla d\phi$ is the tension field of ϕ. The smooth solutions of (2) will be called harmonic maps with potential H. One can refer to [3], [5] and [9] for background on harmonic maps with potential. In [4], the authors calculate the second variation for harmonic maps with potential and they introduce the notion of biharmonic maps with potential. In this paper, we will recalculate the second variation of the H-energy and the first variation of the H-bi-energy (Theorems 1 and 2). As the second result we give the relation between $\tau_{2,H}(\phi)$ and $\tau_2(\phi)$ (Theorem 3) where we study the case of the identity map (Corollary 2 and Theorem 2) and we construct some examples of biharmonic with a potential. Finally, we study the particular of the conformal maps between equidimensional manifolds (Theorems 5 and 6).

2. The second variation of the H-energy functional

Let $\phi : (M^m, g) \to (N^n, h)$ be a harmonic map with potential H between Riemannian manifolds. By a two parameter variation we mean a smooth map $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$ defined by $\Phi(x, t, s) = \phi_{t,s}(x)$, such that $\phi_{0,0} = \phi$. Its variation vector fields are the vector fields v, w along ϕ defined by

$$v = \left.\frac{\partial \phi_{t,s}}{\partial t}\right|_{t=s=0}$$

and

$$w = \left.\frac{\partial \phi_{t,s}}{\partial s}\right|_{t=s=0}.$$

Now suppose that M is compact and let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. By the Leibniz rule,

$$\left(\nabla^\phi\right)^2_{X,Y} v = \nabla^\phi_X \nabla^\phi_Y v - \nabla^\phi_{\nabla^M_X Y} v$$

for any $X, Y \in \Gamma(TM)$ and $v \in \Gamma\left(\phi^{-1}TN\right)$. On taking the trace we obtain

$$Tr_g\left(\nabla^\phi\right)^2 v = \nabla^\phi_{e_i} \nabla^\phi_{e_i} v - \nabla^\phi_{\nabla^M_{e_i} e_i} v,$$

where $\{e_i\}_{1 \le i \le m}$ is an orthonormal frame on M and where we sum over repeated indices. Under the notation above we have the following :

Theorem 1 (*The Second Variation Formula*)**.** *Let $\phi : (M^n, g) \to (N^n, h)$ be a harmonic map with potential H and suppose that M is compact, we have*

$$\left.\frac{\partial^2}{\partial t \partial s} E_H\left(\phi_{t,s}\right)\right|_{t=s=0} = \int_M h\left(J^\phi_H(v), w\right) dv_g, \tag{3}$$

where $J^\phi_H(v) \in \Gamma\left(\phi^{-1}TN\right)$ is given by

$$J^\phi_H(v) = -Tr_g\left(\nabla^\phi\right)^2 v - Tr_g R^N(v, d\phi)\, d\phi - \left(\nabla^N_v grad^N H\right) \circ \phi. \tag{4}$$

Proof of Theorem 1. Let $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$, $(x, t, s) \longmapsto \Phi(x, t, s) = \phi_{t,s}(x)$ be a smooth variation of ϕ with variation vector fields v and w. Let ∇^Φ denote the pull-back connection on $\Phi^{-1}TN$, a vector bundle over $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$. Note that

$$\left[\frac{\partial}{\partial t}, X\right] = \left[\frac{\partial}{\partial s}, X\right] = \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$$

for any vector field X on M considered as a vector field on $M\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$. By Eq. (1), we have

$$\frac{\partial^2}{\partial t\partial s}E_H\left(\phi_{t,s}\right)=\int_M\frac{\partial^2}{\partial t\partial s}\left(e\left(\phi_{t,s}\right)-H\left(\phi_{t,s}\right)\right)dv_g. \tag{5}$$

We evaluate this at $(t,s)=(0,0)$. Calculating in a normal frame at $x\in M$, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}e\left(\phi_{t,s}\right)&=\frac{1}{2}\frac{\partial^2}{\partial t\partial s}h\left(d\Phi\left(e_i\right),d\Phi\left(e_i\right)\right)\\&=\frac{\partial}{\partial t}h\left(\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),d\Phi\left(e_i\right)\right),\end{aligned}$$

then

$$\frac{\partial^2}{\partial t\partial s}e\left(\phi_{t,s}\right)=h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),d\Phi\left(e_i\right)\right)+h\left(\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right)\right). \tag{6}$$

For the first term $h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),d\Phi\left(e_i\right)\right)$ of (6), we have

$$\begin{aligned}h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),d\Phi\left(e_i\right)\right)&=h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}\nabla^{\Phi}_{e_i}d\Phi\left(\frac{\partial}{\partial s}\right),d\Phi\left(e_i\right)\right)\\&=h\left(\nabla^{\Phi}_{e_i}\nabla^{\Phi}_{\frac{\partial}{\partial t}}d\Phi\left(\frac{\partial}{\partial s}\right),d\Phi\left(e_i\right)\right)\\&\quad+h\left(R^N\left(d\Phi\left(\frac{\partial}{\partial t}\right),d\Phi\left(e_i\right)\right)d\Phi\left(\frac{\partial}{\partial s}\right),d\Phi\left(e_i\right)\right).\end{aligned}$$

Define a 1-form on M by

$$\alpha\left(\cdot\right)=h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}d\Phi\left(\frac{\partial}{\partial s}\right)\Big|_{t=s=0},d\Phi\left(\cdot\right)\right).$$

Then

$$div\alpha=h\left(\nabla^{\Phi}_{e_i}\nabla^{\Phi}_{\frac{\partial}{\partial t}}d\Phi\left(\frac{\partial}{\partial s}\right)\Big|_{t=s=0},d\phi\left(e_i\right)\right)+h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}d\Phi\left(\frac{\partial}{\partial s}\right)\Big|_{t=s=0},\tau\left(\phi\right)\right),$$

it follows that

$$\begin{aligned}h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),d\Phi\left(e_i\right)\right)\Big|_{t=s=0}&=div\alpha-h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}}d\Phi\left(\frac{\partial}{\partial s}\right)\Big|_{t=s=0},\tau\left(\phi\right)\right)\\&\quad+h\left(R^N\left(v,d\phi\left(e_i\right)\right)w,d\phi\left(e_i\right)\right).\end{aligned} \tag{7}$$

For the second term $h\left(\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right)\right)$ of (6), we have

$$\begin{aligned}h\left(\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right),\nabla^{\Phi}_{\frac{\partial}{\partial s}}d\Phi\left(e_i\right)\right)&=h\left(\nabla^{\Phi}_{e_i}d\Phi\left(\frac{\partial}{\partial s}\right),\nabla^{\Phi}_{e_i}d\Phi\left(\frac{\partial}{\partial s}\right)\right)\\&=e_i\left(h\left(d\Phi\left(\frac{\partial}{\partial s}\right),\nabla^{\Phi}_{e_i}d\Phi\left(\frac{\partial}{\partial s}\right)\right)\right)\\&\quad-h\left(d\Phi\left(\frac{\partial}{\partial s}\right),\nabla^{\Phi}_{e_i}\nabla^{\Phi}_{e_i}d\Phi\left(\frac{\partial}{\partial s}\right)\right).\end{aligned}$$

Define a 1-form on M by $\beta(\cdot) = h\left(w, \nabla^{\phi}_{\cdot} v\right)$. By calculating the divergence of β, we obtain

$$h\left(\nabla^{\Phi}_{\frac{\partial}{\partial s}} d\Phi(e_i), \nabla^{\Phi}_{\frac{\partial}{\partial s}} d\Phi(e_i)\right)\Big|_{t=s=0} = div\beta - h\left(w, \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} v\right),$$

which gives us

$$\begin{aligned} \frac{\partial^2}{\partial t \partial s} e\left(\phi_{t,s}\right)\Big|_{t=s=0} &= -h\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} v, w\right) - h\left(R^N\left(v, d\phi\left(e_i\right)\right) d\phi\left(e_i\right), w\right) \\ &\quad + div\alpha + div\beta - h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}} d\Phi\left(\frac{\partial}{\partial s}\right)\Big|_{t=s=0}, \tau(\phi)\right). \end{aligned} \tag{8}$$

Finally, a simple calculation gives

$$\begin{aligned} \frac{\partial^2}{\partial t \partial s} H\left(\phi_{t,s}\right)\Big|_{t=s=0} &= h\left(\nabla^{\Phi}_{\frac{\partial}{\partial t}} d\Phi\left(\frac{\partial}{\partial s}\right)\Big|_{t=s=0}, \left(grad^N H\right)\circ\phi\right) \\ &\quad + h\left(\left(\nabla^N_v grad^N H\right)\circ\phi, w\right). \end{aligned} \tag{9}$$

By replacing Eqs. (8) and (9) in (5) and using the divergence theorem, we obtain

$$\frac{\partial^2}{\partial t \partial s} E_H\left(\phi_{t,s}\right)\Big|_{t=s=0} = \int_M h\left(J^{\phi}_H(v), w\right) dv_g,$$

where $J^{\phi}_H(v) \in \Gamma\left(\phi^{-1}TN\right)$ is given by

$$J^{\phi}_H(v) = -Tr_g\left(\nabla^{\phi}\right)^2 v - Tr_g R^N\left(v, d\phi\right) d\phi - \left(\nabla^N_v grad^N H\right)\circ\phi.$$

3. Biharmonic maps with potential

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds and let H be a smooth function on N. A natural generalization of harmonic maps with potential is given by integrating the square of the norm of $\tau_H(\phi)$. More precisely, the H-bi-energy functional of ϕ is defined by

$$E_{2,H}(\phi) = \int_K |\tau_H(\phi)|^2 dv_g \tag{10}$$

for any compact subset $K \subset M$.

Definition 1. The map ϕ is said to be biharmonic with potential H if it is a critical point of the H-bi-energy functional $E_{2,H}(\phi)$.

3.1. *The first variation of the H-bi-energy functional*

Theorem 2. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and H a smooth function on N, K a compact subset of M and let $\{\phi_t\}_{t\in I}$, $I = (-\epsilon, \epsilon)$ be a smooth variation of ϕ with compact support in K. Then*

$$\frac{d}{dt} E_{2,H}\left(\phi_t\right)\Big|_{t=0} = -\int_K h\left(\tau_{2,H}(\phi), V\right) dv_g, \tag{11}$$

where V denotes the variation vector field of $\{\phi_t\}_{t\in I}$ and

$$\tau_{2,H}(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2 \tau_H(\phi) - Tr_g R^N\left(\tau_H(\phi), d\phi\right) d\phi - \left(\nabla^N_{\tau_H(\phi)} grad H\right)\circ\phi.$$

Proof of Theorem 2. Let $\{\phi_t\}_{t\in I}$, $I = (-\epsilon, \epsilon)$, be a smooth variation of ϕ, i.e. a smooth map $\Phi : I \times M \longrightarrow N$, satisfying

$$\begin{cases} \Phi(t,x) = \phi_t(x), \forall (t,x) \in I \times M \\ \Phi(0,x) = \phi_0(x) = \phi(x), \forall x \in M. \end{cases}$$

The variation vector field $V \in \Gamma(\phi^{-1}TN)$ associated to the variation $\{\phi_t\}_{t\in I}$ is given by

$$V(x) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x) = d\Phi_{(0,x)}\left(\frac{\partial}{\partial t}\right) \in T_{\phi(x)}N, \forall x \in M.$$

We have

$$\frac{d}{dt}E_{2,H}(\phi_t) = \int_K h\left(\nabla_{\frac{\partial}{\partial t}}\tau_H(\phi), \tau_H(\phi)\right) dv_g. \tag{12}$$

Let now $\{e_i\}_{i=1}^m$ be a local orthonormal frame field geodesic at $x \in K$, i.e. $\{e_i\}_{i=1}^m$ is a local orthonormal frame field with $\left(\nabla_{e_i}e_j\right)_x = 0, \forall i, j = 1, \ldots, m$. With respect to $\{e_i\}_{i=1}^m$ we have

$$\nabla^\phi_{\frac{\partial}{\partial t}}\tau_H(\phi) = \nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{e_i}d\phi_t(e_i) + \nabla^\phi_{\frac{\partial}{\partial t}}(gradH)\circ\phi_t.$$

For a given $Z \in \Gamma(TM)$, since $\left[\frac{\partial}{\partial t}, Z\right] = 0$, we get

$$\nabla^\phi_{\frac{\partial}{\partial t}}d\phi_t(Z) = \nabla^\phi_Z d\phi_t\left(\frac{\partial}{\partial t}\right) + d\phi_t\left(\left[\frac{\partial}{\partial t}, Z\right]\right) = \nabla^\phi_Z d\phi_t\left(\frac{\partial}{\partial t}\right).$$

By definition of the curvature tensor of (N, h), we obtain

$$\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{e_i}d\phi_t(e_i) = \nabla^\phi_{e_i}\nabla^\phi_{\frac{\partial}{\partial t}}d\phi_t(e_i) + R^N\left(d\phi\left(\frac{\partial}{\partial t}\right), d\phi_t(e_i)\right)d\phi_t(e_i),$$

by the compatibility of ∇^ϕ with h, we have

$$\begin{aligned} h\left(\nabla^\phi_{e_i}\nabla^\phi_{\frac{\partial}{\partial t}}d\phi_t(e_i), \tau_H(\phi_t)\right) &= e_i\left(h\left(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi_t(e_i), \tau_H(\phi_t)\right)\right) \\ &\quad - h\left(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi_t(e_i), \nabla^\phi_{e_i}\tau_H(\phi_t)\right). \end{aligned}$$

Note that

$$\begin{aligned} h\left(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi_t(e_i), \nabla^\phi_{e_i}\tau_H(\phi_t)\right) &= e_i\left(h\left(d\phi_t\left(\frac{\partial}{\partial t}\right), \nabla^\phi_{e_i}\tau_H(\phi_t)\right)\right) \\ &\quad - h\left(d\phi_t\left(\frac{\partial}{\partial t}\right), \nabla^\phi_{e_i}\nabla^\phi_{e_i}\tau_H(\phi_t)\right) \end{aligned}$$

and

$$h\left(\nabla^\phi_{\frac{\partial}{\partial t}}(gradH)\circ\phi_t, \tau_H(\phi_t)\right) = h\left(\left(\nabla^\phi_{\tau_H(\phi_t)}gradH\right)\circ\phi_t, d\phi\left(\frac{\partial}{\partial t}\right)\right).$$

Then, by using the symmetries of the Riemann–Christoffel tensor and the Divergence Theorem, we deduce that

$$\left.\frac{d}{dt}E_{2,H}(\phi_t)\right|_{t=0} = -\int_K h\left(\tau_{2,H}(\phi), V\right) dv_g,$$

where

$$\tau_{2,H}(\phi) = -Tr_g\left(\nabla^\phi\right)^2\tau_H(\phi) - Tr_g R^N(\tau_H(\phi), d\phi)d\phi - \left(\nabla^N_{\tau_H(\phi)}gradH\right)\circ\phi.$$

Theorem 3. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and H a smooth function on N. Then ϕ is biharmonic with potential if and only if*

$$\begin{aligned}\tau_{2,H}(\phi) = & -Tr_g\left(\nabla^{\phi}\right)^2\tau_H(\phi) - Tr_g R^N(\tau_H(\phi), d\phi)\, d\phi \\ & - \left(\nabla^N_{\tau_H(\phi)} gradH\right)\circ\phi = 0.\end{aligned} \tag{13}$$

If we consider $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and let $H \in C^{\infty}(N)$ be a smooth function, the relation between $\tau_{2,H}(\phi)$ and $\tau_2(\phi)$ is given by the following remark.

Remark 1. Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds and H a smooth function on N. Then

$$\begin{aligned}\tau_{2,H}(\phi) = & \ \tau_2(\phi) - J_{\phi}((gradH)\circ\phi) - \left(\nabla_{\tau(\phi)}(gradH)\right)\circ\phi \\ & - \left(\nabla_{(gradH)\circ\phi}(gradH)\right)\circ\phi,\end{aligned} \tag{14}$$

where

$$\begin{aligned}J_{\phi}((gradH)\circ\phi) &= Tr_g\left(\nabla^{\phi}\right)^2((gradH)\circ\phi) + Tr_g R^N((gradH)\circ\phi, d\phi)\, d\phi \\ &= \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}((gradH)\circ\phi) - \nabla^{\phi}_{\nabla^M_{e_i}e_i}((gradH)\circ\phi) \\ &\quad + R^N((gradH)\circ\phi, d\phi(e_i))\, d\phi(e_i),\end{aligned}$$

where we sum over repeated indices.

In the case where ϕ is a harmonic map, we obtain the following corollary.

Corollary 1. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map and let $H \in C^{\infty}(N)$ be a smooth function on N. Then ϕ is biharmonic with potential H if and only if*

$$J_{\phi}((gradH)\circ\phi) + \left(\nabla_{(gradH)\circ\phi}(gradH)\right)\circ\phi = 0.$$

We apply this remark to construct some examples of biharmonic maps with potential.

Example 1. Let the projection $\phi : \left(\mathbb{R}^4, g_{\mathbb{R}^4}\right) \longrightarrow \left(\mathbb{R}^3, g_{\mathbb{R}^3}\right)$ defined by $\phi(t, x_2, x_3, x_4) = (t, x_2, x_3)$. Suppose that the function $\alpha = H\circ\phi$ depends only on t, then by Corollary 1, the projection ϕ is biharmonic with potential H if and only if

$$\alpha''' + \alpha'\alpha'' = 0.$$

Let $\beta = \alpha'$, then the last expression becomes

$$\beta'' + \beta\beta' = 0.$$

A particular solution is given by of the form $\beta = \frac{2}{t+k}$ $(k \in \mathbb{R})$ which gives us

$$\alpha(t) = (H\circ\phi)(t) = 2\ln|C_1 t + C_2|, (C_1, C_2 \in \mathbb{R}).$$

In this case the projection ϕ is biharmonic with potential H, where $(H\circ\phi)(t) = 2\ln|C_1 t + C_2|$.

In particular, if we consider the identity map, we obtain the following result.

Corollary 2. *The identity map* $Id_M : (M^m, g) \longrightarrow (M^m, g)$ *is biharmonic with potential* H *if and only if*

$$grad\Delta H + \frac{1}{2} grad\left(|gradH|^2\right) + 2Ricci^M\left(gradH\right) = 0. \quad (15)$$

Proof of Corollary 2. By Corollary 1, the identity map $Id_M : (M^m, g) \longrightarrow (M^m, g)$ is biharmonic with potential H if and only if

$$J_{Id_M}\left(gradH\right) + \nabla_{gradH} gradH = 0.$$

For the term $J_{Id_M}\left(gradH\right)$, by definition we have

$$J_{Id_M}\left(gradH\right) = Tr_g \nabla^2 gradH + Ricci\left(gradH\right).$$

It is known that (see [6])

$$Tr_g \nabla^2 gradH = grad\Delta H + Ricci\left(gradH\right),$$

then

$$J_{Id_M}\left(gradH\right) = grad\Delta H + 2Ricci\left(gradH\right).$$

Finally, it is easy to see that

$$\nabla_{gradH} gradH = \frac{1}{2} grad\left(|gradH|^2\right).$$

We deduce that the identity map $Id_M : (M^m, g) \longrightarrow (M^m, g)$ is biharmonic with potential H if and only if the function H satisfies the following equation

$$grad\Delta H + \frac{1}{2} grad\left(|gradH|^2\right) + 2Ricci^M\left(gradH\right) = 0.$$

In the following we shall present an example of biharmonic with potential.

Example 2. We consider the identity map $Id : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ when we suppose that H is radial $(H = H(r), r = |x|)$. An orthonormal basis of $\mathbb{R}^m$ is given by $e_1 = \frac{\partial}{\partial r}$ and $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ for $i = 2, \dots, m$. We have (see [7])

$$\sum_{i=2}^{m} \nabla_{e_i} e_i = -\frac{(m-1)}{r} \frac{\partial}{\partial r},$$

$$\Delta H = H'' + \frac{(m-1)}{r} H'$$

and

$$grad\left(\Delta H\right) = \left(H'' + \frac{(m-1)}{r} H'' - \frac{(m-1)}{r^2} H'\right) \frac{\partial}{\partial r}.$$

Then by Corollary 2, we deduce that the identity map Id is biharmonic with potential if and only if the function H satisfies the following differential equation

$$H''' + \frac{m-1}{r} H'' - \frac{m-1}{r^2} H' + H'H'' = 0.$$

Let $\beta = H'$, this equation becomes

$$\beta'' + \frac{m-1}{r}\beta' - \frac{m-1}{r^2}\beta + \beta\beta' = 0.$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, then Id is biharmonic with potential if and only if $a = 4 - 2m$. We obtain $H(r) = \ln\left(Cr^{4-2m}\right)$ $(C > 0)$ and in this case the identity map $Id : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is proper biharmonic with potential $H(r) = \ln\left(Cr^{4-2m}\right)$ $(C > 0)$.

Example 3. We consider $M = S^n$ with parameterization

$$x = (\cos s, \sin s \cdot y), s \in [0, \pi], y \in S^{n-1}.$$

An orthonormal basis of S^n is given by

$$e_1 = \frac{\partial}{\partial s}, e_i = (0, f_i), i = 2, \dots, n$$

where the vectors f_i are tangents to the sphere S^{n-1}. We suppose that $H = H(s)$. A direct calculation gives

$$grad H = H'\frac{\partial}{\partial s},$$
$$|grad H|^2 = \left(H'\right)^2,$$
$$grad\left(|grad H|^2\right) = 2H'H''\frac{\partial}{\partial s},$$
$$\Delta H = H'' + (n-1)(\cot s)H',$$
$$grad \Delta H = \left(H''' + (n-1)(\cot s)H'' - (n-1)\left(1 + \cot^2 s\right)H'\right)\frac{\partial}{\partial s}$$

and

$$Ricci^{S^n}(grad H) = (n-1)H'\frac{\partial}{\partial s}.$$

Then by Corollary 1, we deduce that the map Id_{S^n} is biharmonic with potential H if and only if the function $\beta = H'$ satisfies the following differential equation

$$\beta'' + (n-1)(\cot s)\beta' + (n-1)\left(1 - \cot^2 s\right)\beta + \beta\beta' = 0.$$

For example, if $n = 1$ the map Id_{S^1} is biharmonic with potential H if and only if

$$\beta'' + \beta\beta' = 0.$$

A particular solution is given by of the form $\beta = \frac{2}{s+k}$ $(k \in \mathbb{R})$ which gives us

$$H(s) = 2\ln|C_1 s + C_2|, (C_1, C_2 \in \mathbb{R}).$$

In this case Id_{S^1} is biharmonic with potential H, where $H(s) = 2\ln|C_1 s + C_2|$.

3.2. The case of conformal maps

We study conformal maps between equidimensional manifolds of the same dimension $n \geq 3$. Note that by a result in [2], any such map can have no critical points and so is a local conformal diffeomorphism. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exists a C^∞ function $\lambda : M \to \mathbb{R}_+^*$ such that for any $X, Y \in \Gamma(TM)$:

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y).$$

The function λ is called the dilation for the map ϕ. The tension field for the map ϕ is given by (see [1]):

$$\tau(\phi) = (2-n)d\phi(grad \ln \lambda).$$

Note that the conformal map $\phi : (M^n, g) \to (N^n, h)$ of dilation λ is harmonic if and only if $n = 2$ or the dilation λ is constant. The bi-tension field of the conformal map is given by the following theorem (see [8])

Theorem 4 (*[8]*)**.** *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation λ, then for any function $\gamma \in C^\infty(M)$, we have*

$$\begin{aligned} Tr_g\left(\nabla^\phi\right)^2 d\phi\,(grad\gamma) = {} & d\phi\,(grad\Delta\gamma) + 4d\phi\left(\nabla_{grad \ln\lambda} grad\gamma\right) \\ & + d\phi\left(Ricci^M\,(grad\gamma)\right) \\ & + (\Delta \ln\lambda)\, d\phi\,(grad\gamma) - 2\,(\Delta\gamma)\, d\phi\,(grad \ln\lambda) \\ & - (n-2)\, d\ln\lambda\,(grad\gamma)\, d\phi\,(grad \ln\lambda)\,, \end{aligned} \tag{16}$$

$$\begin{aligned} Tr_g R^N\,(d\phi\,(grad\gamma)\,, d\phi)\, d\phi = {} & d\phi\left(Ricci^M\,(grad\gamma)\right) \\ & - (n-2)\, d\phi\left(\nabla_{grad\gamma} grad \ln\lambda\right) \\ & - \left(\Delta\ln\lambda + (n-2)\,|grad \ln\lambda|^2\right) d\phi\,(grad\gamma) \\ & + (n-2)\, d\ln\lambda\,(grad\gamma)\, d\phi\,(grad\ln\lambda) \end{aligned} \tag{17}$$

and the bi-tension field of ϕ is given by

$$\tau_2(\phi) = (n-2)\, d\phi\,(T_1)$$

where

$$\begin{aligned} T_1 = {} & grad\Delta\ln\lambda - \frac{(n-6)}{2} grad\left(|grad\ln\lambda|^2\right) + 2Ricci^M\,(grad\ln\lambda) \\ & - \left(2\,(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right) grad\ln\lambda. \end{aligned}$$

In the first, we calculate $\tau_{2,H}(\phi)$ for a conformal map ϕ.

Theorem 5. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation λ, then $\tau_{2,H}(\phi)$ is given by :*

$$\tau_{2,H}\,(\phi) = (n-2)\, d\phi\,(A) - \frac{1}{\lambda^2} d\phi\,(B) - \frac{1}{\lambda^4} d\phi\,(C)\,,$$

where

$$\begin{aligned} A = {} & grad\Delta\ln\lambda - \frac{(n-6)}{2} grad\left(|grad\ln\lambda|^2\right) + 2Ricci^M\,(grad\ln\lambda) \\ & - \left(2\,(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right) grad\ln\lambda, \\ B = {} & grad\Delta\,(H\circ\phi) - 2\,(\Delta\ln\lambda)\, grad\,(H\circ\phi) - 2\,(\Delta\,(H\circ\phi))\, grad\ln\lambda \\ & - (n-2)\,\nabla_{grad(H\circ\phi)} grad\ln\lambda - (n-2)\,\nabla_{grad\ln\lambda} grad\,(H\circ\phi) \\ & + 2Ricci^M\,(grad\,(H\circ\phi)) \end{aligned}$$

and

$$C = -|grad\,(H \circ \phi)|^2 grad \ln \lambda + \frac{1}{2} grad \left(|grad\,(H \circ \phi)|^2\right).$$

As a consequence to Theorem 5, we have the following result :

Corollary 3. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation $\lambda = 1$, then ϕ is biharmonic with potential if and only if*

$$grad\,\Delta\,(H \circ \phi) + \frac{1}{2} grad \left(|grad\,(H \circ \phi)|^2\right) + 2 Ricci^M\,(grad\,(H \circ \phi)) = 0.$$

In particular if $\phi = Id_M$, we obtain Eq. (15) of Corollary 2.

To prove Theorem 5, we need two lemmas. In the first lemma, we give a simple formula of the term $Tr_g\left(\nabla^{\phi}\right)^2 (grad H) \circ \phi$ for a conformal map $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ of dilation λ and for any function $H \in C^{\infty}(N)$.

Lemma 1. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, then for any function $H \in C^{\infty}(N)$, we have*

$$Tr_g\left(\nabla^{\phi}\right)^2 (grad H) \circ \phi = \frac{1}{\lambda^2} d\phi\,(T_1),$$

where

$$\begin{aligned} T_1 = {} & grad\,\Delta\,(H \circ \phi) - (\Delta \ln \lambda)\, grad\,(H \circ \phi) - 2\,(\Delta\,(H \circ \phi))\, grad \ln \lambda \\ & - (n-2)\, d \ln \lambda\, (grad\,(H \circ \phi))\, grad \ln \lambda + Ricci^M\,(grad\,(H \circ \phi)). \end{aligned}$$

Proof of Lemma 1. Let $(e_i)_{1 \leq i \leq n}$ be an orthonormal frame on M, by definition, we have

$$Tr_g\left(\nabla^{\phi}\right)^2 (grad H) \circ \phi = \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} (grad H) \circ \phi - \nabla^{\phi}_{\nabla_{e_i} e_i} (grad H) \circ \phi.$$

(Here henceforth we sum over repeated indices.) It is easy to see that

$$(grad H) \circ \phi = \frac{1}{\lambda^2} d\phi\,(grad\,(H \circ \phi)),$$

which gives us

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 (grad H) \circ \phi &= Tr_g\left(\nabla^{\phi}\right)^2 \frac{1}{\lambda^2} d\phi\,(grad\,(H \circ \phi)) \\ &= \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} \frac{1}{\lambda^2} d\phi\,(grad\,(H \circ \phi)) \\ &\quad - \nabla^{\phi}_{\nabla_{e_i} e_i} \frac{1}{\lambda^2} d\phi\,(grad\,(H \circ \phi)). \end{aligned} \tag{18}$$

We have

$$\begin{aligned} \nabla^{\phi}_{e_i} \frac{1}{\lambda^2} d\phi\,(grad\,(H \circ \phi)) &= \frac{1}{\lambda^2} \nabla^{\phi}_{e_i} d\phi\,(grad\,(H \circ \phi)) + e_i \left(\frac{1}{\lambda^2}\right) d\phi\,(grad\,(H \circ \phi)) \\ &= \frac{1}{\lambda^2} \nabla^{\phi}_{e_i} d\phi\,(grad\,(H \circ \phi)) \\ &\quad - \frac{2}{\lambda^2} e_i\,(\ln \lambda)\, d\phi\,(grad\,(H \circ \phi)), \end{aligned}$$

then

$$\begin{aligned}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\frac{1}{\lambda^2}d\phi\left(grad\left(H\circ\phi\right)\right) &= \nabla^{\phi}_{e_i}\left(\frac{1}{\lambda^2}\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\right)\\
&- \nabla^{\phi}_{e_i}\left(\frac{2}{\lambda^2}e_i\left(\ln\lambda\right)d\phi\left(grad\left(H\circ\phi\right)\right)\right)\\
&= \frac{1}{\lambda^2}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&+ e_i\left(\frac{1}{\lambda^2}\right)\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{2}{\lambda^2}e_i\left(\ln\lambda\right)\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{2}{\lambda^2}e_i\left(e_i\left(\ln\lambda\right)\right)d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- e_i\left(\frac{2}{\lambda^2}\right)e_i\left(\ln\lambda\right)d\phi\left(grad\left(H\circ\phi\right)\right)\\
&= \frac{1}{\lambda^2}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{4}{\lambda^2}e_i\left(\ln\lambda\right)\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{2}{\lambda^2}e_i\left(e_i\left(\ln\lambda\right)\right)d\phi\left(grad\left(H\circ\phi\right)\right)\\
&+ \frac{2}{\lambda^2}e_i\left(\ln\lambda\right)e_i\left(\ln\lambda\right)d\phi\left(grad\left(H\circ\phi\right)\right).\end{aligned}$$

It follows that

$$\begin{aligned}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\frac{1}{\lambda^2}d\phi\left(grad\left(H\circ\phi\right)\right) &= \frac{1}{\lambda^2}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{4}{\lambda^2}\nabla^{\phi}_{grad\ln\lambda}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{2}{\lambda^2}e_i\left(e_i\left(\ln\lambda\right)\right)d\phi\left(grad\left(H\circ\phi\right)\right)\\
&+ \frac{4}{\lambda^2}|grad\ln\lambda|^2d\phi\left(grad\left(H\circ\phi\right)\right).\end{aligned}\tag{19}$$

For the term $\nabla^{\phi}_{\nabla_{e_i}e_i}\frac{1}{\lambda^2}d\phi\left(grad\left(H\circ\phi\right)\right)$, we have

$$\begin{aligned}\nabla^{\phi}_{\nabla_{e_i}e_i}\frac{1}{\lambda^2}d\phi\left(grad\left(H\circ\phi\right)\right) &= \frac{1}{\lambda^2}\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&+ \nabla_{e_i}e_i\left(\frac{1}{\lambda^2}\right)d\phi\left(grad\left(H\circ\phi\right)\right)\\
&= \frac{1}{\lambda^2}\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(grad\left(H\circ\phi\right)\right)\\
&- \frac{2}{\lambda^2}\nabla_{e_i}e_i\left(\ln\lambda\right)d\phi\left(grad\left(H\circ\phi\right)\right).\end{aligned}\tag{20}$$

If we replace (19) and (20) in (18), we deduce that

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 (grad H)\circ\phi = &\frac{1}{\lambda^2} Tr_g\left(\nabla^{\phi}\right)^2 d\phi\,(grad\,(H\circ\phi)) \\ &- \frac{4}{\lambda^2}\nabla^{\phi}_{grad\ln\lambda} d\phi\,(grad\,(H\circ\phi)) \\ &- \frac{2}{\lambda^2}\left((\Delta\ln\lambda) - 2|grad\ln\lambda|^2\right) d\phi\,(grad\,(H\circ\phi)). \end{aligned} \tag{21}$$

For the term $Tr_g\left(\nabla^{\phi}\right)^2 d\phi\,(grad\,(H\circ\phi))$, we have by (16)

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 d\phi\,(grad\,(H\circ\phi)) = &\, d\phi\,(grad\,\Delta\,(H\circ\phi)) \\ &+ 4d\phi\left(\nabla_{grad\ln\lambda} grad\,(H\circ\phi)\right) \\ &- ((n-2)\,d\ln\lambda\,(grad\,(H\circ\phi)) \\ &+ 2\,(\Delta\,(H\circ\phi)))\,d\phi\,(grad\ln\lambda) \\ &+ (\Delta\ln\lambda)\,d\phi\,(grad\,(H\circ\phi)) \\ &+ d\phi\left(Ricci^M\,(grad\,(H\circ\phi))\right). \end{aligned} \tag{22}$$

For the last term $\nabla^{\phi}_{grad\ln\lambda} d\phi\,(grad\,(H\circ\phi))$, it is known that (see [2])

$$\begin{aligned} \nabla^{\phi}_{grad\ln\lambda} d\phi\,(grad\,(H\circ\phi)) &= \nabla d\phi\,(grad\ln\lambda, grad\,(H\circ\phi)) \\ &\quad + d\phi\left(\nabla_{grad\ln\lambda} grad\,(H\circ\phi)\right) \\ &= |grad\ln\lambda|^2 d\phi\,(grad\,(H\circ\phi)) \\ &\quad + d\phi\left(\nabla_{grad\ln\lambda} grad\,(H\circ\phi)\right). \end{aligned} \tag{23}$$

Replacing (22) and (23) in (21), we conclude that

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 (grad H)\circ\phi = &\frac{1}{\lambda^2} d\phi\,(grad\,\Delta\,(H\circ\phi)) - \frac{1}{\lambda^2}(\Delta\ln\lambda)\,d\phi\,(grad\,(H\circ\phi)) \\ &- \frac{1}{\lambda^2}((n-2)\,d\ln\lambda\,(grad\,(H\circ\phi)) \\ &+ 2\,(\Delta\,(H\circ\phi)))\,d\phi\,(grad\ln\lambda) \\ &+ \frac{1}{\lambda^2} d\phi\left(Ricci^M\,(grad\,(H\circ\phi))\right). \end{aligned}$$

Then

$$Tr_g\left(\nabla^{\phi}\right)^2 (grad H)\circ\phi = \frac{1}{\lambda^2} d\phi\,(T_1)$$

where

$$\begin{aligned} T_1 = &\, grad\,\Delta\,(H\circ\phi) - (\Delta\ln\lambda)\,grad\,(H\circ\phi) + Ricci^M\,(grad\,(H\circ\phi)) \\ &- ((n-2)\,d\ln\lambda\,(grad\,(H\circ\phi)) + 2\,(\Delta\,(H\circ\phi)))\,grad\ln\lambda. \end{aligned}$$

This completes the proof of Lemma 1. Now, in the second lemma, we prove other properties for conformal maps.

Lemma 2. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, then for any function $H \in C^\infty(N)$ and for $X \in \Gamma(TM)$, we have*

$$\begin{aligned}\left(\nabla_{d\phi(X)} grad H\right) \circ \phi = &-\frac{1}{\lambda^2} X(\ln\lambda)\, d\phi\left(grad\left(H\circ\phi\right)\right)\\ &+ \frac{1}{\lambda^2} d\ln\lambda\left(grad\left(H\circ\phi\right)\right) d\phi(X)\\ &- \frac{1}{\lambda^2} X(H\circ\phi)\, d\phi\left(grad \ln\lambda\right) + \frac{1}{\lambda^2} d\phi\left(\nabla_X grad\left(H\circ\phi\right)\right)\end{aligned} \tag{24}$$

and

$$\begin{aligned}\left(\nabla_{(grad H)\circ\phi}\left(grad H\right)\right)\circ\phi = &-\frac{1}{\lambda^4}\left|grad\left(H\circ\phi\right)\right|^2 d\phi\left(grad\ln\lambda\right)\\ &+ \frac{1}{2\lambda^4} d\phi\left(grad\left(\left|grad\left(H\circ\phi\right)\right|^2\right)\right).\end{aligned} \tag{25}$$

Proof of Lemma 2. For the term $\left(\nabla_{d\phi(X)} grad H\right)\circ\phi$, we have

$$\begin{aligned}\left(\nabla_{d\phi(X)} grad H\right) \circ \phi &= \nabla^\phi_X\left(grad H\circ\phi\right)\\ &= \nabla^\phi_X \frac{1}{\lambda^2} d\phi\left(grad\left(H\circ\phi\right)\right)\\ &= \frac{1}{\lambda^2}\nabla^\phi_X d\phi\left(grad\left(H\circ\phi\right)\right) + X\left(\frac{1}{\lambda^2}\right) d\phi\left(grad\left(H\circ\phi\right)\right)\\ &= \frac{1}{\lambda^2}\nabla^\phi_X d\phi\left(grad\left(H\circ\phi\right)\right) - \frac{2}{\lambda^2} X(\ln\lambda)\, d\phi\left(grad\left(H\circ\phi\right)\right).\end{aligned}$$

It is easy to see that (see [2])

$$\begin{aligned}\nabla^\phi_X d\phi\left(grad\left(H\circ\phi\right)\right) &= \nabla d\phi\left(X, grad\left(H\circ\phi\right)\right) + d\phi\left(\nabla_X grad\left(H\circ\phi\right)\right)\\ &= X(\ln\lambda)\, d\phi\left(grad\left(H\circ\phi\right)\right) + d\ln\lambda\left(grad\left(H\circ\phi\right)\right) d\phi(X)\\ &= -X(H\circ\phi)\, d\phi\left(grad\ln\lambda\right) + \frac{1}{\lambda^2} d\phi\left(\nabla_X grad\left(H\circ\phi\right)\right).\end{aligned}$$

Then

$$\begin{aligned}\left(\nabla_{d\phi(X)} grad H\right) \circ \phi = &-\frac{1}{\lambda^2} X(\ln\lambda)\, d\phi\left(grad\left(H\circ\phi\right)\right)\\ &+ \frac{1}{\lambda^2} d\ln\lambda\left(grad\left(H\circ\phi\right)\right) d\phi(X)\\ &- \frac{1}{\lambda^2} X(H\circ\phi)\, d\phi\left(grad \ln\lambda\right) + \frac{1}{\lambda^2} d\phi\left(\nabla_X grad\left(H\circ\phi\right)\right).\end{aligned}$$

Now we will simplify the term $\left(\nabla_{(grad H)\circ\phi}\left(grad H\right)\right)\circ\phi$, we have

$$(grad H)\circ\phi = \frac{1}{\lambda^2} d\phi\left(grad\left(H\circ\phi\right)\right),$$

it follows that

$$\left(\nabla_{(grad H)\circ\phi}\left(grad H\right)\right)\circ\phi = \frac{1}{\lambda^2}\left(\nabla_{d\phi(grad(H\circ\phi))}\left(grad H\right)\right)\circ\phi.$$

If we replace $X = grad\,(H \circ \phi)$ in (24), we obtain

$$\left(\nabla_{(grad H)\circ\phi}\,(grad H)\right)\circ\phi = -\frac{1}{\lambda^4}|grad\,(H\circ\phi)|^2 d\phi\,(grad\ln\lambda) + \frac{1}{2\lambda^4}d\phi\left(grad\left(|grad\,(H\circ\phi)|^2\right)\right).$$

Proof of Theorem 5. By definition, we have

$$\tau_{2,H}(\phi) = \tau_2(\phi) - J_\phi\left((grad H)\circ\phi\right) - \nabla_{\tau(\phi)}(grad H)\circ\phi - \left(\nabla_{(grad H)\circ\phi}(grad H)\right)\circ\phi. \tag{26}$$

We will study term by term the right-hand of this expression. For the first term $\tau_2(\phi)$, we have by Theorem 4

$$\tau_2(\phi) = (n-2)\,d\phi\,(T_1) \tag{27}$$

where

$$T_1 = grad\,\Delta\ln\lambda - \frac{(n-6)}{2}grad\left(|grad\ln\lambda|^2\right) + 2Ricci^M(grad\ln\lambda) - \left(2\,(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right)grad\ln\lambda.$$

Let us now simplify the term $J_\phi((grad H)\circ\phi)$, we have by definition

$$J_\phi\left((grad H)\circ\phi\right) = Tr_g\left(\nabla^\phi\right)^2(grad H)\circ\phi + Tr_g R^N\left((grad H)\circ\phi, d\phi\right)d\phi. \tag{28}$$

Using Lemma 1, we obtain

$$Tr_g\left(\nabla^\phi\right)^2(grad H)\circ\phi = \frac{1}{\lambda^2}d\phi\,(T_2) \tag{29}$$

where

$$T_2 = grad\,\Delta\,(H\circ\phi) - (\Delta\ln\lambda)\,grad\,(H\circ\phi) + Ricci^M(grad\,(H\circ\phi)) - \left((n-2)\,d\ln\lambda\,(grad\,(H\circ\phi)) + 2\,(\Delta\,(H\circ\phi))\right)grad\ln\lambda.$$

Since

$$(grad H)\circ\phi = \frac{1}{\lambda^2}d\phi\,(grad\,(H\circ\phi)),$$

it follows that

$$Tr_g R^N\left((grad H)\circ\phi, d\phi\right)d\phi = \frac{1}{\lambda^2}Tr_g R^N\left(d\phi\,(grad\,(H\circ\phi)), d\phi\right)d\phi.$$

Using Eq. (16), we get

$$\begin{aligned}Tr_g R^N\left(d\phi\,(grad\,(H\circ\phi)), d\phi\right)d\phi = {} & d\phi\left(Ricci^M(grad\,(H\circ\phi))\right)\\ & + (n-2)\,d\ln\lambda\,(grad\,(H\circ\phi))\,d\phi\,(grad\ln\lambda)\\ & - (n-2)\,d\phi\left(\nabla_{grad(H\circ\phi)}grad\ln\lambda\right)\\ & - \left(\Delta\ln\lambda + (n-2)\,|grad\ln\lambda|^2\right)\\ & \times d\phi\,(grad\,(H\circ\phi)).\end{aligned}$$

Then

$$Tr_g R^N\left(d\phi\,(grad\,(H\circ\phi)), d\phi\right)d\phi = \frac{1}{\lambda^2}d\phi\,(T_3),$$

where

$$T_3 = Ricci^M\left(grad\left(H\circ\phi\right)\right) + (n-2)\,d\ln\lambda\left(grad\left(H\circ\phi\right)\right)grad\ln\lambda \\ -\ (n-2)\,\nabla_{grad(H\circ\phi)}grad\ln\lambda - \left(\Delta\ln\lambda + (n-2)\left|grad\ln\lambda\right|^2\right)grad\left(H\circ\phi\right).$$

Finally, we conclude that

$$J_\phi\left((gradH)\circ\phi\right) = \frac{1}{\lambda^2}d\phi\left(T_2+T_3\right),$$

where

$$T_2+T_3 = grad\Delta\left(H\circ\phi\right) - \left(2\Delta\ln\lambda + (n-2)\left|grad\ln\lambda\right|^2\right)grad\left(H\circ\phi\right) \\ -2\left(\Delta\left(H\circ\phi\right)\right)grad\ln\lambda - (n-2)\,\nabla_{grad(H\circ\phi)}grad\ln\lambda \\ +\ 2Ricci^M\left(grad\left(H\circ\phi\right)\right).$$

Now let us look at the last term $\nabla_{\tau(\phi)}\left(gradH\right)\circ\phi$, we have

$$\tau\left(\phi\right) = (2-n)\,d\phi\left(grad\ln\lambda\right),$$

it follows that

$$\left(\nabla_{\tau(\phi)}\left(gradH\right)\right)\circ\phi = (2-n)\left(\nabla_{d\phi(grad\ln\lambda)}\left(gradH\right)\right)\circ\phi.$$

If we replace $X = grad\ln\lambda$ in (24), we obtain

$$\left(\nabla_{d\phi(X)}gradH\right)\circ\phi = -\frac{1}{\lambda^2}\left|grad\ln\lambda\right|^2d\phi\left(grad\left(H\circ\phi\right)\right) \\ +\frac{1}{\lambda^2}d\phi\left(\nabla_{grad\ln\lambda}grad\left(H\circ\phi\right)\right).$$

We conclude that

$$\left(\nabla_{\tau(\phi)}\left(gradH\right)\right)\circ\phi = \frac{1}{\lambda^2}d\phi\left(T_4\right),$$

where

$$T_4 = (n-2)\left|grad\ln\lambda\right|^2grad\left(H\circ\phi\right) - (n-2)\,\nabla_{grad\ln\lambda}grad\left(H\circ\phi\right).$$

To complete the proof, it remains to investigate the term $\left(\nabla_{(gradH)\circ\phi}\left(gradH\right)\right)\circ\phi$, we have by Lemma 2

$$\left(\nabla_{(gradH)\circ\phi}\left(gradH\right)\right)\circ\phi = -\frac{1}{\lambda^4}\left|grad\left(H\circ\phi\right)\right|^2d\phi\left(grad\ln\lambda\right) \\ +\frac{1}{2\lambda^4}d\phi\left(grad\left(\left|grad\left(H\circ\phi\right)\right|^2\right)\right),$$

then

$$\left(\nabla_{(gradH)\circ\phi}\left(gradH\right)\right)\circ\phi = \frac{1}{\lambda^4}d\phi\left(T_5\right),$$

where

$$T_5 = -\left|grad\left(H\circ\phi\right)\right|^2grad\ln\lambda + \frac{1}{2}grad\left(\left|grad\left(H\circ\phi\right)\right|^2\right).$$

Finally, we deduce that

$$\tau_{2,H}\left(\phi\right) = (n-2)\,d\phi\left(A\right) - \frac{1}{\lambda^2}d\phi\left(B\right) - \frac{1}{\lambda^4}d\phi\left(C\right),$$

where

$$A = grad\Delta \ln\lambda - \frac{(n-6)}{2} grad\left(|grad \ln\lambda|^2\right) + 2Ricci^M\left(grad \ln\lambda\right)$$
$$- \left(2\left(\Delta \ln\lambda\right) + (n-2)\,|grad \ln\lambda|^2\right) grad \ln\lambda,$$
$$B = grad\Delta\left(H\circ\phi\right) - 2\left(\Delta \ln\lambda\right) grad\left(H\circ\phi\right) - 2\left(\Delta\left(H\circ\phi\right)\right) grad \ln\lambda$$
$$- (n-2)\,\nabla_{grad(H\circ\phi)} grad \ln\lambda - (n-2)\,\nabla_{grad \ln\lambda} grad\left(H\circ\phi\right)$$
$$+ 2Ricci^M\left(grad\left(H\circ\phi\right)\right)$$

and

$$C = -|grad\left(H\circ\phi\right)|^2 grad \ln\lambda + \frac{1}{2} grad\left(|grad\left(H\circ\phi\right)|^2\right).$$

An immediate consequence of Theorem 5 is given by the following theorem

Theorem 6. *Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ be a conformal map of dilation λ. Then ϕ is biharmonic with potential if and only if*

$$(n-2)\,A - \frac{1}{\lambda^2} B - \frac{1}{\lambda^4} C = 0.$$

where

$$A = grad\Delta \ln\lambda - \frac{(n-6)}{2} grad\left(|grad \ln\lambda|^2\right) + 2Ricci^M\left(grad \ln\lambda\right)$$
$$- \left(2\left(\Delta \ln\lambda\right) + (n-2)\,|grad \ln\lambda|^2\right) grad \ln\lambda,$$
$$B = grad\Delta\left(H\circ\phi\right) - 2\left(\Delta \ln\lambda\right) grad\left(H\circ\phi\right) - 2\left(\Delta\left(H\circ\phi\right)\right) grad \ln\lambda$$
$$- (n-2)\,\nabla_{grad(H\circ\phi)} grad \ln\lambda - (n-2)\,\nabla_{grad \ln\lambda} grad\left(H\circ\phi\right)$$
$$+ 2Ricci^M\left(grad\left(H\circ\phi\right)\right)$$

and

$$C = -|grad\left(H\circ\phi\right)|^2 grad \ln\lambda + \frac{1}{2} grad\left(|grad\left(H\circ\phi\right)|^2\right).$$

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