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Some new generalized 2D Ostrowski-Grüss type inequalities on time scales

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Abstract. In this paper, we present some new generalized 2D Ostrowski-Grüss type integral inequalities on time scales, which on one hand extend some known results in the literature, and on the other hand unify corresponding continuous and discrete analysis. New bounds for the 2D Ostrowski-Grüss type inequalities are derived, some of which are sharp.

Mathematical subject classification: 26E70; 26D15; 26D10

Keywords: Ostrowski type inequality; Grüss type inequality; Time scales

1. Introduction

Recently, research on the Ostrowski type and Grüss type inequalities has been paid much attention. The Ostrowski type inequality, which was originally presented by Ostrowski in [32], can be used to estimate the absolute deviation of a function from its integral mean. The Grüss inequality, which can be used to estimate the absolute deviation of the integral of the product of two functions from the product of their respective integrals, was originally presented by Grüss in [15].

In the past few decades, various generalizations of the Ostrowski inequality and the Grüss inequality have been established (for example, see [40,8,11,36,2,37,3,10,41,20,1] and the references therein), while some new inequalities are established, one of which is the inequality of Ostrowski-Grüss type (for example, see [12,29,9,39,38,35,43,19,44,31, 13,28,21,42]). The first Ostrowski-Grüss type inequality was presented by Dragomir and Wang in [12] as follows:

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$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leqslant \frac{1}{4}(b-a)(\Gamma - \gamma),$$

where $f' \in L_1[a,b]$, and $\gamma \leqslant f'(x) \leqslant \Gamma$.

The bounds related to the Ostrowski-Grüss type inequality established above got improved in [29,9,39,38,19,44,31].

In [28], Lü extended UJEVIĆ's results [39] to 2D case, and established the following 2D Ostrowski-Grüss type inequality:

$$\begin{aligned}&|(c-a)(d-b)f(x,y) - \left(x - \frac{a+c}{2}\right)\left(y - \frac{b+d}{2}\right)[f(c,d) - f(a,d) - f(c,b)\\&+ f(a,b)] - (d-b)\int_a^c f(s,y)ds - (c-a)\int_b^d f(x,t)dt + \int_a^c \int_b^d f(s,t)dsdt|\\&\leqslant \frac{(7(c-a)(d-b))^{\frac{3}{2}}}{12}\sqrt{\sigma(f'')},\end{aligned}$$

where $f: \ [a,c] \times [b,d] \to \mathbb{R}$ is an absolutely continuous function, whose partial derivative is of order 2 $f'' \in L^2((a,c) \times (b,d))$, and $\sigma(f) = \|f\|^2 - \frac{1}{(c-a)(d-b)}\left(\int_a^c \int_b^d f(s,t)dsdt\right)^2$.

In [21], Liu extended the inequality above, and established a more generalized 2D Ostrowski-Grüss type inequality.

On the other hand, Hilger [16] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied the Ostrowski type and Grüss type inequalities on time scales (see [23,33,5,26,34,18,24,25,17,22,4,27,30]). But we notice that 2D Ostrowski-Grüss type inequality on time scales has been paid little attention in the literature.

Motivated by the above work, in this paper, we will establish some more generalized 2D Ostrowski-Grüss type inequalities on time scales. New bounds related to the Ostrowski-Grüss type inequalities are derived, and some of them are sharp. The established results unify continuous and discrete analysis, and extend some known results in the literature.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}_0$ denotes the set of nonnegative integers. $\mathbb{T}_1, \ \mathbb{T}_2$ denote two arbitrary time scales, and for an interval $[a,b]$, $[a,b]_{\mathbb{T}_i} := [a,b] \bigcap \mathbb{T}_i, \ i = 1, 2$. For the sake of convenience, we denote the forward jump operators on $\mathbb{T}_1, \ \mathbb{T}_2$ by σ uniformly. Finally, a point $t \in \mathbb{T}_i$ is said to be right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}_i$.

Definition 1.1. Let $\mathbb{T}$ be a time scale, then $h_k : \mathbb{T}^2 \to \mathbb{R}, \ k = 0, 1, 2\ldots$ are defined by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau, \ \forall s,t \in \mathbb{T},$$

where $h_0(t,s) = 1$.

Remark 1.2. If $\mathbb{T} = \mathbb{R}$, then $h_2(t,s) = \frac{(t-s)^2}{2}$. If $\mathbb{T} = \mathbb{Z}$, then $h_2(t,s) = \frac{(t-s)(t-s-1)}{2}$. If $\mathbb{T} = q^{\mathbb{N}_0}$, then $h_2(t,s) = \frac{(t-s)(t-qs)}{1+q}$.

Definition 1.3. Let $\mathbb{T}$ be a time scale, then for a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f at t is denoted by $f^{\Delta}(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s| \text{ for all } s \in \mathfrak{U}.$$

Remark 1.4. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = f(t+1) - f(t)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, then $f^{\Delta}(t) = \frac{f(qt)-f(t)}{t(q-1)}$.

For more details about the calculus of time scales, we refer the reader to [6,7].

2. Main results

Lemma 2.1 (*Generalized montgomery identity*). *Let* $a, b, s, x \in \mathbb{T}_1, c, d, t, y \in \mathbb{T}_2$ *with* $a < b$, $c < d$. $f : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \to \mathbb{R}$ *is* $\Delta_1\Delta_2$ *differentiable.* $\lambda \in [0,1]$ *is such that* $a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2} \in \mathbb{T}_1, c + \lambda\frac{d-c}{2}, d - \lambda\frac{d-c}{2} \in \mathbb{T}_2$ *and* $x \in \left[a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2}\right]_{\mathbb{T}_1}$, $y \in \left[c + \lambda\frac{d-c}{2}, d - \lambda\frac{d-c}{2}\right]_{\mathbb{T}_2}$. *Then*

$$\begin{aligned}
&(1-\lambda)^2 f(x,y) + (1-\lambda)\frac{\lambda}{2}[f(a,y) + f(b,y) + f(x,c) + f(x,d)] + \frac{\lambda^2}{4}[f(a,c) + f(b,c) \\
&\quad + f(a,d) + f(b,d)] - \frac{1}{b-a}(1-\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s \\
&\quad - \frac{1}{d-c}(1-\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t - \frac{\lambda}{2}\frac{1}{b-a}\int_a^b [f(\sigma(s),c) + f(\sigma(s),d)]\Delta_1 s \\
&\quad - \frac{\lambda}{2}\frac{1}{d-c}\int_c^d [f(a,\sigma(t)) + f(b,\sigma(t))]\Delta_2 t \\
&\quad + \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&\quad = \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s,
\end{aligned}$$

where $K(x,y,s,t) = K_1(x,s)K_2(y,t)$, *and*

$$K_1(x,s) = \begin{cases} s - \left(a + \lambda\frac{b-a}{2}\right), & s \in [a,x)_{\mathbb{T}_1} \\ s - \left(b - \lambda\frac{b-a}{2}\right), & s \in [x,b]_{\mathbb{T}_1} \end{cases},$$

$$K_2(y,t) = \begin{cases} t - \left(c + \lambda\frac{d-c}{2}\right), & t \in [c,y)_{\mathbb{T}_2} \\ t - \left(d - \lambda\frac{d-c}{2}\right), & t \in [y,d]_{\mathbb{T}_2} \end{cases}. \tag{1}$$

Proof. We have the following observations:

$$
\begin{aligned}
&\int_a^b \int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s = \\
&\quad \int_a^x \int_c^y \left[s-\left(a+\lambda\tfrac{b-a}{2}\right)\right]\left[t-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s \\
&\quad +\int_a^x \int_y^d \left[s-\left(a+\lambda\tfrac{b-a}{2}\right)\right]\left[t-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s \\
&\quad +\int_x^b \int_c^y \left[s-\left(b-\lambda\tfrac{b-a}{2}\right)\right]\left[t-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s \\
&\quad +\int_x^b \int_y^d \left[s-\left(b-\lambda\tfrac{b-a}{2}\right)\right]\left[t-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s \\
&= \int_a^x \left[s-\left(a+\lambda\tfrac{b-a}{2}\right)\right]\left\{\left[y-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,y)}{\Delta_1 s}-\left[c-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,c)}{\Delta_1 s}\right\}\Delta_1 s \\
&\quad -\int_a^x \int_c^y \left[s-\left(a+\lambda\tfrac{b-a}{2}\right)\right]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s \\
&\quad +\int_a^x \left[s-\left(a+\lambda\tfrac{b-a}{2}\right)\right]\left\{\left[d-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,d)}{\Delta_1 s}-\left[y-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,y)}{\Delta_1 s}\right\}\Delta_1 s \\
&\quad -\int_a^x \int_y^d \left[s-\left(a+\lambda\tfrac{b-a}{2}\right)\right]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s \\
&\quad +\int_x^b \left[s-\left(b-\lambda\tfrac{b-a}{2}\right)\right]\left\{\left[y-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,y)}{\Delta_1 s}-\left[c-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,c)}{\Delta_1 s}\right\}\Delta_1 s \\
&\quad -\int_x^b \int_c^y \left[s-\left(b-\lambda\tfrac{b-a}{2}\right)\right]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s \\
&\quad +\int_x^b \left[s-\left(b-\lambda\tfrac{b-a}{2}\right)\right]\left\{\left[d-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,d)}{\Delta_1 s}-\left[y-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\frac{\partial f(s,y)}{\Delta_1 s}\right\}\Delta_1 s \\
&\quad -\int_x^b \int_y^d \left[s-\left(b-\lambda\tfrac{b-a}{2}\right)\right]\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\Delta_1 s \\
&= \left[y-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[x-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(x,y)-\left[a-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(a,y)-\int_a^x f(\sigma(s),y)\Delta_1 s\right\} \\
&\quad -\left[c-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[x-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(x,c)-\left[a-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(a,c)-\int_a^x f(\sigma(s),c)\Delta_1 s\right\} \\
&\quad -\int_c^y \left\{\left[x-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(x,\sigma(t))-\left[a-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(a,\sigma(t))\right\}\Delta_2 t+\int_a^x \int_c^y f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&\quad +\left[d-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[x-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(x,d)-\left[a-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(a,d)-\int_a^x f(\sigma(s),d)\Delta_1 s\right\} \\
&\quad -\left[y-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[x-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(x,y)-\left[a-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(a,y)-\int_a^x f(\sigma(s),y)\Delta_1 s\right\} \\
&\quad -\int_y^d \left\{\left[x-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(x,\sigma(t))-\left[a-\left(a+\lambda\tfrac{b-a}{2}\right)\right]f(a,\sigma(t))\right\}\Delta_2 t+\int_a^x \int_y^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&\quad +\left[y-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[b-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(b,y)-\left[x-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(x,y)-\int_x^b f(\sigma(s),y)\Delta_1 s\right\} \\
&\quad -\left[c-\left(c+\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[b-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(b,c)-\left[x-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(x,c)-\int_x^b f(\sigma(s),c)\Delta_1 s\right\} \\
&\quad -\int_c^y \left\{\left[b-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(b,\sigma(t))-\left[x-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(x,\sigma(t))\right\}\Delta_2 t+\int_x^b \int_c^y f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&\quad +\left[d-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[b-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(b,d)-\left[x-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(x,d)-\int_x^b f(\sigma(s),d)\Delta_1 s\right\} \\
&\quad -\left[y-\left(d-\lambda\tfrac{d-c}{2}\right)\right]\left\{\left[b-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(b,y)-\left[x-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(x,y)-\int_x^b f(\sigma(s),y)\Delta_1 s\right\} \\
&\quad -\int_y^d \left\{\left[b-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(b,\sigma(t))-\left[x-\left(b-\lambda\tfrac{b-a}{2}\right)\right]f(x,\sigma(t))\right\}\Delta_2 t+\int_x^b \int_y^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&= (1-\lambda)^2(b-a)(d-c)f(x,y)+(b-a)(d-c)(1-\lambda)\tfrac{\lambda}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)] \\
&\quad +(b-a)(d-c)\tfrac{\lambda^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]-(d-c)(1-\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s \\
&\quad -(b-a)(1-\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t-\tfrac{\lambda}{2}(d-c)\int_a^b [f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s \\
&\quad -\tfrac{\lambda}{2}(b-a)\int_c^d [f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t+\int_a^b \int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s,
\end{aligned}
$$

which is the desired result. □

Theorem 2.2. *Under the conditions of Lemma* 2.1, *if* $f^{\Delta_1\Delta_2} \in L_2((a,b)_{\mathbb{T}_1} \times (c,d)_{\mathbb{T}_2})$, then we have

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 f(x,y) + (1-\lambda)\frac{\lambda}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)] \\
&+\frac{\lambda^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)] - \frac{1}{b-a}(1-\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s \\
&-\frac{1}{d-c}(1-\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t - \frac{\lambda}{2}\frac{1}{b-a}\int_a^b [f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s \\
&-\frac{\lambda}{2}\frac{1}{d-c}\int_c^d [f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}\times \\
&\left[h_2(x,a+\lambda\frac{b-a}{2}) - h_2\left(a,a+\lambda\frac{b-a}{2}\right) + h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right]\times \\
&\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right) + h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right]\Bigg| \\
&\leqslant \Bigg\{\Bigg[\frac{b^3-a^3}{3} - 2\left(a+\lambda\frac{b-a}{2}\right)\left(h_2\left(x,a+\lambda\frac{b-a}{2}\right) - h_2\left(a,a+\lambda\frac{b-a}{2}\right)\right) - \left(a+\lambda\frac{b-a}{2}\right)^2(x-a) \\
&-2\left(b-\lambda\frac{b-a}{2}\right)\left(h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right) - \left(b-\lambda\frac{b-a}{2}\right)^2(b-x)\Bigg]\times \\
&\Bigg[\frac{d^3-c^3}{3} - 2\left(c+\lambda\frac{d-c}{2}\right)\left(h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right)\right) - \left(c+\lambda\frac{d-c}{2}\right)^2(y-c) \\
&-2\left(d-\lambda\frac{d-c}{2}\right)\left(h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right) - \left(d-\lambda\frac{d-c}{2}\right)^2(d-y)\Bigg] \\
&-\frac{1}{(b-a)(d-c)}\Bigg[\left(h_2\left(x,a+\lambda\frac{b-a}{2}\right) - h_2\left(a,a+\lambda\frac{b-a}{2}\right) + h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right)\times \\
&\left(h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right) + h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right)\Bigg]^2\Bigg\}^{\frac{1}{2}}\sqrt{T(f^{\Delta_1\Delta_2})},
\end{aligned} \tag{2}$$

where

$$T(f) = \int_a^b\int_c^d f^2(s,t)\Delta_2 t\Delta_1 s - \frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d f(s,t)\Delta_2 t\Delta_1 s\right)^2.$$

The inequality Eq. (2) *is sharp in the sense that the coefficient constant 1 of the right-hand side of it cannot be replaced by a smaller one.*

Proof. From the definition of $K(x, y, s, t)$ we obtain

$$\begin{aligned}
&\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s = \int_a^b K_1(x,s)\Delta_1 s\int_c^d K_2(y,t)\Delta_2 t \\
&=\left\{\int_a^x\left[s-\left(a+\lambda\frac{b-a}{2}\right)\right]\Delta_1 s + \int_x^b\left[s-\left(b-\lambda\frac{b-a}{2}\right)\right]\Delta_1 s\right\}\times\left\{\int_c^y\left[t-\left(c+\lambda\frac{d-c}{2}\right)\right]\Delta_2 t + \int_y^d\left[t-\left(d-\lambda\frac{d-c}{2}\right)\right]\Delta_2 t\right\} \\
&=\left[h_2\left(x,a+\lambda\frac{b-a}{2}\right) - h_2\left(a,a+\lambda\frac{b-a}{2}\right) + h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right] \\
&\times\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right) + h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right]
\end{aligned} \tag{3}$$

and

$$\int_a^b \int_c^d K^2(x,y,s,t)\Delta_2 t\Delta_1 s = \int_a^b K_1^2(x,s)\Delta_1 s\int_c^d K_2^2(y,t)\Delta_2 t$$

$$= \left\{\int_a^x \left[s-\left(a+\lambda\frac{b-a}{2}\right)\right]^2\Delta_1 s + \int_x^b \left[s-\left(b-\lambda\frac{b-a}{2}\right)\right]^2\Delta_1 s\right\}\times$$

$$\left\{\int_c^y \left[t-\left(c+\lambda\frac{d-c}{2}\right)\right]^2\Delta_2 t + \int_y^d \left[t-\left(d-\lambda\frac{d-c}{2}\right)\right]^2\Delta_2 t\right\}$$

$$= \left\{\int_a^x \left[s^2-2\left(a+\lambda\frac{b-a}{2}\right)\left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)-\left(a+\lambda\frac{b-a}{2}\right)^2\right]\Delta_1 s\right.$$

$$\left.+\int_x^b \left[s^2-2\left(b-\lambda\frac{b-a}{2}\right)\left(s-\left(b-\lambda\frac{b-a}{2}\right)\right)-\left(b-\lambda\frac{b-a}{2}\right)^2\right]\Delta_1 s\right\}\times$$

$$\left\{\int_c^y \left[t^2-2\left(c+\lambda\frac{d-c}{2}\right)\left(t-\left(c+\lambda\frac{d-c}{2}\right)\right)-\left(c+\lambda\frac{d-c}{2}\right)^2\right]\Delta_2 t\right.$$

$$\left.+\int_y^d \left[t^2-2\left(d-\lambda\frac{d-c}{2}\right)\left(t-\left(d-\lambda\frac{d-c}{2}\right)\right)-\left(b-\lambda\frac{d-c}{2}\right)^2\right]\Delta_2 t\right\}$$

$$\leqslant \left\{\int_a^x \left[\frac{s^2+s\sigma(s)+(\sigma(s))^2}{3}-2\left(a+\lambda\frac{b-a}{2}\right)\left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)-\left(a+\lambda\frac{b-a}{2}\right)^2\right]\Delta_1 s\right.$$

$$\left.+\int_x^b \left[\frac{s^2+s\sigma(s)+(\sigma(s))^2}{3}-2\left(b-\lambda\frac{b-a}{2}\right)\left(s-\left(b-\lambda\frac{b-a}{2}\right)\right)-\left(b-\lambda\frac{b-a}{2}\right)^2\right]\Delta_1 s\right\}\times$$

$$\left\{\int_c^y \left[\frac{t^2+t\sigma(t)+(\sigma(t))^2}{3}-2\left(c+\lambda\frac{d-c}{2}\right)\left(t-\left(c+\lambda\frac{d-c}{2}\right)\right)-\left(c+\lambda\frac{d-c}{2}\right)^2\right]\Delta_2 t\right.$$

$$\left.+\int_y^d \left[\frac{t^2+t\sigma(t)+(\sigma(t))^2}{3}-2\left(d-\lambda\frac{d-c}{2}\right)\left(t-\left(d-\lambda\frac{d-c}{2}\right)\right)-\left(d-\lambda\frac{d-c}{2}\right)^2\right]\Delta_2 t\right\}$$

$$= \left\{\frac{x^3-a^3}{3}-2\left(a+\lambda\frac{b-a}{2}\right)\left[h_2(x,a+\lambda\frac{b-a}{2})-h_2\left(a,a+\lambda\frac{b-a}{2}\right)\right]-\left(a+\lambda\frac{b-a}{2}\right)^2(x-a)\right.$$

$$\left.+\frac{b^3-x^3}{3}-2\left(b-\lambda\frac{b-a}{2}\right)\left[h_2\left(b,b-\lambda\frac{b-a}{2}\right)-h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right]-\left(b-\lambda\frac{b-a}{2}\right)^2(b-x)\right\}\times$$

$$\left\{\frac{y^3-c^3}{3}-2\left(c+\lambda\frac{d-c}{2}\right)\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right)-h_2\left(c,c+\lambda\frac{d-c}{2}\right)\right]-\left(c+\lambda\frac{d-c}{2}\right)^2(y-c)\right.$$

$$\left.+\frac{d^3-y^3}{3}-2\left(d-\lambda\frac{d-c}{2}\right)\left[h_2\left(d,d-\lambda\frac{d-c}{2}\right)-h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right]-\left(d-\lambda\frac{b-a}{2}\right)^2(d-y)\right\}. \tag{4}$$

Furthermore, we have

$$\int_a^b \int_c^d \left\{ \left[K(x,y,s,t) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s \right] \right.$$
$$\left. \times \left[\frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \right] \right\} \Delta_2 t \Delta_1 s$$
$$= \int_a^b \int_c^d K(x,y,s,t) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s$$
$$\times \int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s =$$
$$\int_a^b \int_c^d K(x,y,s,t) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s - \frac{[f(b,d) - f(a,d) - f(b,c) + f(a,c)]}{(b-a)(d-c)}$$
$$\times \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s. \tag{5}$$

On the other hand,

$$\int_a^b \int_c^d \left\{ \left[K(x,y,s,t) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s \right] \times \right.$$
$$\left. \left[\frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \right] \right\} \Delta_2 t \Delta_1 s$$
$$\leqslant \left\| K(x,y,.,.) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s \right\|_2 \times$$
$$\left\| \frac{\partial^2 f(.,.)}{\Delta_1 s \Delta_2 t} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \right\|_2$$
$$= \left[\int_a^b \int_c^d K^2(x,y,s,t)\Delta_2 t \Delta_1 s - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s \right)^2 \right]^{\frac{1}{2}} \times$$
$$\left[\int_a^b \int_c^d \left(\frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t}\right)^2 \Delta_2 t \Delta_1 s - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \right)^2 \right]^{\frac{1}{2}}$$
$$= \left[\int_a^b \int_c^d K^2(x,y,s,t)\Delta_2 t \Delta_1 s - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d K(x,y,s,t)\Delta_2 t \Delta_1 s \right)^2 \right]^{\frac{1}{2}} \sqrt{T(f^{\Delta_1 \Delta_2})}. \tag{6}$$

Combining Eqs. (3)–(6) we can get the desired inequality Eq. (2). To prove the sharpness of Eq. (2), let $\mathbb{T}$ be right dense and $f(s,t) = f_1(s)f_2(t)$, where

$$f_1(s) = \begin{cases} h_2\left(s, a + \lambda \frac{b-a}{2}\right) - h_2\left(x, a + \lambda \frac{b-a}{2}\right), & s \in [a,x)_{\mathbb{T}_1} \\ h_2\left(s, b - \lambda \frac{b-a}{2}\right) - h_2\left(x, b - \lambda \frac{b-a}{2}\right), & s \in [x,b]_{\mathbb{T}_1} \end{cases},$$

$$f_2(t) = \begin{cases} h_2\left(t, c + \lambda \frac{d-c}{2}\right) - h_2\left(y, c + \lambda \frac{d-c}{2}\right), & t \in [c,y)_{\mathbb{T}_2} \\ h_2\left(t, d - \lambda \frac{d-c}{2}\right) - h_2\left(y, d - \lambda \frac{d-c}{2}\right), & t \in [x,b]_{\mathbb{T}_2} \end{cases}.$$

Then

$$\frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} = \begin{cases} \left[s-\left(a+\lambda\frac{b-a}{2}\right)\right]\left[t-\left(c+\lambda\frac{d-c}{2}\right)\right], & s\in[a,x)_{\mathbb{T}_1},\ t\in[c,y)_{\mathbb{T}_2} \\ \left[s-\left(a+\lambda\frac{b-a}{2}\right)\right]\left[t-\left(d-\lambda\frac{d-c}{2}\right)\right], & s\in[a,x)_{\mathbb{T}_1},\ t\in[y,d)_{\mathbb{T}_2} \\ \left[s-\left(b-\lambda\frac{b-a}{2}\right)\right]\left[t-\left(c+\lambda\frac{d-c}{2}\right)\right], & s\in[x,b)_{\mathbb{T}_1},\ t\in[c,y)_{\mathbb{T}_2} \\ \left[s-\left(b-\lambda\frac{b-a}{2}\right)\right]\left[t-\left(d-\lambda\frac{d-c}{2}\right)\right], & s\in[x,b)_{\mathbb{T}_1},\ t\in[y,d)_{\mathbb{T}_2} \end{cases}, \tag{7}$$

which implies $\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t} = K(x,y,s,t)$. So Eqs. (4) and (6) hold equality, which implies Eq. (2) holds equality, and the proof is complete.

In Theorem 2.2, if we take $\mathbb{T}_1,\ \mathbb{T}_2$ for some special time scales, then we immediately obtain the following two corollaries.

Corollary 2.3. *If we take $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 2.2, then we obtain the following inequality*

$$\begin{aligned}
&\Big|(1-\lambda)^2 f(x,y) + (1-\lambda)\tfrac{\lambda}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)] \\
&+\tfrac{\lambda^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)] - \tfrac{1}{b-a}(1-\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s \\
&-\tfrac{1}{d-c}(1-\lambda)\int_c^d f(x,t)dt - \tfrac{\lambda}{2}\tfrac{1}{b-a}\int_a^b [f(s,c)+f(s,d)]ds \\
&-\tfrac{\lambda}{2}\tfrac{1}{d-c}\int_c^d [f(a,t)+f(b,t)]dt + \tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dtds \\
&-\tfrac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}\times \\
&\left[\tfrac{\left(x-\left(a+\lambda\frac{b-a}{2}\right)\right)^2}{2} - \tfrac{\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)^2}{2} + \tfrac{\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)^2}{2} - \tfrac{\left(x-\left(b-\lambda\frac{b-a}{2}\right)\right)^2}{2}\right] \\
&\times\left[\tfrac{\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)^2}{2} - \tfrac{\left(c-\left(c+\lambda\frac{d-c}{2}\right)\right)^2}{2} + \tfrac{\left(d-\left(d-\lambda\frac{d-c}{2}\right)\right)^2}{2} - \tfrac{\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)^2}{2}\right]\Big| \\
&\leqslant \Bigg\{\left[\tfrac{b^3-a^3}{3} - 2\left(a+\lambda\tfrac{b-a}{2}\right)\left(\tfrac{\left(x-\left(a+\lambda\frac{b-a}{2}\right)\right)^2}{2} - \tfrac{\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)^2}{2}\right) - \left(a+\lambda\tfrac{b-a}{2}\right)^2(x-a)\right. \\
&\left.-2\left(b-\lambda\tfrac{b-a}{2}\right)\left(\tfrac{\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)^2}{2} - \tfrac{\left(x-\left(b-\lambda\frac{b-a}{2}\right)\right)^2}{2}\right) - \left(b-\lambda\tfrac{b-a}{2}\right)^2(b-x)\right] \\
&\times\left[\tfrac{d^3-c^3}{3} - 2\left(c+\lambda\tfrac{d-c}{2}\right)\left(\tfrac{\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)^2}{2} - \tfrac{\left(c-\left(c+\lambda\frac{d-c}{2}\right)\right)^2}{2}\right) - \left(c+\lambda\tfrac{d-c}{2}\right)^2(y-c)\right. \\
&\left.-2\left(d-\lambda\tfrac{d-c}{2}\right)\left(\tfrac{\left(d-\left(d-\lambda\frac{d-c}{2}\right)\right)^2}{2} - \tfrac{\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)^2}{2}\right) - \left(d-\lambda\tfrac{d-c}{2}\right)^2(d-y)\right] \\
&\quad -\tfrac{1}{(b-a)(d-c)}\left[\left(\tfrac{\left(x-\left(a+\lambda\frac{b-a}{2}\right)\right)^2}{2} - \tfrac{\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)^2}{2} + \tfrac{\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)^2}{2} - \tfrac{\left(x-\left(b-\lambda\frac{b-a}{2}\right)\right)^2}{2}\right)\right. \\
&\quad\left.\times\left(\tfrac{\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)^2}{2} - \tfrac{\left(c-\left(c+\lambda\frac{d-c}{2}\right)\right)^2}{2} + \tfrac{\left(d-\left(d-\lambda\frac{d-c}{2}\right)\right)^2}{2} - \tfrac{\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)^2}{2}\right)\right]^2\Bigg\}^{\frac{1}{2}}\sqrt{T(f''_{st})},
\end{aligned}$$

where

$$T(f) = \int_a^b\int_c^d f^2(s,t)dtds - \frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d f(s,t)dtds\right)^2.$$

Corollary 2.4. *If we take* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ *in Theorem* 2.2, *then we obtain the following inequality*

$$
\begin{aligned}
&\Big|(1-\lambda)^2 f(x,y)+(1-\lambda)\tfrac{\lambda}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]\\
&+\tfrac{\lambda^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]-\tfrac{1}{b-a}(1-\lambda)\sum_{s=a}^{b-1} f(s+1,y)\\
&-\tfrac{1}{d-c}(1-\lambda)\sum_{t=c}^{d-1} f(x,t+1)-\tfrac{\lambda}{2}\tfrac{1}{b-a}\sum_{s=a}^{b-1}[f(s+1,c)+f(s+1,d)]\\
&-\tfrac{\lambda}{2}\tfrac{1}{d-c}\sum_{t=c}^{d-1}[f(a,t+1)+f(b,t+1)]+\tfrac{1}{(b-a)(d-c)}\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\\
&-\tfrac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}\times\\
&\Bigg[\tfrac{\left(x-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(x-\left(a+\lambda\frac{b-a}{2}\right)-1\right)}{2}-\tfrac{\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(a-\left(a+\lambda\frac{b-a}{2}\right)-1\right)}{2}\\
&+\tfrac{\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(b-\left(b+\lambda\frac{b-a}{2}\right)-1\right)}{2}-\tfrac{\left(x-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(b-\left(b-\lambda\frac{b-a}{2}\right)-1\right)}{2}\Bigg]\times\\
&\Bigg[\tfrac{\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)\left(y-\left(c+\lambda\frac{d-c}{2}\right)-1\right)}{2}-\tfrac{\left(c-\left(c+\lambda\frac{d-c}{2}\right)\right)\left(c-\left(c+\lambda\frac{d-c}{2}\right)-1\right)}{2}\\
&+\tfrac{\left(d-\left(d-\lambda\frac{d-c}{2}\right)\right)\left(d-\left(d+\lambda\frac{d-c}{2}\right)-1\right)}{2}-\tfrac{\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)\left(d-\left(d-\lambda\frac{d-c}{2}\right)-1\right)}{2}\Bigg]\Bigg|\\
&\leqslant\Bigg\{\Bigg[\tfrac{b^3-a^3}{3}-2\left(a+\lambda\tfrac{b-a}{2}\right)\Bigg(\tfrac{\left(x-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(x-\left(a+\lambda\frac{b-a}{2}\right)-1\right)}{2}\\
&-\tfrac{\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(a-\left(a+\lambda\frac{b-a}{2}\right)-1\right)}{2}\Bigg)-\left(a+\lambda\tfrac{b-a}{2}\right)^2(x-a)\\
&-2\left(b-\lambda\tfrac{b-a}{2}\right)\Bigg(\tfrac{\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(b-\left(b+\lambda\frac{b-a}{2}\right)-1\right)}{2}-\tfrac{\left(x-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(b-\left(b-\lambda\frac{b-a}{2}\right)-1\right)}{2}\Bigg)\\
&-\left(b-\lambda\tfrac{b-a}{2}\right)^2(b-x)\Bigg]\times\\
&\Bigg[\tfrac{d^3-c^3}{3}-2\left(c+\lambda\tfrac{d-c}{2}\right)\Bigg(\tfrac{\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)\left(y-\left(c+\lambda\frac{d-c}{2}\right)-1\right)}{2}\\
&-\tfrac{\left(c-\left(c+\lambda\frac{d-c}{2}\right)\right)\left(c-\left(c+\lambda\frac{d-c}{2}\right)-1\right)}{2}\Bigg)-\left(c+\lambda\tfrac{d-c}{2}\right)^2(y-c)\\
&-2\left(d-\lambda\tfrac{d-c}{2}\right)\Bigg(\tfrac{\left(d-\left(d-\lambda\frac{d-c}{2}\right)\right)\left(d-(d+\lambda\frac{d-c}{2})-1\right)}{2}-\tfrac{\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)\left(d-\left(d-\lambda\frac{d-c}{2}\right)-1\right)}{2}\Bigg)\\
&-\left(d-\lambda\tfrac{d-c}{2}\right)^2(d-y)\Bigg]\\
&-\tfrac{1}{(b-a)(d-c)}\Bigg[\Bigg(\Bigg(\tfrac{\left(x-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(x-\left(a+\lambda\frac{b-a}{2}\right)-1\right)}{2}-\tfrac{\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(a-\left(a+\lambda\frac{b-a}{2}\right)-1\right)}{2}\\
&+\tfrac{\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(b-\left(b+\lambda\frac{b-a}{2}\right)-1\right)}{2}-\tfrac{\left(x-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(b-\left(b-\lambda\frac{b-a}{2}\right)-1\right)}{2}\Bigg)\times\\
&\Bigg(\tfrac{\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)\left(y-\left(c+\lambda\frac{d-c}{2}\right)-1\right)}{2}-\tfrac{\left(c-\left(c+\lambda\frac{d-c}{2}\right)\right)\left(c-\left(c+\lambda\frac{d-c}{2}\right)-1\right)}{2}\\
&+\tfrac{\left(d-\left(d-\lambda\frac{d-c}{2}\right)\right)\left(d-\left(d+\lambda\frac{d-c}{2}\right)-1\right)}{2}-\tfrac{\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)\left(d-\left(d-\lambda\frac{d-c}{2}\right)-1\right)}{2}\Bigg)\Bigg]^2\Bigg\}^{\frac{1}{2}}\sqrt{T(\Delta_2\Delta_1 f)},
\end{aligned}
$$

where $\Delta_2\Delta_1 f$ *denotes the difference on f with order two, and* $T(f) = \sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f^2(s,t) - \frac{1}{(b-a)(d-c)}\left(\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s,t)\right)^2$.

Remark 2.5. Corollary 2.3 is equivalent to [21, Theorem 3], and is the generalization of [44, Theorem 5] to 2D case. If we take $\lambda = 0$, then Corollary 2.3 reduces to [28, Theorem 4], and is the 2D generalization of [39, Theorem 4]. If we take $\lambda = \frac{1}{3}$, $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$, then Corollary 2.3 reduces to [28, Theorem 3], and is the 2D generalization of [39, Theorem 1]. So in this way, Theorem 2.2 is the further extension of some known results in the literature to arbitrary time scales.

Theorem 2.6. *Under the conditions of Lemma* 2.1, *if we assume* $f^{\Delta_1\Delta_2} \in L_\infty((a,b)_{\mathbb{T}_1} \times (c,d)_{\mathbb{T}_2})$, *then we have*

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 f(x,y) + (1-\lambda)\frac{\lambda}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)] \\
&+\frac{\lambda^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)] - \frac{1}{b-a}(1-\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s \\
&-\frac{1}{d-c}(1-\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t - \frac{\lambda}{2}\frac{1}{b-a}\int_a^b [f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s \\
&-\frac{\lambda}{2}\frac{1}{d-c}\int_c^d [f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2} \\
&\left[h_2\left(x,a+\lambda\frac{b-a}{2}\right) - h_2\left(a,a+\lambda\frac{b-a}{2}\right) + h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right]\times \\
&\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right) + h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right]\Bigg| \\
&\leqslant \sqrt{b-a}\Bigg\{\Bigg\{\frac{x^3-a^3}{3} - 2\left(a+\lambda\frac{b-a}{2}\right)\left[h_2\left(x,a+\lambda\frac{b-a}{2}\right) - h_2\left(a,a+\lambda\frac{b-a}{2}\right)\right] - \left(a+\lambda\frac{b-a}{2}\right)^2(x-a) \\
&+\frac{b^3-x^3}{3} - 2\left(b-\lambda\frac{b-a}{2}\right)\left[h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right] - \left(b-\lambda\frac{b-a}{2}\right)^2(b-x)\Bigg\}\times \\
&\Bigg\{\frac{y^3-c^3}{3} - 2\left(c+\lambda\frac{d-c}{2}\right)\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right)\right] - \left(c+\lambda\frac{d-c}{2}\right)^2(y-c) \\
&+\frac{d^3-y^3}{3} - 2\left(d-\lambda\frac{d-c}{2}\right)\left[h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right] - \left(d-\lambda\frac{b-a}{2}\right)^2(d-y)\Bigg\} \\
&-\frac{1}{(b-a)(d-c)}\Bigg\{\left[h_2\left(x,a+\lambda\frac{b-a}{2}\right) - h_2\left(a,a+\lambda\frac{b-a}{2}\right) + h_2\left(b,b-\lambda\frac{b-a}{2}\right) - h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right]\times \\
&\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right) - h_2\left(c,c+\lambda\frac{d-c}{2}\right) + h_2\left(d,d-\lambda\frac{d-c}{2}\right) - h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right]\Bigg\}^2\Bigg\}^{\frac{1}{2}}\|f^{\Delta_1\Delta_2}\|_\infty.
\end{aligned} \tag{8}$$

Proof. First we have the following observation:

$$\int_a^b \int_c^d \left[K(x,y,s,t) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\right] \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s$$

$$= \int_a^b \int_c^d K(x,y,s,t) \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s - \frac{1}{(b-a)(d-c)}$$

$$\int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s \int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s$$

$$= \int_a^b \int_c^d K(x,y,s,t) \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s - \frac{[f(b,d) - f(a,d) - f(b,c) + f(a,c)]}{(b-a)(d-c)}$$

$$\int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s. \tag{9}$$

Then

$$\left|\int_a^b \int_c^d \left[K(x,y,s,t) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\right] \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\right|$$

$$\leqslant \int_a^b \int_c^d \left|K(x,y,s,t) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\right|\Delta_2 t\Delta_1 s\|f^{\Delta_1\Delta_2}\|_\infty$$

$$\leqslant \sqrt{(b-a)(d-c)}\left[\int_a^b \int_c^d \left|K(x,y,s,t) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\right|^2 \Delta_2 t\Delta_1 s\right]^{\frac{1}{2}} \|f^{\Delta_1\Delta_2}\|_\infty$$

$$= \sqrt{(b-a)(d-c)}\left[\int_a^b \int_c^d K^2(x,y,s,t)\Delta_2 t\Delta_1 s - \frac{1}{(b-a)(d-c)}\left(\int_a^b \int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\right)^2\right]^{\frac{1}{2}} \|f^{\Delta_1\Delta_2}\|_\infty. \tag{10}$$

Then combining Eqs. (3), (4) and (10) we get the desired result.

Lemma 2.7 (*14*, *Lemma 2.8*). *(2D Grüss' inequality on time scales) Let* $f,\ g \in C_{rd}([a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}, \mathbb{R})$ *such that* $\phi \leqslant f(x,y) \leqslant \Phi$ *and* $\gamma \leqslant g(x,y) \leqslant \Gamma$ *for all* $x \in [a,b]_{\mathbb{T}_1}, y \in [c,d]_{\mathbb{T}_2}$, *where* ϕ, Φ, γ, Γ *are constants. Then we have*

$$\left|\frac{1}{(d-c)(b-a)} \int_a^b \int_c^d f(s,t)g(s,t)\Delta_2 t\Delta_1 s - \frac{1}{(d-c)(b-a)} \int_a^b \int_c^d f(s,t)\Delta_2 t\Delta_1 s \frac{1}{(d-c)(b-a)}\right.$$

$$\left.\int_a^b \int_c^d g(s,t)\Delta_2 t\Delta_1 s\right| \leqslant \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{11}$$

Theorem 2.8. *Under the conditions of* Lemma 2.1, *if there exist constants* K_1, K_2 *such that* $K_1 \leqslant \frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t} \leqslant K_2$, then we have

$$\Bigg|(1-\lambda)^2 f(x,y)+(1-\lambda)\frac{\lambda}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]$$
$$+\frac{\lambda^2}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]-\frac{1}{b-a}(1-\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s$$
$$-\frac{1}{d-c}(1-\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t-\frac{\lambda}{2}\frac{1}{b-a}\int_a^b [f(\sigma(s),c)+f(\sigma(s),d)]\Delta_1 s$$
$$-\frac{\lambda}{2}\frac{1}{d-c}\int_c^d [f(a,\sigma(t))+f(b,\sigma(t))]\Delta_2 t+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$
$$-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}$$
$$\left[h_2(x,a+\lambda\frac{b-a}{2})-h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)-h_2\left(x,b-\lambda\frac{b-a}{2}\right)\right]\times$$
$$\left[h_2\left(y,c+\lambda\frac{d-c}{2}\right)-h_2\left(c,c+\lambda\frac{d-c}{2}\right)+h_2\left(d,d-\lambda\frac{d-c}{2}\right)-h_2\left(y,d-\lambda\frac{d-c}{2}\right)\right]\Bigg|\leqslant\frac{1}{4}(K_2-K_1). \tag{12}$$

Proof. From the definition of $K(x,y,s,t)$ we have $\sup(K(x,y,s,t)) - \inf(K(x,y,s,t)) \leqslant (b-a)(d-c)$. So by Lemma 2.7 we obtain

$$\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s-\frac{1}{(b-a)(d-c)}$$
$$\times\int_a^b\int_c^d K(x,y,s,t)\Delta_2 t\Delta_1 s\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\Bigg|$$
$$\leqslant\frac{1}{4}(b-a)(d-c)(K_2-K_1).$$

The desired result can be obtained by the combination of (3) and Lemma 2.1. □

Remark 2.9. If we take $\lambda = 0$ in Theorem 2.8, then Theorem 2.8 becomes the 2D extension of [23, Theorem 4].

3. Conclusions

In this paper, we establish some generalized 2D Ostrowski-Grüss type inequalities on time scales, and derive some new estimates for them. Some of the estimates are sharp. The established results unify continuous and discrete analysis, and are further extensions of some known results in the literature.

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