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Some fixed point results for a class of *g*-monotone increasing multi-valued mappings☆

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Abstract. In this paper, we introduce the new notion of g-monotone mapping and prove some fixed point theorems for multi-valued and single-valued g-increasing mappings in partially ordered metric spaces. The mappings considered in this paper are assumed to satisfy certain metric inequalities which are established by an altering distance function. The presented results extend and improve the main results of Choudhury and Metiya [B.S. Choudhury, N. Metiya, Multi-valued and single-valued fixed point results in partially ordered metric spaces, Arab J. Math. Sci. 17 (2011) 135–151].

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1. Introduction and preliminaries

Let (X,d) denote a metric space. Denote by $B(X)$ the class of nonempty and bounded subsets of X. For A, $B \in B(X)$, functions $D(A,B)$ and $\delta(A,B)$ are defined below:

$$D(A,B) = \inf\{d(x,y) : x \in A, y \in B\},$$
$$\delta(A,B) = \sup\{d(x,y) : x \in A, y \in B\}.$$

If $A = \{x\}$, then we write $D(A,B) = D(x,B)$ and $\delta(A,B) = \delta(x,B)$. In addition, if $B = \{y\}$, then $D(A,B) = d(x,y)$ and $\delta(A,B) = d(x,y)$. Obviously, $D(A,B) \leqslant \delta(A,B)$. For all A, B, $C \in B(X)$, the definition of $\delta(A,B)$ gives the following formulas:

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(1) $\delta(A,B) = \delta(B,A)$,
(2) $\delta(A,B) \leqslant \delta(A,C) + \delta(C,B)$,
(3) $\delta(A,B) = 0$ iff $A = B = \{x\}$,
(4) $\delta(A,B) = \mathrm{diam}(A)$ (see [11,12]).

The fixed point theory of multi-valued functions is an important part of functional analysis (see [2,7,19,21,22,24]). Recently, the existence of fixed points for multi-valued mappings has been investigated by many authors (see [6,7,9,10,16,23]). It is notable that in these works the function $\delta(A,B)$ has been used.

For the sake of convenience, let us recall some notions.

Definition 1.1 [4]. Let A and B be two nonempty subsets of a partially ordered set $(X,\preceq)$. The relation between A and B is denoted and defined as follows: $A \prec B$, if for every $x \in A$, there exists $y \in B$ such that $x \preceq y$.

Definition 1.2 [14]. A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied: (i) ψ is monotone increasing and continuous, (ii) $\psi(t) = 0$ if and only if $t = 0$.

Some authors have used the above control function to study some fixed point problems in partially ordered metric spaces, in particular, we refer to [1,5–10,16–18,20,23]. Choudhury and Metiya, in partially ordered spaces, considered the existence of fixed point for a class of increasing multi-valued mappings satisfying certain metric inequalities yielded by a control function. Based on the work of Choudhury and Metiya [7], for multi-valued mappings, we introduce the new concept of g-monotone mapping and prove some existence results of fixed point for g-monotone increasing multi-valued mappings or single-valued mappings in partially ordered metric spaces.

It is remarked that recently coincidence fixed points of multi-valued Lipschitzian mappings on a metric space (hyperbolic metric space), without order and the mappings satisfying metric inequalities independent of altering distances have been studied by Khamsi and Khan [13] and Khan et al. [15], respectively.

Let (X,d) be a metric space, $T : X \to B(X)$ be a multi-valued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in T(x)$. An element $x \in X$ is said to be an endpoint (or stationary point) of T, if $T(x) = \{x\}$ (see [3]). Clearly, an endpoint (or stationary point) of T is a fixed point of T.

Definition 1.3 [15]. Let (X,d) be a metric space, $T : X \to B(X)$ be a multi-valued mapping, $g : X \to X$ be a mapping. If there exists $x \in X$ such that $g(x) \in Tx$, then x is said to be a coincidence point of g and T.

If $T : X \to X$ is a single-valued mapping and there exists $z \in X$ such that $g(z) = Tz$, then z is said to be a coincidence point of g and T.

A fixed point of T can be looked as a coincidence point of g and T if we take g as the identity mapping on X. But converse, in general, it is not true.

Definition 1.4. Let (X,d) be a metric space endowed with a partial order $\preceq$, $T : X \to B(X)$ be a multi-valued mapping and $g : X \to X$ a mapping. For any x, $y \in X$, if

$g(x) \preceq g(y)$ implies $Tx \prec Ty$, then T is said to be g-monotone increasing; if $g(x) \preceq g(y)$ implies $Ty \prec Tx$, then T is said to be g-monotone decreasing.

For x, $y \in X$, if $g(x) \preceq g(y)$ or $g(y) \preceq g(x)$, then we call x, y are g-comparable.

Definition 1.5. Let (X, d) be a metric space endowed with a partial order $\preceq$, $S : X \to X$ and $g : X \to X$. If for any sequence $\{x_n\}$ in X, $g(x_n) \to g(x)$ implies $S(x_n) \to S(x)$ as $n \to \infty$, then S is said to be a g-continuous mapping.

Example 1.1. Let $\mathbb{R}$ be the set of all real numbers with the usual metric d, that is, $d(x, y) = |x - y|$ for any $x, y \in \mathbb{R}$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a mapping defined as follows: for any $x \in \mathbb{R}$, if $x \neq 0$, then $g(x) = \sin\frac{1}{x}$ and $g(x) = 0$ if $x = 0$. Let $S(x) = 2g(x)$ for any $x \in \mathbb{R}$. Namely, $S(x) = 2\sin\frac{1}{x}$ if $x \neq 0$ and $S(x) = 0$ if $x = 0$. Obviously, S is g-continuous but not continuous.

Remark 1.1. The concepts of g-monotone mapping and g-continuous mapping are generalizations of the concepts of monotone mapping and continuous mapping, respectively, because if g is just taken as the identity mapping on X, then each g-monotone mapping is monotone. The same concerns g-continuous mapping.

2. Main results

In this section, we will prove some coincidence point results for $g-$ monotone increasing multi-valued mappings from the metric space (X, d) to itself, on which a partial order $\preceq$ is endowed.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $\{g(x_0)\} \prec Tx_0$,
(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \preceq g(x)$, for all n,
(iii) $\psi(\delta(Tx, Ty)) \leqslant \alpha\psi(\max\{d(g(x), g(y)),\ D(g(x), Tx),\ D(g(y), Ty),\ H(x, y)\})$, for all g-comparable x, $y \in X$, where $H(x, y) = \frac{D(g(x),Ty)+D(g(y),Tx)}{2}$, $0 < \alpha < 1$ and ψ is an altering distance function.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

Proof. By the assumption (i) and as g is a surjection, there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$ and $g(x_0) \preceq g(x_1)$. Also since T is a g-monotone increasing mapping, $Tx_0 \prec Tx_1$. Similarly, there exists $x_2 \in X$ such that $g(x_2) \in Tx_1$ and $g(x_1) \preceq g(x_2)$. Continuing this process we construct a monotone increasing sequence $\{g(x_n)\}$ in X such that $g(x_{n+1}) \in Tx_n$, for all $n \geqslant 0$, and

$$g(x_0) \preceq g(x_1) \preceq \cdots \preceq g(x_n) \preceq \cdots$$

If there is a positive integer k such that $g(x_k) = g(x_{k+1})$, then $g(x_k) = g(x_{k+1}) \in Tx_k$, this gives that x_k is a coincidence point of g and T. Hence we may assume $g(x_n) \neq g(x_{n+1})$, for all $n \geqslant 0$.

From the monotone property of ψ, we have, for all $n \geqslant 0$,

$$\begin{aligned}
&\psi(d(g(x_{n+1}), g(x_{n+2}))) \leqslant \psi(\delta(Tx_n, Tx_{n+1})) \\
&\qquad \leqslant \alpha\psi(\max\{d(g(x_n), g(x_{n+1})), D(g(x_n), Tx_n), D(g(x_{n+1}), \\
&Tx_{n+1}), H(x_n, x_{n+1})\}) \leqslant \alpha\psi(\max\{d(g(x_n), g(x_{n+1})), d(g(x_n), g(x_{n+1})), \\
&d(g(x_{n+1}), g(x_{n+2})), I(x_n, x_{n+1}, x_{n+2})\}),
\end{aligned}$$

where $I(x_n, x_{n+1}, x_{n+2}) = \frac{d(g(x_n), g(x_{n+2})) + d(g(x_{n+1}), g(x_{n+1}))}{2}$. Since

$$\frac{d(g(x_n), g(x_{n+2}))}{2} \leqslant \max\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2}))\},$$

it follows that

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \leqslant \alpha\psi(\max\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2}))\}). \tag{2.1}$$

Suppose that there exists some positive integer n such that

$$d(g(x_n), g(x_{n+1})) \leqslant d(g(x_{n+1}), g(x_{n+2})).$$

It follows from (2.1) that $\psi(d(g(x_{n+1}), g(x_{n+2}))) \leqslant \alpha\psi(d(g(x_{n+1}), g(x_{n+2})))$. Since $0 < \alpha < 1$, we get $d(g(x_{n+1}), g(x_{n+2})) = 0$ or $g(x_{n+1}) = g(x_{n+2})$, which contradicts our assumption that $g(x_{n+1}) \neq g(x_{n+2})$ for each n. Therefore

$$d(g(x_{n+1}), g(x_{n+2})) < d(g(x_n), g(x_{n+1}))$$

for all $n \geqslant 0$ and $\{d(g(x_n), g(x_{n+1}))\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geqslant 0$ such that $d(g(x_n), g(x_{n+1})) \to r$ as $n \to \infty$. Taking limit in (2.1) as $n \to \infty$, and from the continuity of ψ, we get $\psi(r) \leqslant \alpha\psi(r)$, which is a contradiction unless $r = 0$. Thus

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0. \tag{2.2}$$

Next we show that $\{g(x_n)\}$ is a Cauchy sequence. Suppose on the contrary that there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(g(x_{m(k)}), g(x_{n(k)})) \geqslant \varepsilon$. Assuming that $n(k)$ is the smallest such positive integer, we obtain $n(k) > m(k) > k$, $d(g(x_{n(k)}), g(x_{m(k)})) \geqslant \varepsilon$ and $d(g(x_{n(k)-1}), g(x_{m(k)})) < \varepsilon$. Now,

$$\varepsilon \leqslant d(g(x_{n(k)}), g(x_{m(k)})) \leqslant d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(x_{n(k)-1}), g(x_{m(k)})),$$

hence, $\varepsilon \leqslant d(g(x_{n(k)}), g(x_{m(k)})) < d(g(x_{n(k)}), g(x_{n(k)-1})) + \varepsilon$. Taking limit as $k \to \infty$ in the above inequality and using (2.2), we get

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)})) = \varepsilon. \tag{2.3}$$

Again

$$\begin{aligned}
d(g(x_{m(k)}), g(x_{n(k)})) \leqslant{}& d(g(x_{m(k)}), g(x_{m(k)+1})) + d(g(x_{m(k)+1}), g(x_{n(k)+1})) \\
&+ d(g(x_{n(k)+1}), g(x_{n(k)}))
\end{aligned}$$

and

$$d(g(x_{m(k)+1}), g(x_{n(k)+1})) \leqslant d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(x_{m(k)}), g(x_{n(k)})) + d(g(x_{n(k)}), g(x_{n(k)+1})).$$

Taking limit as $k \to \infty$ in the above inequalities and using (2.2) and (2.3), we have

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)+1})) = \varepsilon. \tag{2.4}$$

Similarly, we have

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)})) = \varepsilon \tag{2.5}$$

and

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)+1})) = \varepsilon. \tag{2.6}$$

For each positive integer k, we have $g(x_{m(k)}) \preceq g(x_{n(k)})$, so $g(x_{m(k)})$ and $g(x_{n(k)})$ are comparable. Then using the monotone property of ψ, we have

$$\begin{aligned}&\psi(d(g(x_{n(k)+1}), g(x_{m(k)+1}))) \leqslant \psi(\delta(Tx_{n(k)}, Tx_{m(k)}))\\&\quad \leqslant \alpha\psi(\max\{d(g(x_{n(k)}), g(x_{m(k)})), D(g(x_{n(k)}), Tx_{n(k)}),\\&D(g(x_{m(k)}), Tx_{m(k)}), H(x_{n(k)}, x_{m(k)})\}) \leqslant \alpha\psi(\max\{d(g(x_{n(k)}), g(x_{m(k)})), d(g(x_{n(k)}),\\&g(x_{n(k)+1})), d(g(x_{m(k)}), g(x_{m(k)+1})), J(n(k), m(k))\}),\end{aligned}$$

where $J(n(k), m(k)) = \frac{d(g(x_{m(k)}), g(x_{n(k)+1})) + d(g(x_{n(k)}), g(x_{m(k)+1}))}{2}$. Letting $k \to \infty$ in the above inequality, using (2.2)–(2.6) and the continuity of ψ, we have $\psi(\varepsilon) \leqslant \alpha\psi(\varepsilon)$ which contradicts a property of ψ. Hence $\{g(x_n)\}$ is a Cauchy sequence. From the completeness of X and the surjective property of g, there exists $z \in X$ such that

$$g(x_n) \to g(z)(n \to \infty). \tag{2.7}$$

By the assumption (ii), $g(x_n) \preceq g(z)$, for all n.

Then from the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}&\psi(\delta(g(x_{n+1}), Tz)) \leqslant \psi(\delta(Tx_n, Tz))\\&\quad \leqslant \alpha\psi(\max\{d(g(x_n), g(z)), D(g(x_n), Tx_n), D(g(z), Tz), H(x_n, z)\})\\&\quad \leqslant \alpha\psi(\max\{d(g(x_n), g(z)), d(g(x_n), g(x_{n+1})), \delta(g(z), Tz), K(x_n, x_{n+1}, z)\}),\end{aligned}$$

where $K(x_n, x_{n+1}, z) = \frac{\delta(g(x_n), Tz) + d(g(z), g(x_{n+1}))}{2}$. Taking the limit as $n \to \infty$ in the above inequality, using (2.2) and (2.7) and the continuity of ψ, we have

$$\psi(\delta(g(z), Tz)) \leqslant \alpha\psi(\delta(g(z), Tz)),$$

which implies that $\delta(g(z), Tz) = 0$ or $\{g(z)\} = Tz$. Thus z is a coincidence point of g and T.

In addition, suppose that g is an injection. Now we prove the uniqueness of the coincidence point of g and T.

Assume that z_1 is also a coincidence point of g and T, that is $g(z_1) \in Tz_1$, then

$$\begin{aligned}\psi(d(g(z), g(z_1))) &\leqslant \psi(\delta(Tz, Tz_1)) \\ &\leqslant \alpha\psi\left(\max\left\{d(g(z), g(z_1)), D(g(z), Tz), D(g(z_1), Tz_1), \frac{D(g(z), Tz_1) + D(g(z_1), Tz)}{2}\right\}\right) \\ &\leqslant \alpha\psi\left(\max\left\{d(g(z), g(z_1)), \frac{D(g(z), Tz_1) + D(g(z_1), Tz)}{2}\right\}\right), \\ &\leqslant \alpha\psi\left(\max\left\{d(g(z), g(z_1)), \frac{d(g(z), g(z_1)) + d(g(z_1), g(z))}{2}\right\}\right), \leqslant \alpha\psi(d(g(z), g(z_1))).\end{aligned}$$

The assumption $0 < \alpha < 1$ implies $d(g(z), g(z_1)) = 0$ or $g(z) = g(z_1)$. As g is an injection, obviously, $z = z_1$. $\square$

Remark 2.1. Taking mapping g as the identity mapping on X, we get ([7], Theorem 2.1). In addition, we prove that the coincidence point of g and T is unique, which is not discussed in [7]. Therefore, this result extends and improves Theorem 2.1 of Choudhury and Metiya [7].

Putting ψ the identity function in Theorem 2.1, we get the following result.

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping such that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\{g(x_0)\} \prec Tx_0$,
(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \preceq g(x)$, for all n,
(iii) $\delta(Tx, Ty) \leqslant \alpha \max\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), H(x, y)\}$, for all g-comparable $x, y \in X$, where $0 < \alpha < 1$.

Then there exists a coincidence point of g and T in X. Moreover, if g is an injection, then the coincidence point of g and T is unique.

The following corollary is a special case of Theorem 2.1 when T is a single-valued mapping.

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to X$ be a g-monotone increasing mapping such that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $g(x_0) \preceq Tx_0$,
(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \preceq g(x)$, for all n,
(iii) $\psi(d(T(x), T(y))) \leqslant \alpha\psi\,(\max\{d(g(x), g(y)), d(g(x), T(x)), d(g(y), T(y)), H(x, y)\})$, for all g-comparable $x, y \in X$, where $0 < \alpha < 1$.

Then there exists a coincidence point of g and T in X. Moreover, if g is an injection, then the coincidence point of g and T is unique.

In the following theorem we replace condition (ii) of the above corollary by "T is g-continuous".

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to X$ be g-monotone increasing and g-continuous such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $g(x_0) \preceq Tx_0$,
(ii) $\psi(d(T(x), T(y))) \leqslant \alpha\psi\ (\max\{d(g(x), g(y)),\ d(g(x), T(x)),\ d(g(y), T(y)),\ L(x, y)\})$, for all g-comparable $x, y \in X$, where $L(x, y) = \frac{d(g(x), T(y)) + d(g(y), T(x))}{2}$, $0 < \alpha < 1$.

Then there exists a coincidence point of g and T in X. Moreover, if g is an injection, then the coincidence point of g and T is unique.

Proof. We can consider T as a multi-valued mapping in which case $T(x)$ is a singleton set for every $x \in X$. Then considering the sequence $\{g(x_n)\}$ in the proof of Theorem 2.1 and arguing exactly as in the proof of Theorem 2.1, we get that $\{g(x_n)\}$ is a Cauchy sequence and $\lim_{n\to\infty} g(x_n) = g(z)$. Then, the g-continuity of T implies that

$$g(z) = \lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} T(x_n) = T(z).$$

This proves that z is a coincidence point of g and T. The uniqueness of the coincidence point of g and T can be proved as in Theorem 2.1 provided g is an injection. □

Theorem 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $\{g(x_0)\} \prec Tx_0$,*
(ii) *if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \preceq g(x)$, for all n,*
(iii) $\psi(\delta(Tx, Ty)) \leqslant \psi(\max\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), H(x, y)\})$

$$-\phi(\max\{d(g(x), g(y)), \delta(g(y), Ty)\})$$

for all g-comparable $x, y \in X$, where $0 < \alpha < 1$ and ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

Proof. We consider the sequence $\{g(x_n)\}$ of Theorem 2.1. If there exists a positive integer k such that $g(x_k) = g(x_{k+1})$, then x_k is a coincidence point of g and T. Hence we assume that $g(x_n) \neq g(x_{n+1})$, for all $n \geqslant 0$.

Using the monotone property of ψ, we have for all $n \geqslant 0$,

$$\begin{aligned}\psi(d(g(x_{n+1}), g(x_{n+2}))) &\leqslant \psi(\delta(Tx_n, Tx_{n+1})) \\ &\leqslant \psi(\max\{d(g(x_n), g(x_{n+1})), D(g(x_n), Tx_n), D(g(x_{n+1}), Tx_{n+1}), H(x_n, x_{n+1})\}) \\ &\quad - \phi(\max\{d(g(x_n), g(x_{n+1})), \delta(g(x_{n+1}), Tx_{n+1})\}) \\ &\leqslant \psi\left(\max\left\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2})), \frac{d(g(x_n), g(x_{n+2})) + d(g(x_{n+1}), g(x_{n+1}))}{2}\right\}\right) \\ &\quad - \phi(\max\{d(g(x_n), g(x_{n+1}), d(g(x_{n+1}), g(x_{n+2}))\}).\end{aligned}$$

Since $\frac{d(g(x_n),g(x_{n+2}))}{2} \leqslant \max\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2}))\}$, it follows that

$$\begin{aligned}\psi(d(g(x_{n+1}), g(x_{n+2}))) \leqslant \psi(&\max\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2}))\} \\ &- \phi(\max\{d(g(x_n), g(x_{n+1}), d(g(x_{n+1}), g(x_{n+2}))\}).\end{aligned}$$

Assume that $d(g(x_k), g(x_{k+1})) \leqslant d(g(x_{k+1}), g(x_{k+2}))$ for some positive integer k. Then from the above inequality, we have

$$\psi(d(g(x_{k+1}), g(x_{k+2}))) \leqslant \psi(d(g(x_{k+1}), g(x_{k+2})) - \phi(d(g(x_{k+1}), g(x_{k+2}))$$

this yields $\phi(d(g(x_{k+1}), g(x_{k+2}))) \leqslant 0$, which gives that $d(g(x_{k+1}), g(x_{k+2})) = 0$, that is $g(x_{k+1}) = g(x_{k+2})$, it contradicts our assumption that $g(x_n) \neq g(x_{n+1})$, for all $n \geqslant 0$. Therefore

$$d(g(x_{n+1}), g(x_{n+2})) < d(g(x_n), g(x_{n+1})), \quad \text{forall } n \geqslant 0$$

and $\{d(g(x_n), g(x_{n+1}))\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geqslant 0$ such that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = r. \tag{2.8}$$

From the above facts, we have, for all $n \geqslant 0$,

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \leqslant \psi(d(g(x_n), g(x_{n+1})) - \phi(d(g(x_n), g(x_{n+1}))).$$

Taking limit as $n \to \infty$ in the above inequality, and from the continuity of ψ and ϕ and by (2.8), we get $\psi(r) \leqslant \psi(r) - \phi(r)$, which is a contradiction unless $r = 0$. Thus

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0. \tag{2.9}$$

Next we show that $\{g(x_n)\}$ is a Cauchy sequence. If $\{g(x_n)\}$ is not a Cauchy sequence, then using an argument similar to that given in Theorem 2.1, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ and a positive constant ε, for which

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)})) = \varepsilon. \tag{2.10}$$

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)+1})) = \varepsilon. \tag{2.11}$$

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)})) = \varepsilon. \tag{2.12}$$

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)+1})) = \varepsilon. \tag{2.13}$$

For each positive integer k, $g(x_{n(k)})$ and $g(x_{m(k)})$ are comparable. Then using the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}
&\psi(d(g(x_{m(k)+1}), g(x_{n(k)+1}))) \leqslant \psi(\delta(Tx_{m(k)}, Tx_{n(k)}))\\
&\qquad \leqslant \psi(\max\{d(g(x_{m(k)}), g(x_{n(k)})), D(g(x_{m(k)}), Tx_{m(k)}),\\
&D(g(x_{n(k)}), Tx_{n(k)}), H(x_{m(k)}, x_{n(k)})\})\\
&\ - \phi(\max\{d(g(x_{m(k)}), g(x_{n(k)}), \delta(g(x_{n(k)}), Tx_{n(k)})\})\\
&\ \leqslant \psi(\max\{d(g(x_{m(k)}), g(x_{n(k)})), d(g(x_{m(k)}), g(x_{m(k)+1})), d(g(x_{n(k)}), g(x_{n(k)+1}),\\
&J(n(k), m(k))\}) - \phi(\max\{d(g(x_{m(k)}), g(x_{n(k)}), d(g(x_{n(k)}), g(x_{n(k)+1}))\}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using the continuity of ψ and ϕ, we have

$$\psi(\varepsilon) \leqslant \psi(\varepsilon) - \phi(\varepsilon),$$

which implies $\phi(\varepsilon) < 0$. It is in contradiction with the property of ϕ. Hence $\{g(x_n)\}$ is a Cauchy sequence. From the completeness of X and the surjection of g, there exists a $z \in X$ such that $g(x_n) \to g(z)$ as $n \to \infty$. By the assumption (ii), $g(x_n) \leqslant g(z)$, for all n. Then by the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}
&\psi(\delta(g(x_{n+1}), Tz) \leqslant \psi(\delta(Tx_n, Tz)\\
&\quad \leqslant \psi\Bigg(\max\Bigg\{d(g(x_n), g(z)), D(g(x_n), Tx_n), D(g(z), Tz),\\
&\frac{D(g(x_n), Tz) + D(g(z), Tx_n)}{2}\Bigg\}\Bigg) - \phi(\max\{d(g(x_n), g(z), \delta(g(z), Tz)\})\\
&\ \leqslant \psi\Bigg(\max\Bigg\{d(g(x_n), g(z)), d(g(x_n), g(x_{n+1})), D(g(z), Tz),\\
&\frac{D(g(x_n), Tz) + d(g(z), g(x_{n+1}))}{2}\Bigg\}\Bigg) - \phi(\max\{d(g(x_n), g(z)), \delta(g(z), Tz)\}).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, using the continuity of ψ and ϕ, we have

$$\psi(\delta(g(z), Tz)) \leqslant \psi(D(g(z), Tz)) - \phi(\delta(g(z), Tz)),$$

which implies that

$$\psi(\delta(g(z), Tz)) \leqslant \psi(\delta(g(z), Tz)) - \phi(\delta(g(z), Tz)).$$

This gives a contradiction unless $\delta(g(z), Tz)) = 0$ or $\{g(z)\} = Tz$, that is $\{g(z)\} = Tz$, so z is a coincidence point of g and T.

In addition, suppose that g is an injection. Now we prove the uniqueness of the coincidence point of g and T.

Assume that z_1 is also a coincidence point of g and T, that is $g(z_1) \in Tz_1$, then

$$\begin{aligned}
&\psi(\delta(g(z), g(z_1))) \leqslant \psi(\delta(Tz, Tz_1))\\
&\quad \leqslant \psi\left(\max\left\{d(g(z), g(z_1)), D(g(z), Tz), D(g(z_1), Tz_1),\right.\right.\\
&\left.\left.\frac{D(g(z), Tz_1) + D(g(z_1), Tz)}{2}\right\}\right) - \phi(\max\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\})\\
&\quad \leqslant \psi\left(\max\left\{d(g(z), g(z_1)), \frac{d(g(z), g(z_1)) + d(g(z_1), g(z))}{2}\right\}\right)\\
&- \phi(\max\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\})\\
&\quad \leqslant \psi(d(g(z), g(z_1))) - \phi(\max\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}).
\end{aligned}$$

Therefore, $\phi(\max\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}) = 0$. Suppose that $g(z) \neq g(z_1)$, then by a property of ϕ, we have $\phi(\max\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}) \neq 0$, which yields a contradiction to the above inequality. Thus $g(z) = g(z_1)$. Since g is an injection, $z = z_1$, that is, the coincidence point of g and T is unique. $\square$

Taking ψ as identity function in Theorem 2.3, we have the following result.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $\{g(x_0)\} \prec Tx_0$,
(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \preceq g(x)$, for all n,
(iii) $\delta(Tx, Ty) \leqslant \max\left\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), \frac{D(g(x),Ty)+D(g(y),Tx)}{2}\right\}$
$-\phi(\max\{d(g(x), g(y)), \delta(g(y), Ty)\})$

for all g-comparable x, $y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

The following corollary is a special case of Theorem 2.3 when T is a single-valued mapping.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to X$ be a g-monotone increasing single-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $g(x_0) \preceq T(x_0)$,
(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \preceq g(x)$, for all n,
(iii) $\psi(d(T(x), T(y))) \leqslant \psi(\max\{d(g(x), g(y)),\ d(g(x), T(x)),\ d(g(y), T(y)),\ L(x, y)\})$
$-\phi(\max\{d(g(x), g(y)), d(g(x), T(y))\})$

for all g-comparable $x, y \in X$, where ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

In the following theorem we replace condition (ii) of the above corollary by "T is g-continuous".

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $g : X \to X$ be a surjection and $T : X \to X$ be a g-continuous and g-monotone increasing single-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $g(x_0) \preceq T(x_0)$,
(ii) $\psi(d(T(x), T(y))) \leqslant \psi(\max\{d(g(x), g(y)),\ d(g(x), T(x)),\ d(g(y), T(y)),\ L(x, y)\})$
$-\phi(\max\{d(g(x), g(y)), d(g(y), T(y))\})$

for all g-comparable $x,y \in X$, where ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

Proof. We can treat T as a multi-valued mapping in which case $T(x)$ is a singleton set for every $x \in X$. Then we consider the sequence $\{g(x_n)\}$ as in the proof of Theorem 2.3 and arguing exactly as in the proof of Theorem 2.3, we have that $\{g(x_n)\}$ is a Cauchy sequence and $\lim_{n\to\infty} g(x_n) = g(z)$. Then, the g-continuity of T implies that

$$g(z) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} Tx_n = T(z).$$

This proves that z is a coincidence point of g and T. The uniqueness of z follows as before. $\square$

3. Applications

Let $X = \left\{(0,0), \left(0, -\frac{1}{\lambda_1}\right), \left(-\frac{1}{\lambda_2}, 0\right)\right\}$ be a subset of $\mathbb{R}^2$, $\lambda_2 \geqslant \lambda_1 \geqslant 1$, with the order $\preceq$ defined as : for $(x_1, y_1), (x_2, y_2) \in X$, $(x_1, y_1) \preceq (x_2, y_2)$ iff $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. Assume that $d : X \times X \to \mathbb{R}$ is a metric defined on X as

$$d(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}, \quad \text{for} \quad x = (x_1, y_1), y = (x_2, y_2).$$

Then (X, d) is a complete metric space.Let $T : X \to B(X)$ be defined as follows: $T((0,0)) = \{(0,0)\}$, $T\left(\left(0, -\frac{1}{\lambda_1}\right)\right) = \left\{(0,0), \left(-\frac{1}{\lambda_2}, 0\right)\right\}$, $T\left(\left(-\frac{1}{\lambda_2}, 0\right)\right) = \{(0,0)\}$.Define $g : X \to X$ as $g((0,0)) = (0,0)$, $g\left(\left(0, -\frac{1}{\lambda_1}\right)\right) = \left(-\frac{1}{\lambda_2}, 0\right)$, $g\left(\left(-\frac{1}{\lambda_2}, 0\right)\right) = \left(0, -\frac{1}{\lambda_1}\right)$.Then g is a continuous surjective mapping and T satisfies the properties mentioned in

Corollary 2.1. In fact, T is g-monotone increasing multi-valued mapping and we take $(0,0)$ as x_0, and so the condition (i) of Corollary 2.1 holds. The condition (ii) holds clearly. Now we show the condition (iii) of Corollary 2.1 is satisfied. From the definition of g, we get that only $(0,0)$ and $\left(-\frac{1}{\lambda_2},0\right)$, $\left(0,-\frac{1}{\lambda_1}\right)$ and $\left(-\frac{1}{\lambda_2},0\right)$ are g-comparable. Thus if $x=(0,0)$, $y=\left(-\frac{1}{\lambda_2},0\right)$, then $Tx=Ty=\{(0,0)\}$, which gives that the condition (iii) of Corollary 2.1 is satisfied. When $x=\left(0,-\frac{1}{\lambda_1}\right)$, $y=\left(-\frac{1}{\lambda_2},0\right)$,$\delta(Tx,Ty)=\frac{1}{\lambda_2}$. $\max\{d(g(x),g(y)),D(g(x),Tx),D(g(y),Ty),H(x,y)\}=\max\left\{\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right\}$. As $\lambda_2>\lambda_1$, so $0<\frac{\lambda_1}{\lambda_2}<1$. By putting $\frac{\lambda_1}{\lambda_2}<\alpha<1$, the condition (iii) holds. Hence from Corollary 2.1, we obtain that there exists a unique coincidence point of g and T in X.

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