ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 23 (2017) 124–132

Some congruences modulo 2 and 5 for bipartition with 5-core

NIPEN SAIKIA*, CHAYANIKA BORUAH

Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791112, Arunachal Pradesh, India

Received 13 October 2015; received in revised form 14 May 2016; accepted 24 May 2016
Available online 2 June 2016

Abstract. We find some congruences modulo 2 and 5 for the number of bipartitions with 5-core for a positive integer n in the spirit of Ramanujan.

Keywords: Bipartition; 5-core partition; Partition congruence

2010 Mathematics Subject Classification: 11P83; 05A15; 05A17

1. INTRODUCTION

A bipartition of a positive integer n is a pair of partitions (λ, μ) such that the sum of all of the parts is n. A bipartition with t-core is a pair of partitions (λ, μ) such that λ and μ are both t-cores. If $A_t(n)$ denotes the number of bipartitions with t-core of n, then $A_t(n)$ is defined by

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{(q^t;q^t)_\infty^{2t}}{(q;q)_\infty^2}, \tag{1.1}$$

where $(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^n)$. We note the following well known congruence property which can be proved by using binomial theorem: For any prime p and positive integer k,

$$(q^k;q^k)_\infty^p \equiv (q^{pk};q^{pk})_\infty \pmod p. \tag{1.2}$$

The function $A_t(n)$ defined in (1.1) have been studied by many mathematicians. Lin [8] discovered some interesting congruences modulo 4, 5, 7, and 8 for $A_3(n)$. Yao [10] established several infinite families of congruences modulo 3 and 9 for $A_9(n)$. Xia [9]

* Corresponding author.
E-mail addresses: nipennak@yahoo.com (N. Saikia), cboruah123@gmail.com (C. Boruah).
Peer review under responsibility of King Saud University.

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http://dx.doi.org/10.1016/j.ajmsc.2016.05.003

established several infinite families of congruences modulo 4, 8 and $\frac{4^k-1}{3}$ $(k \geq 2)$ for $A_3(n)$ and also generalized some results due to Lin and Yao. Baruah and Nath [1] also proved some results on $A_3(n)$.

In this paper, we are concerned with the function $A_5(n)$ which denotes the number of bipartition with 5-core of n and is given by

$$\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}. \tag{1.3}$$

In Section 3, we find some congruences modulo 2 and 5 for $A_5(n)$ in the spirit of Ramanujan. Section 2 is devoted to record some preliminary results.

2. Preliminaries

Ramanujan's general theta-function $f(a,b)$ [3, p. 35, Entry 19] is defined by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1. \tag{2.1}$$

Lemma 2.1 (*[4, Theorem 2.2]*). *For any prime $p \geq 5$, we have*

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty, \tag{2.2}$$

where $\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6}, & if\ p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6}, & if\ p \equiv -1 \pmod 6. \end{cases}$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p-1}{2}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

Lemma 2.2 (*[7, Theorem 1]*). *We have*

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^{20};q^{20})_\infty}.$$

Lemma 2.3 (*[6]*). *We have*

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}(F^{-4}(q^5) + qF^{-3}(q^5)$$
$$+ 2q^2F^{-2}(q^5) + 3q^3F^{-1}(q^5) + 5q^4 - 3q^5F(q^5)$$
$$+ 2q^6F^2(q^5) - q^7F^3(q^5) + q^8F^4(q^5)),$$

where $F(q) := q^{-1/5}R(q)$ and $R(q)$ is Rogers-Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}_{+\cdots}, \qquad |q| < 1.$$

Lemma 2.4 *([3, p. 39, Entry 24(ii)]). We have*

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}.$$

Lemma 2.5 *([2, p. 648, Theorem 2.1; Eqns. (2.1), (2.5) & (2.13)]). If*

$$\sum_{n=0}^{\infty} p_3(n)q^n = (q;q)_\infty^3, \tag{2.3}$$

then for any positive integer k,

$$p_3\left(3^{2k}n + \frac{3^{2k}-1}{8}\right) = (-3)^k p_3(n), \tag{2.4}$$

$$p_3\left(5^{2k}n + \frac{5^{2k}-1}{8}\right) = 5^k p_3(n) \tag{2.5}$$

and

$$p_3\left(7^{2k}n + \frac{7^{2k}-1}{8}\right) = (-7)^k p_3(n). \tag{2.6}$$

3. Congruences modulo 2 and 5 for $A_5(n)$

Theorem 3.1. *We have*

(i) $A_5(2n+1) \equiv 0 \pmod 2$.
(ii) $A_5(8n+4) \equiv 0 \pmod 2$.

Proof. Using (1.2) with $p = 2$ in (1.3), we find that

$$\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2} \equiv \frac{(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty} \pmod 2. \tag{3.1}$$

The right hand side of (3.1) contains no term involving odd power of q, so extracting the terms involving q^{2n+1} from (3.1), we arrive at (i).

Extracting the terms involving q^{2n} from (3.1) and replacing q^2 by q and simplifying using (1.2), we obtain

$$\sum_{n=0}^{\infty} A_5(2n)q^n \equiv \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} \equiv \frac{(q^{20};q^{20})_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \pmod 2. \tag{3.2}$$

Employing Lemma 2.2 in (3.2) and simplifying using (1.2), we deduce that

$$\sum_{n=0}^{\infty} A_5(2n)q^n \equiv (q^4;q^4)_\infty(q^{20};q^{20})_\infty + q\frac{(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty} \pmod 2. \tag{3.3}$$

The right hand side of (3.3) contains no term involving q^{4n+2}, so extracting the terms involving q^{4n+2} from (3.3), we complete the proof of (ii). $\square$

Theorem 3.2. *Let $p \geq 5$ be a prime with $\left(\frac{-5}{p}\right) = -1$. Then for non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} A_5\left(8p^{2\alpha}n + 2p^{2\alpha} - 2\right) q^n \equiv (q;q)_\infty (q^5;q^5)_\infty \pmod 2, \tag{3.4}$$

where, here and throughout the paper $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol.

Proof. Extracting the terms involving q^{4n} from (3.3) and replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} A_5(8n)q^n \equiv (q;q)_\infty (q^5;q^5)_\infty \pmod 2, \tag{3.5}$$

which is the case $\alpha = 0$.

Assume (3.4) holds for α. Employing Lemma 2.1 in (3.4), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} A_5\left(8p^{2\alpha}n + 2p^{2\alpha} - 2\right) q^n \\
&\equiv \Bigg[\sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) \\
&\quad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty \Bigg] \\
&\times \Bigg[\sum_{\substack{m=-\frac{p-1}{2} \\ m \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{5(3m^2+m)/2} f\left(-q^{5\frac{3p^2+(6m+1)p}{2}}, -q^{5\frac{3p^2-(6m+1)p}{2}}\right) \\
&\quad + (-1)^{\frac{\pm p-1}{6}} q^{5\frac{p^2-1}{24}} (q^{5p^2};q^{5p^2})_\infty \Bigg] \pmod 2.
\end{aligned} \tag{3.6}$$

Consider the congruence

$$\frac{3k^2+k}{2} + 5\left(\frac{3m^2+m}{2}\right) \equiv 6\left(\frac{p^2-1}{24}\right) \pmod p. \tag{3.7}$$

The congruence (3.7) is equivalent to

$$(6k+1)^2 + 5(6m+1)^2 \equiv 0 \pmod p. \tag{3.8}$$

For $\left(\frac{-5}{p}\right) = -1$ the congruence (3.8) has a unique solution $k = m = \frac{\pm p-1}{6}$. So extracting the terms involving $q^{pn+(p^2-1)/4}$ from (3.6), dividing by $q^{(p^2-1)/4}$ and replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} A_5\left(8p^{2\alpha+1}n+2p^{2\alpha+2}-2\right)q^n \equiv (q^p;q^p)_\infty(q^{5p};q^{5p})_\infty \pmod 2. \tag{3.9}$$

Extracting the terms involving q^{pn} from (3.9) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} A_5\left(8p^{2\alpha+2}n+2p^{2\alpha+2}-2\right)q^n \equiv (q;q)_\infty(q^5;q^5)_\infty \pmod 2, \tag{3.10}$$

which is the case $\alpha+1$ of (3.4). Hence, the proof is complete. $\square$

Corollary 3.3. *Let $p \geq 5$ be a prime with $\left(\frac{-5}{p}\right) = -1$. Then for non-negative integers α and n, we have*

$$A_5\left(8p^{2\alpha+2}n+2p^{2\alpha+1}(4j+p)-2\right) \equiv 0 \pmod 2, \tag{3.11}$$

where $j = 1, 2, 3, \ldots, p-1$.

Proof. Extracting the terms involving q^{pn+j} for $j = 1, 2, 3, \ldots, p-1$ from (3.9), we arrive at the desired result. $\square$

Theorem 3.4. *For any positive integer k, we have*

$$A_5\left(2^k n + 2^k - 2\right) \equiv A_5(2n) \pmod 2.$$

Proof. Extracting the terms involving q^{2n+1} from (3.3), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} A_5(2^2n+2)q^n \equiv \frac{(q^{20};q^{20})_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \pmod 2. \tag{3.12}$$

Combining (3.2) and (3.12), we deduce that

$$A_5(2^2n+2) \equiv A_5(2n) \pmod 2. \tag{3.13}$$

Iterating (3.13) by replacing n by $2n+1$ and for any positive integer k, we obtain

$$A_5\left(2^k n + 2^{k-1} + 2^{k-2} + \cdots + 2\right) \equiv A_5(2n) \pmod 2. \tag{3.14}$$

Simplifying (3.14), we arrive at the desired result. $\square$

Theorem 3.5. *We have*

(i) $\sum_{n=0}^{\infty} A_5(16n)\,q^n \equiv (q;q)_\infty^3 \pmod 2$,
(ii) $A_5(80n+16i+8) \equiv 0 \pmod 2$, *where* $i = 1, 2, 3$ *and* 4.

Proof. Simplifying (3.5) using (1.2), we obtain

$$\sum_{n=0}^{\infty} A_5(8n)q^n \equiv \frac{(q;q)_\infty^2(q^5;q^5)_\infty}{(q;q)_\infty} \equiv (q^2;q^2)_\infty \frac{(q^5;q^5)_\infty}{(q;q)_\infty} \pmod 2. \tag{3.15}$$

Employing Lemma 2.2 in (3.15) and simplifying using (1.2), we deduce that

$$\sum_{n=0}^{\infty} A_5(8n)q^n \equiv (q^2;q^2)_\infty^3 + q(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty \pmod 2. \tag{3.16}$$

Extracting the terms involving q^{2n} from (3.16) and replacing q^2 by q, we arrive at (i). Again, extracting the terms involving q^{2n+1} in (3.16), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} A_5(16n+8)q^n \equiv (q^5;q^5)_\infty(q^{10};q^{10})_\infty \pmod 2. \tag{3.17}$$

Extracting the terms involving q^{5n+i} for $i = 1, 2, 3,$ and 4 from (3.17), we arrive at (ii). □

Theorem 3.6. *For any positive integer k, then*

(i) $A_5\left(16 \cdot 3^{2k}n + 2 \cdot 3^{2k} - 2\right) \equiv A_5(16n) \pmod 2$,
(ii) $A_5\left(16 \cdot 5^{2k}n + 2 \cdot 5^{2k} - 2\right) \equiv A_5(16n) \pmod 2$,
(iii) $A_5\left(16 \cdot 7^{2k}n + 2 \cdot 7^{2k} - 2\right) \equiv A_5(16n) \pmod 2$.

Proof. Employing (2.3) in Theorem 3.5(i), we deduce that

$$A_5(16n) \equiv p_3(n) \pmod 2. \tag{3.18}$$

Employing (3.18) in (2.4), (2.5), and (2.6), we arrive at (i), (ii), and (iii), respectively. □

Corollary 3.7. *If n is not a triangular number, then*

$$A_5(16n) \equiv 0 \pmod 2.$$

Proof. Employing Lemma 2.4 in Theorem 3.5(i), we obtain

$$\sum_{n=0}^{\infty} A_5(16n)\, q^n \equiv \sum_{n=0}^{\infty} (-1)^n(2n+1)q^{n(n+1)/2} \pmod 2. \tag{3.19}$$

The desired result now follows easily from (3.19). □

Corollary 3.8. *If n is not a triangular number, we have*

$$A_5\left(16 \cdot 3^{2k}n + 2 \cdot 3^{2k} - 2\right) \equiv 0 \pmod 2,$$
$$A_5\left(16 \cdot 5^{2k}n + 2 \cdot 5^{2k} - 2\right) \equiv 0 \pmod 2,$$
$$A_5\left(16 \cdot 7^{2k}n + 2 \cdot 7^{2k} - 2\right) \equiv 0 \pmod 2.$$

Proof. We employ Corollary 3.7 in Theorem 3.6 to complete the proof. □

Theorem 3.9. *We have*

(i) $A_5(5n+2) \equiv 0 \pmod 5$,
(ii) $A_5(5n+3) \equiv 0 \pmod 5$,
(iii) $A_5(5n+4) \equiv 0 \pmod 5$.

Proof. Squaring the identity in Lemma 2.3, we find that

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^{25};q^{25})_\infty^{10}}{(q^5;q^5)_\infty^{12}}\{F^{-8}(q^5)+2qF^{-7}(q^5) \\ +5q^2F^{-6}(q^5)+10q^3F^{-5}(q^5)+20q^4F^{-4}(q^5) \\ +16q^5F^{-3}(q^5)+27q^6F^{-2}(q^5)+20q^7F^{-1}(q^5)+15q^8-20q^9F(q^5) \\ +27q^{10}F^2(q^5)-16q^{11}F^3(q^5)+20q^{12}F^4(q^5)-10q^{13}F^5(q^5) \\ +5q^{14}F^6(q^5)-2q^{15}F^7(q^5)+q^{16}F^8(q^5)\}. \tag{3.20}$$

Employing (3.20) in (1.3), we find that

$$\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^{25};q^{25})_\infty^{10}}{(q^5;q^5)_\infty^{2}}\{F^{-8}(q^5)+2qF^{-7}(q^5) \\ +5q^2F^{-6}(q^5)+10q^3F^{-5}(q^5)+20q^4F^{-4}(q^5) \\ +16q^5F^{-3}(q^5)+27q^6F^{-2}(q^5)+20q^7F^{-1}(q^5)+15q^8-20q^9F(q^5) \\ +27q^{10}F^2(q^5)-16q^{11}F^3(q^5)+20q^{12}F^4(q^5)-10q^{13}F^5(q^5) \\ +5q^{14}F^6(q^5)-2q^{15}F^7(q^5)+q^{16}F^8(q^5)\}. \tag{3.21}$$

Extracting the terms involving q^{5n+2} from (3.21), then dividing by q^2 and replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} A_5(5n+2)q^n = 5\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}\left(F^{-6}(q)+4qF^{-1}(q)+4q^2F^4(q)\right). \tag{3.22}$$

Now (i) follows easily from (3.22).

Extracting the terms involving q^{5n+3} from (3.21), then dividing by q^3 and replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} A_5(5n+3)q^n = 5\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}\left(2F^{-5}(q)+3q-2q^2F^5(q)\right). \tag{3.23}$$

Now (ii) follows easily from (3.23).

Extracting the terms involving q^{5n+4} from (3.21), then dividing by q^4 and replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} A_5(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}\left(4F^{-4}(q)-4qF(q)+q^2F^6(q)\right). \tag{3.24}$$

Now (iii) follows easily from (3.24). □

Remark 3.10. Theorem 3.9(ii) also follows as a particular case of a general result in [5, p. 4, Theorem 8].

Theorem 3.11. *For any positive integer k, we have*

$$A_5\left(5^k n + 2\cdot 5^k - 2\right) \equiv 5^k A_5(n) \pmod{10}.$$

Proof. From (3.23), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} A_5(5n+3)q^n &= \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}\left\{10\left(F^{-5}(q)+q-q^2F^5(q)\right)+5q\right\}\\ &\equiv 5q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2} \pmod{10}. \end{aligned} \tag{3.25}$$

Employing (1.3) in (3.25), we obtain

$$\sum_{n=0}^{\infty} A_5(5n+3)q^n \equiv 5\sum_{n=0}^{\infty} A_5(n)q^{n+1} \pmod{10}. \tag{3.26}$$

Extracting the term involving q^{n+1} on both sides of (3.26), we obtain

$$A_5(5n+8) \equiv 5A_5(n) \pmod{10}. \tag{3.27}$$

Iterating (3.27) by replacing n by $5n+8$ k times, we deduce that

$$A_5\left(5^k n + \left(5^{k-1}+5^{k-2}+\cdots+5+1\right)8\right) \equiv 5^k A_5(n) \pmod{10}. \tag{3.28}$$

Simplifying (3.28), we arrive at the desired result. □

Corollary 3.12. *For any positive integer k, we have*

(i) $A_5\left(5^k n + 2\cdot 5^k - 2\right) \equiv 0 \pmod{5}$,
(ii) $A_5\left(5^k n + 2\cdot 5^k - 2\right) \equiv A_5(n) \pmod{2}$.

Proof. Proof follows from Theorem 3.11. □

Acknowledgment

The first author (Nipen Saikia) is thankful to Council of Scientific and Industrial Research of India for partially supporting the research work under the Research Scheme No. 25(0241)/15/EMR-II (F. No. 25(5498)/15).

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