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Some companions of Ostrowski type inequality for functions whose second derivatives are convex and concave with applications

M. Emin Özdemir [a], Merve Avci Ardic [b,*]

[a] Atatürk University, K.K. Education Faculty, Department of Mathematics, Erzurum 25240, Turkey

[b] Adiyaman University, Faculty of Science and Arts, Department of Mathematics, Adiyaman 02040, Turkey

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Abstract. In this paper, we obtain some companions of Ostrowski type inequality for absolutely continuous functions whose second derivatives absolute values are convex and concave. Finally, we give some applications for special means.

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1. Introduction

In this section, preliminary information and some studies available in the literature on the subject will be discussed. We shall begin with the definition of convex function.

Definition 1. [23] Let I be an interval in R. Then $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y). \tag{1.1}$$

If (1.1) is strict for all $x \neq y$ and $\lambda \in (0, 1)$, then f is said to be strictly convex. If the inequality in (1.1) is reversed, then f is said to be concave. If it is strict for all $x \neq y$ and $\lambda \in (0, 1)$, then f is said to be strictly concave.

* Corresponding author. Tel.: +90 05368890273.
E-mail addresses: emos@atauni.edu.tr (M.E. Özdemir), mavci@posta.adiyaman.edu.tr (M. Avci Ardic).
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For recent results and generalizations concerning convex functions, see [14–22].

The following inequality is well-known as Ostrowski's inequality in the literature for differentiable functions. This inequality was established by Ostrowski in 1938, see [3]:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leqslant M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leqslant M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

Many papers have been written on Ostrowski's inequality concerning some results, generalizations and applications, see [1–16].

In [4], Set et al. have proved some inequalities for s-convex and s-concave functions via following Lemma. They have obtained the second inequality for concave functions in Corollary 1 by putting $s = 1$ in the first inequality in Corollary 1.

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a,b]$, then*

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x) \\ &= \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 f''(tx + (1-t)a)dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 f''(tx + (1-t)b)dt. \end{aligned}$$

Theorem 2. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$ with $a < b$. If $|f''|^q$ is s-concave in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ &\leqslant \frac{2^{\frac{(s-1)}{q}}}{(2p+1)^{\frac{1}{p}}(b-a)} \left(\frac{(x-a)^3 \left| f''\left(\frac{x+a}{2}\right)\right| + (b-x)^3 \left| f''\left(\frac{b+x}{2}\right)\right|}{2} \right) \end{aligned} \tag{1.2}$$

for each $x \in [a,b]$.

Corollary 1. *If in (1.2), we choose $x = \frac{a+b}{2}$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{2^{\frac{(s-1)}{q}}(b-a)^2}{16(2p+1)^{\frac{1}{p}}} \left[\left| f''\left(\frac{3a+b}{4}\right)\right| + \left| f''\left(\frac{a+3b}{4}\right)\right| \right].$$

For instance, if $s = 1$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}}\left[\left|f''\left(\frac{3a+b}{4}\right)\right| + \left|f''\left(\frac{a+3b}{4}\right)\right|\right].$$

In [1], some companions of an Ostrowski type inequality have been introduced by Liu for functions whose derivatives are absolutely continuous. In [2], Barnett et al. have established some companions for the Ostrowski inequality and the generalized trapezoid inequality. In [14], Alomari et al. have introduced some companions of Ostrowski inequality for functions whose first derivatives absolute values are convex. In [5], Alomari has obtained a companion of Ostrowski's integral inequality for functions with bounded first derivatives and has given applications to a composite quadrature rule and probability density functions.

In this paper, we establish some companions of Ostrowski type inequality for absolutely continuous functions whose second derivatives absolute values are convex and concave. Then we give some applications for special means.

In order to prove our main results, we need the following Lemma from [1] :

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have the equality*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\\&\quad+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\\&=\frac{1}{2(b-a)}\left[\int_a^x (t-a)^2 f''(t)dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 f''(t)dt\right.\\&\quad\left.+\int_{a+b-x}^b (t-b)^2 f''(t)dt\right]\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

2. Main results

We will start with the following theorem:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a function such that f' is absolutely continuous on $[a,b]$, $f''\in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right.\\&\quad\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|\\&\leqslant\frac{(x-a)^3}{24(b-a)}[|f''(a)|+|f''(b)|]\\&\quad+\frac{6(x-a)^3+(a+b-2x)^3}{48(b-a)}[|f''(x)|+|f''(a+b-x)|]\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Using Lemma 2 and the property of the modulus we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|$$
$$\leqslant \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2|f''(t)|dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2|f''(t)|dt + \int_{a+b-x}^b (t-b)^2|f''(t)|dt\right].$$

Since $|f''|$ is convex on $[a,b]$, we have

$$|f''(t)| \leqslant \frac{t-a}{x-a}|f''(x)| + \frac{x-t}{x-a}|f''(a)|, \ \ t \in [a,x];$$
$$|f''(t)| \leqslant \frac{t-x}{a+b-2x}|f''(a+b-x)| + \frac{a+b-x-t}{a+b-2x}|f''(x)|, \ \ t \in (x, a+b-x]$$

and

$$|f''(t)| \leqslant \frac{t-a-b+x}{x-a}|f''(b)| + \frac{b-t}{x-a}|f''(a+b-x)|, \ \ t \in (a+b-x, b].$$

Therefore we can write

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|$$
$$\leqslant \frac{1}{2(b-a)}\left\{\int_a^x (t-a)^2\left[\frac{t-a}{x-a}|f''(x)| + \frac{x-t}{x-a}|f''(a)|\right]dt\right.$$
$$+ \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[\frac{t-x}{a+b-2x}|f''(a+b-x)| + \frac{a+b-x-t}{a+b-2x}|f''(x)|\right]dt$$
$$\left. + \int_{a+b-x}^b (t-b)^2\left[\frac{t-a-b+x}{x-a}|f''(b)| + \frac{b-t}{x-a}|f''(a+b-x)|\right]dt\right\}$$
$$= \frac{1}{2(b-a)}\left\{\frac{1}{4}(x-a)^3|f''(x)| + \frac{1}{12}(x-a)^3|f''(a)|\right.$$
$$+ \frac{1}{24}(a+b-2x)^3|f''(a+b-x)| + \frac{1}{24}(a+b-2x)^3|f''(x)|$$
$$\left. + \frac{1}{12}(x-a)^3|f''(b)| + \frac{1}{4}(x-a)^3|f''(a+b-x)|\right\}$$
$$= \frac{(x-a)^3}{24(b-a)}[|f''(a)| + |f''(b)|]$$
$$+ \frac{6(x-a)^3 + (a+b-2x)^3}{48(b-a)}[|f''(x)| + |f''(a+b-x)|],$$

which is the desired result. □

Corollary 2. *Let f as in Theorem 3. Additionally, if $f'(x) = f'(a+b-x)$, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right|\\&\leqslant \frac{(x-a)^3}{24(b-a)}[|f''(a)|+|f''(b)|]\\&\quad+\frac{6(x-a)^3+(a+b-2x)^3}{48(b-a)}[|f''(x)|+|f''(a+b-x)|].\end{aligned}$$

Corollary 3. *In Corollary 2, if f is symmetric function, $f(a+b-x) = f(x)$, for all $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right|\\&\leqslant \frac{(x-a)^3}{24(b-a)}[|f''(a)|+|f''(b)|]\\&\quad+\frac{6(x-a)^3+(a+b-2x)^3}{48(b-a)}[|f''(x)|+|f''(a+b-x)|],\end{aligned}$$

which is an Ostrowski type inequality.

Corollary 4. *In Corollary 3, if we choose $x = \frac{a+b}{2}$, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right|\\&\quad\leqslant \frac{(b-a)^2}{192}\left[|f''(a)|+6\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right].\end{aligned} \tag{1.3}$$

Corollary 5. *In Theorem 3, if we choose $x = \frac{3a+b}{4}$, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right|\\&\quad\leqslant \frac{(b-a)^2}{1536}\left[|f''(a)|+7\left|f''\left(\frac{3a+b}{4}\right)\right|+7\left|f''\left(\frac{a+3b}{4}\right)\right|+|f''(b)|\right].\end{aligned} \tag{1.4}$$

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a function such that f' is absolutely continuous on $[a,b]$, $f'' \in L_1[a,b]$. If $|f''|^q$ is convex on $[a,b]$, for all $x \in \left[a, \frac{a+b}{2}\right]$ and $q > 1$, then we have the following inequality:*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right. \\
&\quad \left. +\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right| \\
&\leqslant \frac{1}{2^{1+\frac{1}{q}}(b-a)(2p+1)^{\frac{1}{p}}}\Big[(x-a)^3(|f''(a)|^q+|f''(x)|^q)^{\frac{1}{q}} \\
&\quad +\frac{(a+b-2x)^3}{4}(|f''(x)|^q+|f''(a+b-x)|^q)^{\frac{1}{q}} \\
&\quad +(x-a)^3(|f''(a+b-x)|^q+|f''(b)|^q)^{\frac{1}{q}}\Big],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2, Hölder inequality and convexity of $|f''|^q$, we have

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right. \\
&\quad \left. +\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right| \\
&\leqslant \frac{1}{2(b-a)}\left\{\left(\int_a^x (t-a)^{2p}dt\right)^{\frac{1}{p}}\left(\int_a^x |f''(t)|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad +\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^{2p}dt\right)^{\frac{1}{p}}\left(\int_x^{a+b-x}|f''(t)|^q dt\right)^{\frac{1}{q}} \\
&\quad \left.+\left(\int_{a+b-x}^b (t-b)^{2p}dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b |f''(t)|^q dt\right)^{\frac{1}{q}}\right\} \\
&\leqslant \frac{1}{2(b-a)}\left\{\left(\int_a^x (t-a)^{2p}dt\right)^{\frac{1}{p}}\left(\int_a^x\left[\frac{t-a}{x-a}|f''(x)|^q+\frac{x-t}{x-a}|f''(a)|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad +\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^{2p}dt\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_x^{a+b-x}\left[\frac{t-x}{a+b-2x}|f''(a+b-x)|^q+\frac{a+b-x-t}{a+b-2x}|f''(x)|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad \left.+\left(\int_{a+b-x}^b (t-b)^{2p}dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b\left[\frac{t-a-b+x}{x-a}|f''(b)|^q+\frac{b-t}{x-a}|f''(a+b-x)|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
&= \frac{1}{2(b-a)}\left\{\left(\frac{(x-a)^{2p+1}}{(2p+1)}\right)^{\frac{1}{p}}\left(\frac{x-a}{2}\right)^{\frac{1}{q}}(|f''(a)|^q+|f''(x)|^q)^{\frac{1}{q}}\right. \\
&\quad +\left(\frac{2}{2p+1}\left(\frac{a+b}{2}-x\right)^{2p+1}\right)^{\frac{1}{p}}\left(\frac{a+b}{2}-x\right)^{\frac{1}{q}}(|f''(x)|^q+|f''(a+b-x)|^q)^{\frac{1}{q}} \\
&\quad \left.+\left(\frac{(x-a)^{2p+1}}{(2p+1)}\right)^{\frac{1}{p}}\left(\frac{x-a}{2}\right)^{\frac{1}{q}}(|f''(a+b-x)|^q+|f''(b)|^q)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

When we arrange the statements above, we obtain the desired result. □

Corollary 6. *Let f as in Theorem* 4 *. Additionally, if* $f'(x) = f'(a+b-x)$*, we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right| \\
&\leqslant \frac{1}{2^{1+\frac{1}{q}}(b-a)(2p+1)^{\frac{1}{p}}}\Big[(x-a)^3(|f''(a)|^q+|f''(x)|^q)^{\frac{1}{q}} \\
&\quad+\frac{(a+b-2x)^3}{4}(|f''(x)|^q+|f''(a+b-x)|^q)^{\frac{1}{q}} \\
&\quad+(x-a)^3(|f''(a+b-x)|^q+|f''(b)|^q)^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Corollary 7. *In Corollary* 6*, if f is symmetric function,* $f(a+b-x) = f(x)$*, we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right| \\
&\leqslant \frac{1}{2^{1+\frac{1}{q}}(b-a)(2p+1)^{\frac{1}{p}}}\Big[(x-a)^3(|f''(a)|^q+|f''(x)|^q)^{\frac{1}{q}} \\
&\quad+\frac{(a+b-2x)^3}{4}(|f''(x)|^q+|f''(a+b-x)|^q)^{\frac{1}{q}} \\
&\quad+(x-a)^3(|f''(a+b-x)|^q+|f''(b)|^q)^{\frac{1}{q}}\Big],
\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Corollary 8. *In Corollary* 6*, if we choose* $x = a$ *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{(b-a)^2}{2^{3+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}[|f''(a)|^q+|f''(b)|^q]^{\frac{1}{q}}.$$

Corollary 9. *In Theorem* 4 *, if we choose*

(1) $x = \frac{a+b}{2}$, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right| \\
&\leqslant \frac{(b-a)^2}{2^{4+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left[\left(|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

(2) $x = \frac{3a+b}{4}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leqslant \frac{(b-a)^2}{2^{7+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left\{\left(|f''(a)|^q + \left|f''\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}} + 2\left(\left|f''\left(\frac{3a+b}{4}\right)\right|^q + \left|f''\left(\frac{a+3b}{4}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f''\left(\frac{a+3b}{4}\right)\right|^q + |f''(b)|^q\right)^{\frac{1}{q}}\right\}.$$

The following results are obtained using the well-known power-mean integral inequality:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a function such that f' is absolutely continuous on $[a,b]$, $f''\in L_1[a,b]$. If $|f''|^q$ is convex on $[a,b]$, for all $x\in\left[a,\frac{a+b}{2}\right]$ and $q\geqslant 1$, then we have the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|$$
$$\leqslant \frac{1}{2(b-a)}\left\{\frac{1}{3\times 4^{\frac{1}{q}}}(x-a)^3[3|f''(x)|^q + |f''(a)|^q]^{\frac{1}{q}} + \frac{1}{3\times 2^{2+\frac{1}{q}}}(a+b-2x)^3[|f''(x)|^q + |f''(a+b-x)|^q]^{\frac{1}{q}} + \frac{1}{3\times 4^{\frac{1}{q}}}(x-a)^3[|f''(b)|^q + 3|f''(a+b-x)|^q]^{\frac{1}{q}}\right\}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Using Lemma 2, the property of the modulus and power-mean integral inequality we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|$$
$$\leqslant \frac{1}{2(b-a)}\left\{\left(\int_a^x (t-a)^2dt\right)^{1-\frac{1}{q}}\left(\int_a^x (t-a)^2|f''(t)|^q dt\right)^{\frac{1}{q}} + \left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\right)^{1-\frac{1}{q}}\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2|f''(t)|^q dt\right)^{\frac{1}{q}} + \left(\int_{a+b-x}^b (t-b)^2dt\right)^{1-\frac{1}{q}}\left(\int_{a+b-x}^b (t-b)^2|f''(t)|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $|f''|^q$ is convex on $[a,b]$, we have

$$|f''(t)|^q \leqslant \frac{t-a}{x-a}|f''(x)|^q + \frac{x-t}{x-a}|f''(a)|^q, \ t \in [a,x];$$
$$|f''(t)|^q \leqslant \frac{t-x}{a+b-2x}|f''(a+b-x)|^q + \frac{a+b-x-t}{a+b-2x}|f''(x)|^q, \ t \in (x, a+b-x]$$

and

$$|f''(t)|^q \leqslant \frac{t-a-b+x}{x-a}|f''(b)|^q + \frac{b-t}{x-a}|f''(a+b-x)|^q, \ t \in (a+b-x, b].$$

Hence we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)] \right.\\
&\left. + \frac{1}{2}\left(x - \frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|\\
&\leqslant \frac{1}{2(b-a)}\left\{\left(\int_a^x (t-a)^2 dt\right)^{1-\frac{1}{q}}\left(\int_a^x (t-a)^2\left[\frac{t-a}{x-a}|f''(x)|^q + \frac{x-t}{x-a}|f''(a)|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[\frac{t-x}{a+b-2x}|f''(a+b-x)|^q + \frac{a+b-x-t}{a+b-2x}|f''(x)|^q\right]dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{a+b-x}^b (t-b)^2 dt\right)^{1-\frac{1}{q}}\left(\int_{a+b-x}^b (t-b)^2\left[\frac{t-a-b+x}{x-a}|f''(b)|^q + \frac{b-t}{x-a}|f''(a+b-x)|^q\right]dt\right)^{\frac{1}{q}}\right\}\\
&= \frac{1}{2(b-a)}\left\{\left(\frac{1}{3}(x-a)^3\right)^{1-\frac{1}{q}}\left(\frac{1}{4}(x-a)^3|f''(x)|^q + \frac{1}{12}(x-a)^3|f''(a)|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{12}(a+b-2x)^3\right)^{1-\frac{1}{q}}\left(\frac{1}{24}(a+b-2x)^3|f''(a+b-x)|^q + \frac{1}{24}(a+b-2x)^3|f''(x)|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{1}{3}(x-a)^3\right)^{1-\frac{1}{q}}\left(\frac{1}{12}(x-a)^3|f''(b)|^q + \frac{1}{4}(x-a)^3|f''(a+b-x)|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By editing the above expression, we get the desired result. □

Corollary 10. *Let f as in Theorem 5. Additionally, if $f'(x) = f'(a+b-x)$, we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right|\\
&\leqslant \frac{1}{2(b-a)}\left\{\frac{1}{3\times 4^{\frac{1}{q}}}(x-a)^3[3|f''(x)|^q + |f''(a)|^q]^{\frac{1}{q}}\right.\\
&+\frac{1}{3\times 2^{2+\frac{1}{q}}}(a+b-2x)^3[|f''(x)|^q + |f''(a+b-x)|^q]^{\frac{1}{q}}\\
&\left.+\frac{1}{3\times 4^{\frac{1}{q}}}(x-a)^3[|f''(b)|^q + 3|f''(a+b-x)|^q]^{\frac{1}{q}}\right\}
\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Corollary 11. *In Corollary* 10*, if f is symmetric function,* $f(a+b-x)=f(x)$*, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-f(x)\right|\\&\leqslant\frac{1}{2(b-a)}\left\{\frac{1}{3\times 4^{\frac{1}{q}}}(x-a)^3[3|f''(x)|^q+|f''(a)|^q]^{\frac{1}{q}}\right.\\&\quad+\frac{1}{3\times 2^{2+\frac{1}{q}}}(a+b-2x)^3[|f''(x)|^q+|f''(a+b-x)|^q]^{\frac{1}{q}}\\&\quad\left.+\frac{1}{3\times 4^{\frac{1}{q}}}(x-a)^3[|f''(b)|^q+3|f''(a+b-x)|^q]^{\frac{1}{q}}\right\}\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

We obtain the following result for concave functions.

Theorem 6. *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* f' *is absolutely continuous on* $[a,b]$, $f''\in L_1[a,b]$. *If* $|f''|^q$ *is concave on* $[a,b]$, *for all* $x\in\left[a,\frac{a+b}{2}\right]$ *and* $q>1$, *then we have the following inequality:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}[f(x)+f(a+b-x)]\right.\\&\quad\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|\\&\leqslant\frac{1}{2(b-a)(2p+1)^{\frac{1}{p}}}\left[(x-a)^3\left|f''\left(\frac{x+a}{2}\right)\right|\right.\\&\quad\left.+\frac{(a+b-2x)^3}{4}\left|f''\left(\frac{a+b}{2}\right)\right|+(x-a)^3\left|f''\left(\frac{a+2b-x}{2}\right)\right|\right]\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and using Hölder inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}[f(x)+f(a+b-x)]\right.\\&\quad\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|\\&\leqslant\frac{1}{2(b-a)}\left\{\left(\int_a^x(t-a)^{2p}dt\right)^{\frac{1}{p}}\left(\int_a^x|f''(t)|^q dt\right)^{\frac{1}{q}}\right.\\&\quad+\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^{2p}dt\right)^{\frac{1}{p}}\left(\int_x^{a+b-x}|f''(t)|^q dt\right)^{\frac{1}{q}}\\&\quad\left.+\left(\int_{a+b-x}^b(t-b)^{2p}dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b|f''(t)|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Let us write,

$$\int_a^x |f''(t)|^q dt = (x-a)\int_0^1 |f''(\lambda x+(1-\lambda)a)|^q d\lambda,$$

$$\int_x^{a+b-x} |f''(t)|^q dt = (a+b-2x)\int_0^1 |f''(\lambda(a+b-x)+(1-\lambda)x)|^q d\lambda$$

and

$$\int_{a+b-x}^b |f''(t)|^q dt = (x-a)\int_0^1 |f''(\lambda b+(1-\lambda)(a+b-x))|^q d\lambda.$$

Since $|f''|^q$ is concave on $[a,b]$, we use the Jensen integral inequality to obtain

$$\begin{aligned}(x-a)\int_0^1 |f''(\lambda x+(1-\lambda)a)|^q d\lambda &= (x-a)\int_0^1 \lambda^0 |f''(\lambda x+(1-\lambda)a)|^q d\lambda \\ &\leqslant (x-a)\left(\int_0^1 \lambda^0 d\lambda\right)\left|f''\left(\frac{1}{\int_0^1 \lambda^0 d\lambda}\int_0^1 (\lambda x+(1-\lambda)a)d\lambda\right)\right|^q \\ &= (x-a)\left|f''\left(\frac{x+a}{2}\right)\right|^q\end{aligned}$$

and analogously

$$(a+b-2x)\int_0^1 |f''(\lambda(a+b-x)+(1-\lambda)x)|^q d\lambda \leqslant (a+b-2x)\left|f''\left(\frac{a+b}{2}\right)\right|^q,$$

$$(x-a)\int_0^1 |f''(\lambda b+(1-\lambda)(a+b-x))|^q d\lambda \leqslant (x-a)\left|f''\left(\frac{a+2b-x}{2}\right)\right|^q.$$

Combining all above inequalities, we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)] \right. \\ &\quad \left. + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right| \\ &\leqslant \frac{1}{2(b-a)}\left\{\left(\frac{(x-a)^{2p+1}}{(2p+1)}\right)^{\frac{1}{p}}(x-a)^{\frac{1}{q}}\left|f''\left(\frac{x+a}{2}\right)\right| \right. \\ &\quad + \left(\frac{2}{2p+1}\left(\frac{a+b}{2}-x\right)^{2p+1}\right)^{\frac{1}{p}}(a+b-2x)^{\frac{1}{q}}\left|f''\left(\frac{a+b}{2}\right)\right| \\ &\quad \left. + \left(\frac{(x-a)^{2p+1}}{(2p+1)}\right)^{\frac{1}{p}}(x-a)^{\frac{1}{q}}\left|f''\left(\frac{a+2b-x}{2}\right)\right|\right\} \\ &\leqslant \frac{1}{2(b-a)(2p+1)^{\frac{1}{p}}}\left[(x-a)^3\left|f''\left(\frac{x+a}{2}\right)\right| \right. \\ &\quad \left. + \frac{(a+b-2x)^3}{4}\left|f''\left(\frac{a+b}{2}\right)\right| + (x-a)^3\left|f''\left(\frac{a+2b-x}{2}\right)\right|\right]\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$ and $\frac{1}{p}+\frac{1}{q}=1$. □

Corollary 12. *Let f as in Theorem* 6. *Additionally, if* $f'(x) = f'(a+b-x)$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x)+f(a+b-x)]\right|$$
$$\leqslant \frac{1}{2(b-a)(2p+1)^{\frac{1}{p}}}\left[(x-a)^3\left(\left|f''\left(\frac{x+a}{2}\right)\right| + \left|f''\left(\frac{a+2b-x}{2}\right)\right|\right)\right.$$
$$\left. + \frac{(a+b-2x)^3}{4}\left|f''\left(\frac{a+b}{2}\right)\right|\right],$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Corollary 13. *In Corollary* 12, *if f is symmetric function,* $f(a+b-x) = f(x)$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right|$$
$$\leqslant \frac{1}{2(b-a)(2p+1)^{\frac{1}{p}}}\left[(x-a)^3\left(\left|f''\left(\frac{x+a}{2}\right)\right| + \left|f''\left(\frac{a+2b-x}{2}\right)\right|\right)\right.$$
$$\left. + \frac{(a+b-2x)^3}{4}\left|f''\left(\frac{a+b}{2}\right)\right|\right],$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Corollary 14. *In Theorem* 6, *if we choose* $x = \frac{3a+b}{4}$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leqslant \frac{(b-a)^2}{128(2p+1)^{\frac{1}{p}}}\left[\left|f''\left(\frac{7a+b}{8}\right)\right| + 2\left|f''\left(\frac{a+b}{2}\right)\right| + \left|f''\left(\frac{a+7b}{8}\right)\right|\right].$$

Remark 1. In Theorem 6, if we choose $x = \frac{a+b}{2}$, we have the second inequality in Corollary 1.

3. Applications for special means

We consider the means for nonnegative real numbers $\alpha < \beta$ as follows:

(1) The arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \ \alpha,\beta \in \mathbb{R}^+.$$

(2) The logarithmic mean:

$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta - \ln\alpha}, \ \alpha,\beta \in \mathbb{R}^+, \ \alpha \neq \beta.$$

(3) The generalized logarithmic mean:

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(\beta-\alpha)(n+1)}\right]^{\frac{1}{n}},\ \alpha,\beta\in\mathbb{R}^+,\ \alpha\neq\beta,\ n\in\mathbb{Z}\setminus\{-1,0\}.$$

(4) The identric mean:

$$I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha\neq\beta\\ \alpha, & \alpha=\beta\end{cases}\ \alpha,\beta\in\mathbb{R}^+.$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let $a,b\in\mathbb{R}^+,\ a<b$. Then we have*

$$\left|L^{-1}(a,b)-A\left(\frac{4}{3a+b},\frac{4}{a+3b}\right)\right|\leqslant\frac{(b-a)^2}{768}\left[\frac{a^3+b^3}{a^3b^3}+448\left(\frac{1}{(3a+b)^3}+\frac{1}{(a+3b)^3}\right)\right].$$

Proof. The assertion follows from Corollary 5 applied to the convex mapping $f\colon[a,b]\to\mathbb{R},\ f(x)=\frac{1}{x}$. □

Proposition 2. *Let $a,b\in\mathbb{R}^+,\ a<b$ and $n\in\mathbb{Z},\ |n|\geqslant 2$. Then we have*

$$\left|L_n^n(a,b)-A^n(a,b)\right|\leqslant\frac{n(n-1)(b-a)^2}{192}\left[a^{n-2}+6A^{n-2}(a,b)+b^{n-2}\right].$$

Proof. The assertion follows from Corollary 4 applied to the convex mapping $f\colon[a,b]\to\mathbb{R},\ f(x)=x^n$. □

Proposition 3. *Let $a,b\in\mathbb{R}^+,\ a<b$. Then we have*

$$|A(\ln a,\ln b)-\ln I|\leqslant\frac{(b-a)^2}{2^{3+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left[\frac{1}{a^{2q}}+\frac{1}{b^{2q}}\right].$$

Proof. The assertion follows from Corollary 8 applied to the convex mapping $f\colon[a,b]\to[0,\infty),\ f(x)=-\ln x$. □

Proposition 4. *Let $a,b\in\mathbb{R}^+,\ a<b$. Then for all $q>1$, we have*

$$\begin{aligned}&\left|L^{-1}(a,b)-A\left(\frac{4}{3a+b},\frac{4}{a+3b}\right)\right|\\&\leqslant\frac{(b-a)^2}{2^{7+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left\{\left(\frac{2^q}{a^{3q}}+\frac{128^q}{(3a+b)^{3q}}\right)^{\frac{1}{q}}\right.\\&\left.+2\left(\frac{128^q}{(3a+b)^{3q}}+\frac{128^q}{(a+3b)^{3q}}\right)^{\frac{1}{q}}+\left(\frac{128^q}{(a+3b)^{3q}}+\frac{2^q}{b^{3q}}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. The assertion follows from second inequality in Corollary 9 applied to the convex mapping $f: [a,b] \to \mathbb{R}, f(x) = \frac{1}{x}$. □

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