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Solving partial fractional differential equations using the $\mathcal{F}_A$-transform

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Abstract. In this article, we introduce the generalized Fourier transform ($\mathcal{F}_A$-transform) and derive an inversion formula and convolution product for this transform. Furthermore, the fundamental solutions of the single-order and distributed-order Cauchy type fractional diffusion equations are given by means of the appropriate $\mathcal{F}_A$-transform in terms of the Wright functions. Also, applicability of this transform for the explicit solution of the generalized Hilbert type singular integral equation is discussed.

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1. Introduction

We consider the Fourier-type integral transform called the $\mathcal{F}_A$-transform as follows

$$\mathcal{F}_A\{f(x);p\} = \int_{-\infty}^{\infty} A'(x)e^{-ipA(x)}f(x)dx, \quad p \in \mathbb{R}, \tag{1-1}$$

where $f(x)$ is piecewise continuous and absolutely integrable on $\mathbb{R}$ (i.e. $\int_{-\infty}^{\infty} |f(A^{-1}(x))|dx < \infty$) and the function $A(x)$ is strictly increasing function with asymptotic behaviors $\lim_{x\to\pm\infty}A(x) = \pm\infty$.

It is obvious that the $\mathcal{F}_A$-transform corresponds to the Fourier transform when it is $A(x) = x$.

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In the recent years, Mainardi et al. [15–19], Gorenflo et al. [11,12] and other researchers (e.g. [8,25]) have investigated on the diffusion- wave type equations and other equations of this type with constant coefficients containing fractional derivatives (in the Riemann–Liouville or Caputo senses of single and distributed orders) in time and/or in space to describe the models of anomalous transport in physics and engineering. They applied the joint transform method to boundary value problems to find the fundamental solutions of these equations in terms of higher transcendental functions such as Fox H-function, the Wright and the Mittag–Leffler functions. For the linear partial fractional differential equations (LPFDEs) with non-constant coefficients the existing integral transform methods (such as the joint Laplace-Fourier transform) are not applicable, therefore, importance of the $\mathcal{F}_A$-transform for solving some fractional-type equations with non-constant coefficients is emphasized. In this regard, we introduced a Laplace-type integral transform ($\mathcal{L}_A$-transform) for solving LPFDEs with non-constant coefficients, see [1–3].

In this work, we focus our attention on the LPFDEs with non-constant coefficients in terms of the ${}_A\delta_x$-derivatives which occur in physical phenomena such as *fractional diffusion with non-constant coefficients*, *distributed-order fractional diffusion* which can be easily solved by applying the $\mathcal{F}_A$-transform by choosing the appropriate $A(x)$. Furthermore, effectiveness of the $\mathcal{F}_A$-transform in solving the singular integral equations with convolution-type kernels is treated.

In Section 2, we derive a new inversion formula for the $\mathcal{F}_A$-transform in terms of the Fourier's integral. Two lemmas in the $\mathcal{F}_A$-transform of the ${}_A\delta_x$-derivatives and the convolution property are also established. These properties can be useful for obtaining the solutions of fractional diffusion with non-constant coefficients and distributed-order fractional diffusion.

In Section 3, we find the fundamental solution of the fractional diffusion equation on fractals introduced by Giona and Roman by applying the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform ($\beta \geqslant 0$). These solutions can be expressed in terms of the higher transcendental functions of the Wright type.

In Section 4, Moshinskii's diffusion equation is generalized to a fractional diffusion equation of distributed order and by using the $\mathcal{F}_{\sinh^{-1}(x)}$- transform the fundamental solution of this equation is given as an integral representation in terms of the Laplace type integral.

In Section 5, applicability of the $\mathcal{F}_{x^{2n+1}}$-transform in solving the generalized Hilbert singular integral equation is discussed and finally the main conclusions are set in Section 6.

2. Elementary properties of the $\mathcal{F}_A$-transform

In this section, we establish some lemmas on the $\mathcal{F}_A$-transform which can be useful for solving LPFDEs. First, related to the classical Fourier transform and inverse Fourier transform

$$\mathcal{F}\{f(x);p\} = \int_{-\infty}^{\infty} e^{-ipx} f(x)dx, \tag{2-1}$$

$$\mathcal{F}^{-1}\{F(p);x\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} F(p)dp, \tag{2-2}$$

we derive an inversion formula for the $\mathcal{F}_A$-transform.

Lemma 2.1 (The inversion formula for the $\mathcal{F}_A$-transform). *Let f, f′ be piecewise continuous in every finite interval and f be absolutely integrable on* $\mathbb{R}$ *i.e.*

$$\int_{-\infty}^{\infty} |f(A^{-1}(x))|dx < \infty.$$

Let us assume that

$$\mathcal{F}_A\{f(x); p\} = F(p) = \int_{-\infty}^{\infty} A'(x)e^{-ipA(x)}f(x)dx,$$

then

$$\mathcal{F}_A^{-1}\{F(p); x\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipA(x)}F(p)dp. \tag{2-3}$$

Proof. By definition of the $\mathcal{F}_A$-transform (1-1), letting $t = A(x)$, we have

$$F(p) = \int_{-\infty}^{\infty} e^{-ipt}f(A^{-1}(t))dt = \mathcal{F}\{f(A^{-1}(t)); p\}.$$

At this point, by the inversion formula for the Fourier transform and setting back $A^{-1}(t) = x$, we get finally

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipA(x)}F(p)dp. \qquad \square$$

Lemma 2.2 (The $\mathcal{F}_A$-transform of ${}_A\delta_x$-derivatives). Let f, f′, ..., $f^{(n-1)}$ *be continuous functions on* $\mathbb{R}$ *and*

$$\lim_{|x|\to\infty} f^{(k)}(x) = 0, \quad k = 0, 1, \ldots, n-1,$$

then

$$\mathcal{F}_A\{{}_A\delta_x^n f(x); p\} = (ip)^n \mathcal{F}_A\{f(x); p\}, \tag{2-4}$$

where the ${}_A\delta_x$*-derivative operator is defined as follows*

$${}_A\delta_x = \frac{1}{A'(x)}\frac{d}{dx}.$$

We also define

$${}_A\delta_x^2 = {}_A\delta_x {}_A\delta_x = \frac{1}{A'^2(x)}\frac{d^2}{dx^2} - \frac{A''(x)}{A'^3(x)}\frac{d}{dx}.$$

The ${}_A\delta_x$*-derivative for any positive integer power can be found.*

Proof. Using the definitions of the $\mathcal{F}_A$-transform (1-1), the ${}_A\delta_x$-derivative and integration by parts, we obtain

$$\begin{aligned}\mathcal{F}_A\{{}_A\delta_x f(x); p\} &= \int_{-\infty}^{\infty} e^{-ipA(x)}f'(x)dx \\ &= e^{-ipA(x)}f(x)|_{-\infty}^{\infty} + ip\int_{-\infty}^{\infty} A'(x)e^{-ipA(x)}f(x)dx.\end{aligned}$$

Since $\lim_{|x|\to\infty} f(x) = 0$, it follows that

$$\mathcal{F}_A\{{}_A\delta_x f(x); p\} = (ip)\mathcal{F}_A\{f(x); p\}.$$

Similarly by repeated application of the above relation, we get

$$\mathcal{F}_A\{{}_A\delta_x^2 f(x); p\} = (ip)\mathcal{F}_A\{{}_A\delta_x f(x); p\} = (ip)^2\mathcal{F}_A\{f(x); p\}$$

and by repeating the above scheme for ${}_A\delta_x^n f(x)$, we obtain (2-4).

Lemma 2.3 (The convolution product for the $\mathcal{F}_A$-transform). *If $F(p)$, $G(p)$ are the $\mathcal{F}_A$-transform of the functions $f(x)$, $g(x)$, respectively, then*

$$F(p)G(p) = \mathcal{F}_A\{f * g\} = \mathcal{F}_A\left\{\int_{-\infty}^{\infty} A'(t)g(t)f(A^{-1}(A(x) - A(t)))dt\right\}. \tag{2-5}$$

Proof. Using the definition of the $\mathcal{F}_A$-transform for $F(p)$ and $G(p)$, we have

$$\begin{aligned} F(p)G(p) &= \left(\int_{-\infty}^{\infty} A'(y)e^{-ipA(y)}f(y)dy\right)\left(\int_{-\infty}^{\infty} A'(t)e^{-ipA(t)}g(t)dt\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'(y)A'(t)e^{-ip(A(t)+A(y))}f(y)g(t)dy\,dt. \end{aligned}$$

Now, by substitution $A(t) + A(y) = A(x)$ and changing the order of integration in the double integral, we get

$$\begin{aligned} F(p)G(p) &= \int_{-\infty}^{\infty} A'(x)e^{-ipA(x)}dx\int_{-\infty}^{\infty} A'(t)g(t)f(A^{-1}(A(x) - A(t)))dt \\ &= \mathcal{F}_A\left\{\int_{-\infty}^{\infty} A'(t)g(t)f(A^{-1}(A(x) - A(t)))dt\right\}. \end{aligned}$$

In view of the theorems of the $\mathcal{F}_A$-transform expressed in this section we may apply the $\mathcal{F}_A$-transform transform to LPFDEs in the next sections. First, in connection with initial-value problems, we consider a partial fractional differential equation in the Riemann–Liouville sense [10,23]. □

3. The time-fractional diffusion equation with non-constant coefficients of single order

Theorem 3.1. *The explicit solution of the following partial fractional differential equation in the Riemann–Liouville sense*

$$\begin{aligned} &{}_tD_{0^+}^{\alpha}u(x,t) = -Cx^{-\beta}\frac{\partial u(x,t)}{\partial x} - ku(x,t), \quad C > 0,\ \beta \geqslant 0,\ k \in \mathbb{R}, \\ &\quad 0 < \alpha \leqslant \frac{1}{2}, \end{aligned} \tag{3-1}$$

with Cauchy type initial and boundary conditions

$$\begin{aligned} &{}_tD_{0^+}^{\alpha-1}u(x,0^+)=f(x)\\ &\lim_{x\to+\infty}x^{\beta+1}=+\infty,\ \lim_{x\to-\infty}x^{\beta+1}=-\infty,\ x\in\mathbb{R},\ t\in\mathbb{R}^+, \end{aligned}\tag{3-2}$$

is

$$u(x,t)=\int_{-\infty}^{\infty}\tau^\beta G^\alpha(x^{\beta+1}-\tau^{\beta+1},t)f(\tau)d\tau.\tag{3-3}$$

The Green function G^α and the Wright function are given by the following relations

$$G^\alpha(x,t)=\frac{1}{Ct}e^{-\frac{kx}{C(\beta+1)}}W\left(-\alpha,0;-\frac{x}{C(\beta+1)}t^{-\alpha}\right)H(x),\tag{3-4}$$

$$W(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)},\quad \alpha>-1,\ \beta\in\mathbb{C},\ z\in\mathbb{C}.\tag{3-5}$$

Proof. Since the Eq. (3-1) is a LPFDE with non-constant coefficients, we set $A(x)=\frac{x^{\beta+1}}{\beta+1}$ in the integral (1-1) and apply this new integral transform (the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform) in space and the Laplace transform in time as follows

$$\mathcal{L}\{u(x,t);s\}=\tilde{u}(x,s)=\int_0^\infty e^{-st}u(x,t)dt,\quad \Re s>0,$$

$$\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}\{u(x,t);p\}=\hat{u}(p,t)=\int_{-\infty}^{\infty}x^\beta e^{-ip\frac{x^{\beta+1}}{\beta+1}}u(x,t)dx.$$

Then, using the Laplace transform of the Riemann–Liouville derivative [13,22,24]

$$\mathcal{L}\{{}_tD_{0^+}^\alpha u(x,t);s\}=s^\alpha\tilde{u}(x,s)-{}_tD_{0^+}^{\alpha-1}u(x,0^+)$$

and the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform of the Eq. (3-1), we obtain

$$\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}\{x^{-\beta}u(x,t);p\}=ip\hat{u}(p,t),$$

which by utilizing the Cauchy type initial conditions (3-2), leads to

$$\hat{\tilde{u}}(p,s)=\frac{1}{s^\alpha+Cip+k}F(p),\tag{3-6}$$

where $F(p)$ is the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform of the initial condition $f(x)$. At this point, by considering the inversion formula for the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform (2-3) and the convolution product (2-5), we obtain

$$\begin{aligned}\tilde{u}(x,s)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\tilde{u}}(p,s)e^{ip\frac{x^{\beta+1}}{\beta+1}}dp=\frac{1}{C}\mathcal{F}^{-1}_{\frac{x^{\beta+1}}{\beta+1}}\left\{\frac{1}{\frac{s^\alpha+k}{C}+ip}\right\}*f(x)\\ &=\frac{1}{C}[e^{-(s^\alpha+k)\frac{x^{\beta+1}}{C(\beta+1)}}H(x^{\beta+1})]*f(x),\end{aligned}$$

where H is the Heaviside unit step function and symbol * is the convolution of the two functions f, g for the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform expressed by the relation (2-5) as follows

$$f*g=\int_{-\infty}^{\infty}t^\beta g(t)f\left(\sqrt[\beta+1]{x^{\beta+1}-t^{\beta+1}}\right)dt.$$

Now, in regard to the inverse Laplace transform of the functions $e^{-s^{\alpha}\frac{x^{\beta+1}}{C(\beta+1)}}$ via the Wright function we have [13,22]

$$\mathcal{L}^{-1}\left\{e^{-s^{\alpha}\frac{x^{\beta+1}}{C(\beta+1)}}\right\}=\frac{1}{t}W\left(-\alpha,0;-\frac{x^{\beta+1}}{C(\beta+1)}t^{-\alpha}\right).$$

Consequently, we get the explicit solution of the Cauchy type problem (3-1) and (3-2) as follows

$$u(x,t)=\int_{-\infty}^{\infty}\tau^{\beta}G^{\alpha}(x^{\beta+1}-\tau^{\beta+1},t)f(\tau)d\tau, \tag{3-7}$$

where the Green function G^{α} is given by the following relation

$$G^{\alpha}(x,t)=\frac{1}{Ct}e^{-\frac{kx}{C(\beta+1)}}W\left(-\alpha,0;-\frac{x}{C(\beta+1)}t^{-\alpha}\right)H(x), \tag{3-8}$$

provided that the integral on the right-hand side of (3-7) is convergent. □

4. The time-fractional diffusion equation of distributed order

The earlier idea of fractional derivatives of distributed order go back to Volterra and was developed by Caputo [6,7]. Later other researchers e.g. Atanackovic et al. [4], Chechkin et al. [9] and Umarov et al. [26] study some linear and non-linear fractional differential equations of distributed order by analyzing some interesting cases of the order-density function. In this paper, by the notion of fractional derivative of distributed order, we consider a generalization of Moshinskii's equation [20] which describes diffusion of impurity in narrow channels.

Theorem 4.1. *The explicit solution of the fractional diffusion of distributed order equation subject to initial condition $u(x,0) = f(x)$ and order-density function $b(\alpha)$*

$$\int_0^1 b(\alpha)\left[{}_t^C D_{0^+}^{\alpha}u(x,t)\right]d\alpha=(1+x^2)\frac{\partial^2 u(x,t)}{\partial x^2}+x\frac{\partial u(x,t)}{\partial x},\quad x\in\mathbb{R},\ t>0,\ b(\alpha)\geqslant 0,\quad \int_0^1 b(\alpha)d\alpha=1 \tag{4-1}$$

is

$$u(x,t)=\frac{1}{2\pi}\int_0^{\infty}\frac{e^{-rt}}{r}dr\int_{-\infty}^{\infty}\sqrt{\rho}e^{-\left|\ln\left(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}}\right)\right|\sqrt{\rho}\cos\left(\frac{\pi}{2}\gamma\right)}\times\sin\left(\frac{\pi\gamma}{2}-\left|\ln\left(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}}\right)\right|\sqrt{\rho}\sin\left(\frac{\pi\gamma}{2}\right)\right)f(\tau)d\tau. \tag{4-2}$$

The parameters ρ and γ are given by relations (4-4) and (4-8).

Proof. In order to solve the Eq. (4-1), we extend the approach by Naber [21] and Mainardi et al. [19] to find a general representation of the fundamental solution related to a generic order-density function $b(\alpha)$. In this respect, by applying the Laplace transform of fractional derivative in the Caputo sense with respect to t

$$\mathcal{L}\left\{ {}_t^C D_{0^+}^{\alpha} u(x,t); s \right\} = s^{\alpha}\tilde{u}(x,s) - s^{\alpha-1}u(x,0^+)$$

and the $\mathcal{F}_{\sinh^{-1}(x)}$-transform with respect to x and setting $n = 2$ in relation (2-4)

$$\mathcal{F}_{\sinh^{-1}(x)}\left\{ (1+x^2)\frac{\partial^2 u(x,t)}{\partial x^2} + x\frac{\partial u(x,t)}{\partial x}; p \right\} = \mathcal{F}_{\sinh^{-1}(x)}\left\{ {}_{\sinh^{-1}(x)}\delta_x^2 u(x,t); p \right\}$$
$$= -p^2 \hat{u}(p,t),$$

we obtain

$$\left(\int_0^1 b(\alpha)s^{\alpha}d\alpha\right)\hat{\tilde{u}}(p,s) + p^2\hat{\tilde{u}}(p,s) = \frac{1}{s}\left(\int_0^1 b(\alpha)s^{\alpha}d\alpha\right)F(p).$$

From which

$$\hat{\tilde{u}}(p,s) = \frac{B(s)}{s(B(s)+p^2)}F(p), \quad \Re s > 0, \tag{4-3}$$

where $F(p)$ is the $\mathcal{F}_{\sinh^{-1}(x)}$-transform of the function $f(x)$ and

$$B(s) = \int_0^1 b(\alpha)s^{\alpha}d\alpha. \tag{4-4}$$

By inverting the $\mathcal{F}_{\sinh^{-1}(x)}$-transform of (4-3), we get the remaining Laplace transform as the following expression

$$\tilde{u}(x,s) = f(x) * \frac{\sqrt{B(s)}}{2s}e^{-|\sinh^{-1}(x)|\sqrt{B(s)}}, \quad \Re\sqrt{B(s)} > 0, \tag{4-5}$$

where the convolution of the two functions f, g for the $\mathcal{F}_{\sinh^{-1}(x)}$-transform is given by relation (2-5) (using the fact that $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$)

$$f * g = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+t^2}}g(t)f(\sinh(\sinh^{-1}(x) - \sinh^{-1}(t))dt,$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+t^2}}g(t)f\left(\frac{1}{2}\left(\frac{x+\sqrt{x^2+1}}{t+\sqrt{t^2+1}} - \frac{t+\sqrt{t^2+1}}{x+\sqrt{x^2+1}}\right)\right)dt. \tag{4-6}$$

By virtue of the Titchmarsh theorem for the inverse Laplace transform of the function

$$\tilde{u}_1(x,s) = \frac{\sqrt{B(s)}}{2s}e^{-|\sinh^{-1}(x)|\sqrt{B(s)}},$$

which has branch cut on the real negative semiaxis [5], we have

$$u_1(x,t) = -\frac{1}{\pi}\int_0^{\infty} e^{-rt}\Im\{\tilde{u}_1(x, re^{i\pi})\}dr. \tag{4-7}$$

In order to simplify the above relation (4-7), we need to evaluate the imaginary part of the function $-\tilde{u}_1(x, re^{i\pi})$ along the ray $s = re^{i\pi}$, $r > 0$, where the branch cut of the function s^α is defined. In this regard, by writing

$$B(re^{i\pi}) = \rho\cos\gamma\pi + i\rho\sin\gamma\pi, \qquad \begin{cases} \rho = \rho(r) = |B(re^{i\pi})| \\ \gamma = \gamma(r) = \frac{1}{\pi}\arg[B(re^{i\pi})] \end{cases} \tag{4-8}$$

and substituting in the above relation, the relation (4-7) leads to

$$u_1(x,t) = \frac{1}{2\pi}\int_0^\infty \frac{\sqrt{\rho}}{r} e^{-rt-|\sinh^{-1}(x)|\sqrt{\rho}\cos\left(\frac{\pi}{2}\gamma\right)} \times \sin\left(\frac{\pi\gamma}{2} - |\sinh^{-1}(x)|\sqrt{\rho}\sin\left(\frac{\pi\gamma}{2}\right)\right)dr.$$

Finally, using the convolution product given by relation (4-6), $u(x,t)$ is expressed as an integral representation

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}f(x) \\ &\quad * \left\{\int_0^\infty \frac{\sqrt{\rho}}{r} e^{-rt-|\sinh^{-1}(x)|\sqrt{\rho}\cos\left(\frac{\pi}{2}\gamma\right)} \sin\left(\frac{\pi\gamma}{2} - |\sinh^{-1}(x)|\sqrt{\rho}\sin\left(\frac{\pi\gamma}{2}\right)\right)dr\right\} \\ &= \frac{1}{2\pi}\int_0^\infty \frac{e^{-rt}}{r}dr \int_{-\infty}^\infty \sqrt{\rho}e^{-\left|\ln\left(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}}\right)\right|\sqrt{\rho}\cos\left(\frac{\pi}{2}\gamma\right)} \\ &\quad \times \sin\left(\frac{\pi\gamma}{2} - \left|\ln\left(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}}\right)\right|\sqrt{\rho}\sin\left(\frac{\pi\gamma}{2}\right)\right)f(\tau)d\tau, \end{aligned} \tag{4-9}$$

provided that the integrals on the right-hand side of (4-9) are convergent.

The explicit solution (4-9) of the time-fractional diffusion equation of distributed order (4-1) can be simplified in particular cases. For example, if we set the order density function with respect to the Dirac delta function $b(\alpha) = \delta(\alpha - n)$, $0 < n < 1$, the time-fractional diffusion equation of distributed order (4-1) is converted to time-fractional disturbance equation of single order n, so that

$$B(s) = s^n, \quad \rho = \rho(r) = r^n, \quad \gamma = n.$$

In this case, the inverse Laplace transform of $s^{\frac{n}{2}-1}e^{-|\sinh^{-1}(x)|s^{\frac{n}{2}}}$ in (4-5) can be easily obtained in terms of the Wright function

$$\mathcal{L}^{-1}\left\{s^{\frac{n}{2}-1}e^{-|\sinh^{-1}(x)|s^{\frac{n}{2}}}\right\} = \frac{1}{t^{\frac{n}{2}}}W\left(-\frac{n}{2}, 1-\frac{n}{2}; -|\sinh^{-1}(x)|t^{-\frac{n}{2}}\right).$$

The formal solution $u(x,t)$ (4-9) takes the form

$$u(x,t) = \frac{1}{2\pi t^{\frac{n}{2}}}\int_{-\infty}^\infty W\left(-\frac{n}{2}, 1-\frac{n}{2}; -\left|\ln\left(\frac{x+\sqrt{x^2+1}}{\tau+\sqrt{\tau^2+1}}\right)\right|t^{-\frac{n}{2}}\right)f(\tau)d\tau. \quad \square \tag{4-10}$$

5. Singular integral equations

In this section, we apply the $\mathcal{F}_A$-transform for solving singular integral equations with convolution-type kernel. For this purpose, let us consider the singular integral equation as follows

$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} k(x,t)f(t)dt, \quad \lambda \in \mathbb{C} \tag{5-1}$$

where, $k(x,t)$ is the convolution-type kernel in the form

$$k(x,t) = A'(t)h(A^{-1}(A(x) - A(t))), \tag{5-2}$$

and g, h are the known functions.

By applying the $\mathcal{F}_A$-transform to the both sides of (5-1) and using the convolution product (2-5), we get

$$F(p) = \frac{G(p)}{1 - \lambda H(p)},$$

and by the inversion of $F(p)$ we finally obtain

$$f(x) = \mathcal{F}_A^{-1}\left\{\frac{G(p)}{1 - \lambda H(p)}\right\}, \quad \lambda H(p) \neq 1. \tag{5-3}$$

As an application of the $\mathcal{F}_A$-transform for solving singular integral equations of convolution-type kernel, in next example we consider the generalized Hilbert type integral equation with a kernel satisfied by the convolution theorem of the $\mathcal{F}_{x^{2n+1}}$-transform.

Theorem 5.1. *The Solution of the generalized Hilbert type singular integral equation*

$$f(x) = g(x) + \lambda \mathcal{H}_{x^{2n+1}} f(x), \quad \lambda \in \mathbb{C}, \ n = 0, 1, 2, \ldots \tag{5-4}$$

is written in terms of the k-th iterate of generalized Hilbert transforms $\mathcal{H}^k_{x^{2n+1}}$, $k = 1, 2, \ldots$ *as follows*

$$f(x) = g(x) + \sum_{k=1}^{\infty} (-\lambda)^k \mathcal{H}^k_{x^{2n+1}} g(x). \tag{5-5}$$

The function g is the known function and the generalized Hilbert transform $\mathcal{H}_{x^{2n+1}}$ *is given by* [14]

$$\mathcal{H}_{x^{2n+1}} f(x) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{t^{2n}}{\sqrt[2n+1]{x^{2n+1} - t^{2n+1}}} f(t)dt. \tag{5-6}$$

Proof. By applying the $\mathcal{F}_{x^{2n+1}}$-transform on the singular integral Eq. (5-4) and knowing the operator $\mathcal{H}_{x^{2n+1}}$ is convolution of functions $f(x)$ and $\frac{1}{x}$ the transformed equation takes the form

$$\begin{aligned} F(p) &= G(p) + \lambda \mathcal{F}_{x^{2n+1}}\left\{\frac{1}{x}; p\right\} F(p) = G(p) + \lambda F(p) \int_{-\infty}^{\infty} x^{2n} \frac{e^{-ipx^{2n+1}}}{x} dx \\ &= G(p) - 2\lambda i F(p) \frac{\text{sgn}(p)}{p^{\frac{2n}{2n+1}}} \int_0^{\infty} \frac{\sin(u)}{u^{\frac{1}{2n+1}}} du, \end{aligned} \tag{5-7}$$

or equivalently

$$F(p) = \frac{1}{1 + \lambda i \frac{\mathrm{sgn}(p)}{p^{\frac{2n}{2n+1}} \Gamma\left(\frac{1}{2n+1}\right) \sin\left(\frac{\pi}{4n+2}\right)}} G(p). \tag{5-8}$$

Now, by expansion of the geometric series in (5-8) and using the obtained result in (5-7)

$$\mathcal{H}_{x^{2n+1}} g(x) = \mathcal{F}^{-1}_{x^{2n+1}} \left\{ \frac{-i\mathrm{sgn}(p)}{p^{\frac{2n}{2n+1}} \Gamma\left(\frac{1}{2n+1}\right) \sin\left(\frac{\pi}{4n+2}\right)} G(p) \right\},$$

the solution of the singular integral equation (5-4) can be written in terms of the k-th iterate of the generalized Hilbert transform $\mathcal{H}^k_{x^{2n+1}}$ as follows

$$f(x) = g(x) + \sum_{k=1}^{\infty} (-\lambda)^k \mathcal{H}^k_{x^{2n+1}} g(x), \tag{5-9}$$

provided that the k-th iterate of generalized Hilbert transforms $\mathcal{H}^k_{x^{2n+1}}$, $k = 1, 2, \ldots$ and series converge absolutely. □

6. Conclusions

This paper provides some new results in the areas of singular integral equations and fractional calculus. Furthermore, the implementation of the new integral transform ($\mathcal{F}_A$-transform) for solving two Cauchy type fractional diffusion equations of single and distributed order was discussed.

It may be concluded that the $\mathcal{F}_A$-transform method is very efficient technique for finding exact solution for PFDEs. Although the $\mathcal{F}_A$-transform method described in this paper is well-suited to solve the time fractional diffusion equations in terms of higher transcendental functions, the method could lead to a promising approach for many applications in applied sciences.

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