Arab Journal of Mathematical Sciences

Arab J Math Sci 22 (2016) 207–231

Solvability of Sturm–Liouville boundary value problems for multiple order fractional differential equations☆

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Received 7 December 2013; received in revised form 12 August 2015; accepted 29 August 2015
Available online 7 September 2015

Abstract. In this paper, definitions of L^1_α function space, β-well function space and quasi-Carathéodory function are given, existence results for solutions of a class of Sturm–Liouville boundary value problems of fractional differential equations with multiple order derivatives $D^\alpha_{0+}[\Phi(\rho(t)D^\beta_{0+}x(t))]$ are established. The analysis relies on a well known fixed point theorem. Some examples are given to illustrate the efficiency of main theorems.

Keywords: L^1_α function space; β-well function; Multiple order fractional differential equation; Sturm–Liouville boundary value problems; Leray–Schauder Alternative principle

2000 MR subject classification: 92D25; 34A37; 34K15

1. Introduction

It is well known that the following boundary value problem (BVP for short) for second order ordinary differential equation

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & t \in (0,1), \\ ax(0) - bx'(0) = 0, \\ cx'(1) + dx(1) = 0, \end{cases} \tag{1}$$

is called a Sturm–Liouville boundary value problem [5,6], where $f(t,u,v)$ is continuous on $[0,1] \times R \times R$, $a \geq 0$, $b \geq 0$, $c \geq 0$ and $d \geq 0$ with $ac + ad + bc > 0$. BVP(1) comes from a situation involving nonlinear elliptic problems in annular regions, one may see [5].

☆ Supported by the Natural Science Foundation of Guangdong province (No. S2011010001900) and Natural science research project for colleges and universities of Guangdong Province (No: 2014KTSCX126).

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.08.001

In order to generalize the results obtained in [5,6] to boundary value problems for fractional differential equations, in recent years, there exist many papers concerned with the existence of positive solutions of boundary value problems for fractional differential equations (see [1–3,7–9,12–14,16,10,15]).

In [8], Kaufmann and Mboumi studied the following boundary value problem for the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}x(t)+a(t)f(x(t))=0, & 0<t<1, \\ x(0)=0, \qquad x'(1)=0, \end{cases} \tag{2}$$

by using the properties of Green's function of the corresponding BVP, where $f:[0,+\infty)\to[0,+\infty)$ is continuous, $a:[0,1]\to(0,+\infty)$ is a continuous function and D_{0+}^{α} is the standard Riemann–Liouville fractional differential derivative of order $\alpha\in(1,2)$. Using the Leggett–Williams fixed point theorem and Krasnoselskii fixed point theorem, the authors in [8] proved that BVP(2) has at least one or three positive solutions.

Dehghant and Ghanbari [4] studied the existence of solutions of the following boundary value problem for the nonlinear fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}x(t)+a(t)f(t,x(t),x'(t))=0, & 0<t<1, \\ x(0)=0, \qquad x'(1)=0, \end{cases}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional differential derivative of order $\alpha\in(1,2)$, f is defined on $[0,1]\times[0,+\infty)\times R$, continuous and nonnegative, $a(t)$ does not identically vanish on any subinterval of $(0,1)$ and $a(t)$ satisfies the following inequality:

$$0<\int_0^1[t(1-t)]^{\alpha}a(t)dt<+\infty.$$

It is easy to see that the following boundary value problem for the ordinary differential equation

$$x''(t)=-1, \quad t\in(0,1), \quad x'(0)=0, \quad x(1)=0$$

has a unique continuous solution $u(t)=\frac{1}{2}(1-t^2)$ on $[0,1]$. However, we find that the similar boundary value problem for the multiple order fractional differential equation

$$D_{0+}^{\alpha}D_{0+}^{\beta}x(t)=-1, \quad t\in(0,1), \quad \lim_{t\to0}t^{1-\alpha}D_{0+}^{\beta}x(t)=0,$$
$$\lim_{t\to1}t^{1-\beta}x(t)=0, \quad \alpha,\beta\in(0,1)$$

has a unique solution

$$x(t)=[-t^{\alpha+\beta}+t^{\beta-1}]\frac{1}{\Gamma(\alpha+1)\Gamma(\beta)}\mathbf{B}(\beta,\alpha+1),$$

which is not continuous at $t=0$. This fact shows that there exist differences between BVPs for ordinary differential equations and BVPs for fractional differential equations.

Liu [11] studied the existence of solutions of the following boundary value problem for the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t))=0, & 0<t<1, \\ \lim_{t\to0}t^{2-\alpha}x(t)=0, \qquad D_{0+}^{\alpha-1}x(1)=0, \end{cases}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional differential derivative of order $\alpha \in (1,2)$, f is defined on $[0,1] \times [0,+\infty)$, nonnegative and is a Carathéodory function. Note that the boundary condition $x(0) = 0$ used in [4,8] is replaced by $\lim_{t\to 0} t^{2-\alpha}x(t) = 0$. Hence the solutions obtained in [11] may be unbounded or singular at $t = 0$.

In this paper, we discuss the existence of solutions to the following Sturm–Liouville boundary value problem for the nonlinear fractional differential equation with the nonlinearity depending on $D_{0+}^{\beta}x$

$$\begin{cases} D_{0+}^{\alpha}[\Phi(\rho(t)D_{0+}^{\beta}x(t))] + f(t,x(t),D_{0+}^{\beta}x(t)) = 0, \quad t \in (0,1), \\ a \lim\limits_{t\to 0} t^{1-\beta}x(t) - b \lim\limits_{t\to 0} \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t) = 0, \\ c \lim\limits_{t\to 1} \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t) + d \lim\limits_{t\to 1} t^{1-\beta}x(t) = 0, \end{cases} \tag{3}$$

where

- $a, b, c, d \in R$,
- D_{0+}^{α} (or D_{0+}^{β}) is the left-side Riemann–Liouville fractional derivative of order α (or β) with $\alpha, \beta \in (0,1)$,
- $\rho : (0,1) \to [0,+\infty)$ is continuous and may be singular at $t = 0$ and $t = 1$,
- f defined on $(0,1) \times R \times R$ is a quasi-Carathéodory function (see Definition 2.4) that may be singular at $t = 0$ and $t = 1$,
- $\Phi(s) = |s|^{p-2}s$ with $p > 1$ is called a p-Laplacian and its inverse function denoted by $\Phi^{-1}\Phi^{-1}(x) = |x|^{q-2}x$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We obtain some results on the existence of solutions of BVP(3) (see Definition 2.5). Two examples are given to illustrate the efficiency of the main theorems. It is obvious that BVP(3) is more general than those studied in the mentioned papers. Our results generalize and enrich known literatures.

The remainder of this paper is as follows: in Section 2, we present preliminary results and main theorems. In Section 3, two examples are given to illustrate the main results.

2. Main results

For convenience, we present some necessary definitions from fractional calculus theory. These definitions and results can be found in the literatures [3,7,12,14]. Let the Gamma function and Beta function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \qquad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Definition 2.1 (*[14]*)**.** The left-side Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,+\infty) \to R$ is given by

$$I_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2 ([14]). Let n be a positive integer. The left-side Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,+\infty) \to R$ is given by

$$D_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 \le \alpha < n$, provided that the right-hand side exists.

Lemma 2.1 ([14]). *Let $n-1 \le \mu < n$, $x \in C^0(0,+\infty) \bigcap L^1(0,+\infty)$. Then*

$$I_{0+}^{\mu} D_{0+}^{\mu} x(t) = x(t) + C_1 t^{\mu-1} + C_2 t^{\mu-2} + \cdots + C_n t^{\mu-n},$$

where $C_i \in R$, $i = 1, 2, \ldots, n$.

Definition 2.3. Let $\sigma > 0$. $x : (0,1) \to R$ is called a L_σ^1 function if $t \to t^{1-\sigma} \int_0^t (t-s)^{\sigma-1} |x(s)| ds$ is continuous on $[0,1]$. The set of all L_σ^1 functions on $(0,1)$ is denoted by $L_\sigma^1(0,1)$.

- $x : (0,1) \to R$ is called a σ-well function if it is a L_σ^1 function and for each $0 \le t_2 \le t_1 \le 1$, it holds that

$$t_1^{1-\sigma} \int_0^{t_1} (t_1-s)^{\sigma-1} x(s) ds + t_2^{1-\sigma} \int_0^{t_2} (t_2-s)^{\sigma-1} x(s) ds \\ - 2t_1^{1-\sigma} \int_0^{t_2} (t_1-s)^{\sigma-1} x(s) ds \to 0$$

as $t_1 \to t_2$.

Remark 2.1. Note that $L_\sigma^1(0,1)$ is a generalization of $L^1(0,1)$. It is easy to check that all bounded functions on $[0,1]$ are σ-well functions and the functions that satisfy that there exists $\mu > -1$ such that $|x(t)| \le t^\mu$ for all $t \in [0,1]$ are σ-well functions too.

Definition 2.4. $f : (0,1) \times R^2 \to R$ is called a quasi-Carathéodory function if

- $t \to f\left(t, t^{\beta-1}u, \frac{1}{\Phi^{-1}(t^{1-\alpha})\rho(t)} v\right)$ is a L^α function for every $(u,v) \in R^2$ (see Definition 2.3),
- $(u,v) \to f\left(t, t^{\beta-1}u, \frac{1}{\Phi^{-1}(t^{1-\alpha})\rho(t)} v\right)$ is continuous for each $t \in (0,1)$,
- for each $r > 0$ there exists a nonnegative β-well L_α^1 function (see Definition 2.3) ϕ_r such that

$$\left| f\left(t, t^{\beta-1}u, \frac{1}{\Phi^{-1}(t^{1-\alpha})\rho(t)} v\right)\right| \le \phi_r(t), \quad t \in (0,+\infty), \\ (u,v) \in [-r,r] \times [-r,r].$$

Definition 2.5. $x : (0,1) \to R$ is called a solution of BVP(3) if $x \in C(0,1) \bigcap L^1(0,1)$ satisfies that $D_{0+}^{\alpha}[\Phi(\rho D_{0+}^{\beta} x)] \in L_\alpha^1(0,1)$ and all equations in (3) are satisfied.

For our construction, we let

$$X = \left\{ x:(0,1)\to R \quad \begin{matrix} x\in C(0,1),\ D_{0+}^{\beta}x\in C(0,1), \\ \text{the following limits are finite} \\ \lim_{t\to 0} t^{1-\beta}x(t),\ \lim_{t\to 1} x(t), \\ \lim_{t\to 0} \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t),\ \lim_{t\to 1}\rho(t)D_{0+}^{\beta}x(t) \end{matrix} \right\}.$$

For $x\in X$, let

$$\|x\| = \max\left\{ \sup_{t\in(0,1)} t^{1-\beta}|x(t)|,\ \sup_{t\in(0,1)} \Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}x(t)| \right\}.$$

Lemma 2.2. *X is a Banach space with the norm $\|\cdot\|$ defined.*

Proof. Note that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X, then $\|x_u - x_v\|\to 0,\ u,v\to+\infty$. It follows that

$$\begin{aligned}
& x_u \in C^0(0,1), \quad u\in N, \\
& \lim_{t\to 0} t^{1-\beta}x_u(t), \lim_{t\to 1} x_u(t) \text{ exist}, \quad u\in N, \\
& \lim_{t\to 0} \Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t), \lim_{t\to 1}\rho(t)D_{0+}^{\beta}x_u(t) \text{ exist}, \quad u\in N, \\
& \sup_{t\in(0,1)} t^{1-\beta}|x_u(t)-x_v(t)|\to 0, \quad u,v\to+\infty, \\
& \sup_{t\in(0,1)} \Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}x(t) - D_{0+}^{\beta}x_v(t)|\to 0, \quad u,v\to+\infty.
\end{aligned}$$

Define

$$t^{1-\beta}x_u(t) = \begin{cases} \lim_{t\to 0} t^{1-\beta}x_u(t), & t=0, \\ t^{1-\beta}x_u(t), & t\in(0,1), \\ \lim_{t\to 1} t^{1-\beta}x_u(t) \end{cases}$$

$$\Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t) = \begin{cases} \lim_{t\to 0} \Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t), & t=0, \\ \Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t), & t\in(0,1), \\ \lim_{t\to 1} \Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t). \end{cases}$$

Then $t\to \Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t)$ and $t\to t^{1-\beta}x_u(t)$ are continuous on $[0,1]$. Thus there exist two functions x_0, y_0 defined on $[0,1]$ such that

$$\lim_{u\to+\infty} t^{1-\beta}x_u(t) = x_0(t), \qquad \lim_{u\to+\infty} \Phi(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t) = y_0(t).$$

It follows from $\|x_u - x_v\|\to 0,\ u,v\to+\infty$ that

$$\begin{aligned}
& \sup_{t\in(0,1)} |t^{1-\beta}x_u(t) - x_0(t)|\to 0, \quad u\to+\infty, \\
& \sup_{t\in(0,1)} |\Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t) - y_0(t)|\to 0, \quad u\to+\infty.
\end{aligned}$$

Denote $z_0(t) = t^{\beta-1}x_0(t)$ and $w_0(t) = \frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}$ for $t \in (0,1)$. This means that functions $z_0, w_0 : (0,1) \to R$ are well defined.

Step 1. Prove that the limits $\lim_{t\to 0} t^{1-\beta}z_0(t)$, $\lim_{t\to 1} t^{1-\beta}z_0(t)$, $\lim_{t\to 0} \Phi^{-1}(t^{1-\alpha})\rho(t) w_0(t)$ and $\lim_{t\to 1} \Phi^{-1}(t^{1-\alpha})\rho(t)w_0(t)$ exist.

Since both $t \to t^{1-\beta}z_0(t) = x_0(t)$ and $t \to \Phi^{-1}(t^{1-\alpha})\rho(t)w_0(t) = y_0(t)$ are continuous on $[0,1]$, the limits $\lim_{t\to 0} t^{1-\beta}z_0(t)$, $\lim_{t\to 1} t^{1-\beta}z_0(t)$, $\lim_{t\to 0} \Phi^{-1}(t^{1-\alpha})\rho(t)w_0(t)$ and $\lim_{t\to 1} \Phi^{-1}(t^{1-\alpha})\rho(t)w_0(t)$ exist.

Step 2. Prove that $\sup_{t\in(0,1)} t^{1-\beta}|x_u(t)-z_0(t)| \to 0$ and $\sup_{t\in(0,1)} \Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta} x_u(t) - w_0(t)| \to 0$ as $u \to +\infty$ in X.

We have that

$$\sup_{t\in(0,1)} |t^{1-\beta}x_u(t) - x_0(t)| \to 0, \quad u \to +\infty,$$
$$\sup_{t\in(0,1)} |\Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t) - y_0(t)| \to 0, \quad u \to +\infty.$$

The results follow.

Step 3. Prove that $y_0(t) = \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}z_0(t)$ for all $t \in (0,1)$.

We have that there exists constant $c_u \in R$ such that

$$\begin{aligned}
&\left|x_u(t) + c_u t^{\beta-1} - I_{0+}^{\beta}\frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}\right| = \left|I_{0+}^{\beta}D_{0+}^{\beta}x_u(t) - I_{0+}^{\beta}\frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}\right| \\
&\quad = I_{0+}^{\beta}\left|D_{0+}^{\beta}x_u(t) - \frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}\right| \\
&\quad = I_{0+}^{\beta}\frac{1}{\Phi^{-1}(t^{1-\alpha})\rho(t)}\left|\Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t) - y_0(t)\right| \\
&\quad \le \sup_{t\in(0,1)}\left|\Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x_u(t) - y_0(t)\right| I_{0+}^{\beta}\frac{1}{\Phi^{-1}(t^{1-\alpha})\rho(t)} \to 0 \\
&\qquad \text{as } u \to +\infty.
\end{aligned}$$

Then $\lim_{u\to+\infty}[x_u(t) + c_u t^{\beta-1}] = I_{0+}^{\beta}\frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}$. Hence $t^{\beta-1}[x_0(t) + c_0 t^{\beta-1}] = I_{0+}^{\beta}\frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}$. Thus $z_0(t) + c_0 t^{\beta-1} = I_{0+}^{\beta}\frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}$. It follows that $D_{0+}^{\beta}z_0(t) = \frac{y_0(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}$.

It follows that X is a Banach space. The proof is completed.

We list the following assumption:

(H). f is a quasi-Carathéodory function, ρ is nonnegative and continuous, $a, b, c, d \ge 0$ are constants that satisfy

$$t \to \frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} \text{ is a } \beta\text{-well function on } (0,1),$$
$$\Delta = ac + bd + ad\int_0^1 (1-s)^{\beta-1}\frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)}ds \ne 0.$$

Lemma 2.3. *Suppose that (H) holds. Then for each $x \in X$, there exists a unique $A_x \in R$ satisfying*

$$|A_x| \le \max_{t\in[0,1]} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s),D_{0+}^{\beta}x(s))|ds \tag{4}$$

such that

$$\begin{aligned} &bd\Phi^{-1}(A_x) + ac\Phi^{-1}\left(A_x - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),D_{0+}^{\beta}x(s))ds\right) \\ &\quad + ad\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \\ &\quad \times \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha)A_x - s^{1-\alpha}\int_0^s (s-\nu)^{\alpha-1} f(\nu,x(\nu),D_{0+}^{\beta}x(\nu))d\nu\right)}{\rho(s)} ds = 0. \end{aligned} \tag{5}$$

Proof. For $x \in X$, we find $\|x\| =: r < +\infty$. Since f is a quasi-Carathéodory function (see Definition 2.4), there exists nonnegative β-well L_α^1 function ϕ_r such that

$$\begin{aligned} |f(t,x(t),D_{0+}^{\beta}x(t))| &= \left| f\left(t, t^{\beta-1}[t^{1-\beta}x(t)], \frac{\Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t)}{\Phi^{-1}(t^{1-\alpha})\rho(t)}\right)\right| \\ &\le \phi_r(t). \end{aligned} \tag{6}$$

Let

$$\begin{aligned} G(u) &= bd\Phi^{-1}(u) + ac\Phi^{-1}\left(u - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),D_{0+}^{\beta}x(s))ds\right) \\ &\quad + ad\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \\ &\quad \times \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha)u - s^{1-\alpha}\int_0^s (s-\nu)^{\alpha-1} f(\nu,x(\nu),D_{0+}^{\beta}x(\nu))d\nu\right)}{\rho(s)} ds. \end{aligned}$$

Then G is well defined on R. Since $\Delta > 0$, G is continuous and strictly increasing on R. One sees that

$$G\left(\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} |f(t,x(t),D_{0+}^{\beta}x(t))|\right) \ge 0$$

and

$$G\left(-\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} |f(t,x(t),D_{0+}^{\beta}x(t))|\right) \le 0.$$

Hence there exists a unique

$$A_x \in \left[-\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} |f(t,x(t),D_{0+}^{\beta}x(t))|,\ \max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} |f(t,x(t),D_{0+}^{\beta}x(t))|\right]$$

such that $G(A_x) = 0$. Furthermore, A_x satisfies (4). Then (5) is proved.

Define the nonlinear operator T on X by

$$(Tx)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \times \frac{\Phi^{-1}\left(-\Gamma(\alpha) I_{0+}^{\alpha} f(s, x(s), D_{0+}^{\beta} x(s)) + \Gamma(\alpha) A_x s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds + B_x t^{\beta-1} \tag{7}$$

for $x \in X$, where A_x satisfies (5) and B_x is defined by

$$B_x = \begin{cases} \frac{b}{a}\Phi^{-1}(A_x), \quad a \neq 0, \\ \frac{c}{d}\Phi^{-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), D_{0+}^{\beta} x(s)) ds\right) \\ \quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \\ \quad \times \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha) s^{1-\alpha} I_{0+}^{\alpha} f(s, x(s), D_{0+}^{\beta} x(s))\right)}{\rho(s)} ds, \quad a = 0. \end{cases} \tag{8}$$

Remark 2.2. It is easy to see from (8) that

$$|B_x| \leq \begin{cases} \frac{b}{a}\Phi^{-1}\left(\max_{t\in[0,1]} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}|f(s, x(s), D_{0+}^{\beta} x(s))| ds\right) & \text{if } a \neq 0, \\ \left[\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds + \frac{c}{d}\right] & \\ \quad \times \Phi^{-1}\left(\max_{t\in[0,1]} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}|f(s, x(s), D_{0+}^{\beta} x(s))| ds\right) & \text{if } a = 0. \end{cases} \tag{9}$$

Lemma 2.4. *Suppose that (H) holds. Then $T : X \to X$ is completely continuous.*

Proof. For $x \in X$, we find $\|x\| \leq r < +\infty$. Since f is a quasi-Carathéodory function (see Definition 2.4), there exists nonnegative β-well L_α^1 function ϕ_r such that (6) holds. Hence A_x is uniquely determined by (5). Hence B_x is uniquely determined by (8) too. Then $(Tx)(t)$ is well defined by (7). It is easy to see from (7) that

$$D_{0+}^{\beta}(Tx)(t) = \frac{\Phi^{-1}\left(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), D_{0+}^{\beta} x(s)) ds + A_x t^{\alpha-1}\right)}{\rho(t)}.$$

Then

$$\Phi^{-1}(t^{1-\alpha})\rho(t) D_{0+}^{\beta}(Tx)(t) = \Phi^{-1}\left(-t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), D_{0+}^{\beta} x(s)) ds + A_x\right).$$

It follows that

$$D_{0+}^{\alpha}[\Phi(\rho(t)D_{0+}^{\beta}(Tx)(t))] = -f(t,x(t),D_{0+}^{\beta}x(t)). \tag{10}$$

Furthermore, we get

$$\begin{aligned}
&(Tx) \in C^0(0,1), \quad D_{0+}^{\beta}(Tx) \in C^0(0,1),\\
&\lim_{t\to 0} t^{1-\beta}(Tx)(t) = B_x,\\
&\lim_{t\to 0} \ \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}(Tx)(t) = \Phi^{-1}(A_x),\\
&\lim_{t\to 1} \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}(Tx)(t)\\
&\quad = \Phi^{-1}\left(-\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s),D_{0+}^{\beta}x(s))ds + A_x\right),\\
&\lim_{t\to 1} t^{1-\beta}(Tx)(t) = B_x + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\\
&\quad \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu,x(\nu),D_{0+}^{\beta}x(\nu))d\nu + \Gamma(\alpha)A_x s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds.
\end{aligned}$$

From (5) and (8), we see

$$\begin{aligned}
&a \lim_{t\to 0} t^{1-\beta}(Tx)(t) - b \lim_{t\to 0} \ \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}(Tx)(t) = 0,\\
&c \lim_{t\to 1} \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}(Tx)(t) + d \lim_{t\to 1} t^{1-\beta}(Tx)(t) = 0.
\end{aligned} \tag{11}$$

Hence $T : X \to X$ is well defined and $x \in X$ is a solution of BVP(3) if and only if x is a fixed point of T in X.

To prove that T is completely continuous, we address the following three steps.

Step 1. T is continuous.

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence such that $x_n \to x_0$ in X. We will show $Tx_n \to Tx_0$ as $n \to +\infty$. We see

$$r = \sup_{n=0,1,2,\dots} \|x_n\| < \infty.$$

Since f is a quasi-Carathéodory function (see Definition 2.4), there exists nonnegative β-well L_{α}^1 function ϕ_r such that (6) holds with x being replaced by x_n.

First we prove that

$$A_{x_n} \to A_{x_0} \quad \text{as } n \to +\infty. \tag{12}$$

From Lemma 2.2, we get

$$\begin{aligned}
|A_{x_n}| &\le \max_{t\in[0,1]} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x_n(s),D_{0+}^{\beta}x_n(s))|ds\\
&\le \max_{t\in[0,1]} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_r(s)ds.
\end{aligned}$$

So $\{A_{x_n}\}$ is bounded. If (12) does not hold, then there exist two subsequences $\{A^{(1)}_{x_{n_k}}\}$ and $\{A^{(2)}_{x_{n_k}}\}$ of $\{A_{x_n}\}$ such that

$$A^{(i)}_{x_{n_k}} \to c_i (i = 1, 2) \quad \text{as } n \to +\infty$$

with $c_1 \neq c_2$. Since $A^{(i)}_{x_{n_k}}$ satisfies

$$\begin{aligned}
& bd\Phi^{-1}(A^{(i)}_{x_{n_k}}) + ac\Phi^{-1}\left(A^{(i)}_{x_{n_k}} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{n_k^{(i)}}(s), D^{\beta}_{0+}x_{n_k^{(i)}}(s))ds\right) \\
& \quad + ad\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \\
& \qquad \times \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha)A^{(i)}_{x_{n_k}} - s^{1-\alpha}\int_0^s (s-\nu)^{\alpha-1} f(\nu, x_{n_k^{(i)}}(\nu), D^{\beta}_{0+}x_{n_k^{(i)}}(\nu))d\nu\right)}{\rho(s)} ds \\
& \quad = 0
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \right. \\
& \qquad \left. \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha)A^{(i)}_{x_{n_k}} - s^{1-\alpha}\int_0^s (s-\nu)^{\alpha-1} f(\nu, x_{n_k^{(i)}}(\nu), D^{\beta}_{0+}x_{n_k^{(i)}}(\nu))d\nu\right)}{\rho(s)} ds \right| \\
& \quad \le \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds \Phi^{-1} \\
& \qquad \times \left((\Gamma(\alpha)+1) \max_{t\in[0,1]} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\phi_r(s)ds\right),
\end{aligned}$$

by Lebesgue's dominant theorem, let $k \to +\infty$, then

$$\begin{aligned}
& bd\Phi^{-1}(c_i) + ac\Phi^{-1}\left(c_i - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_0(s), D^{\beta}_{0+}x_0(s))ds\right) \\
& \quad + ad\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \\
& \quad \times \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha)c_i - s^{1-\alpha}\int_0^s (s-\nu)^{\alpha-1} f(\nu, x_0(\nu), D^{\beta}_{0+}x_0(\nu))d\nu\right)}{\rho(s)} ds = 0.
\end{aligned}$$

Since

$$\begin{aligned}
& bd\Phi^{-1}(A_{x_0}) + ac\Phi^{-1}\left(A_{x_0} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_0(s), D^{\beta}_{0+}x_0(s))ds\right) \\
& \quad + ad\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \\
& \quad \times \frac{\Phi^{-1}(s^{\alpha-1})\Phi^{-1}\left(\Gamma(\alpha)A_{x_0} - s^{1-\alpha}\int_0^s (s-\nu)^{\alpha-1} f(\nu, x_0(\nu), D^{\beta}_{0+}x_0(\nu))d\nu\right)}{\rho(s)} ds = 0,
\end{aligned}$$

together with Lemma 2.2, we get that $c_1 = c_2 = A_{x_0}$, a contradiction. So (12) holds.

Now by the definition of B_x, one has

$$B_{x_n} \to B_{x_0} \quad \text{as } n \to +\infty. \tag{13}$$

From

$$\begin{aligned}
&\Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}(Tx_n)(t) \\
&\quad = \Phi^{-1}\left(-t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(s), D_{0+}^{\beta}x_n(s))ds + A_{x_n}\right)
\end{aligned}$$

and

$$\begin{aligned}
t^{1-\beta}(Tx_n)(t) &= t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, x_n(\nu), D_{0+}^{\beta}x_n(\nu))d\nu + \Gamma(\alpha)A_{x_n}s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds + B_{x_n},
\end{aligned}$$

since f is a quasi-Carathéodory function (see Definition 2.4) and (12), (13), $Tx_n \to Tx_0$ as $n \to +\infty$. Hence T is continuous.

Step 2. T maps bounded sets into bounded sets in X.

It suffices to show that for each $r > 0$, there exists a positive number $L > 0$ such that for each $x \in M = \{y \in X : \|y\| \leq r\}$, we have $\|Ty\| \leq L$. By the assumption, f is a quasi-Carathéodory function (see Definition 2.4), there exists a nonnegative β-well L_α^1 function ϕ_r (see Definition 2.3) such that (6) holds. By the definition of T, we have

$$\begin{aligned}
&\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}(Ty)(t)| \\
&\quad = \left|\Phi^{-1}\left(-t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0+}^{\beta}y(s))ds + A_{x_n}\right)\right| \\
&\quad \leq \left|\Phi^{-1}\left(t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_r(s)ds + \max_{t\in[0,1]} t^{1-\alpha}\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\phi_r(u)du\right)\right| \\
&\quad \leq \Phi^{-1}\left(2\max_{t\in[0,1]} t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_r(s)ds\right).
\end{aligned}$$

Now we consider $t^{1-\beta}|(Ty)(t)|$.

Case 1. $a \neq 0$. We have

$$\begin{aligned}
t^{1-\beta}|(Ty)(t)| &= \left| t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \right. \\
&\quad \left. \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta}y(\nu))d\nu + \Gamma(\alpha)A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds + B_y \right| \\
&\leq |B_y| + \left| t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \right. \\
&\quad \left. \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta}y(\nu))d\nu + \Gamma(\alpha)A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \right|
\end{aligned}$$

$$\leq \left(\frac{b}{a} + \max_{t\in[0,1]} t^{1-\beta} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds\right)$$
$$\times \Phi^{-1}\left(2 \max_{t\in[0,1]} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_r(s) ds\right).$$

Case 2. $a = 0$. We have

$$t^{1-\beta}|(Ty)(t)| = \left| t^{1-\beta} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \right.$$
$$\left. \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu)) d\nu + \Gamma(\alpha) A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds + B_y \right|$$
$$\leq |B_y| + \left| t^{1-\beta} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \right.$$
$$\left. \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu)) d\nu + \Gamma(\alpha) A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \right|$$
$$\leq \left(\frac{c}{d} + 2 \max_{t\in[0,1]} t^{1-\beta} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds\right)$$
$$\times \Phi^{-1}\left(2 \max_{t\in[0,1]} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_r(s) ds\right).$$

It follows that there exists a constant $M > 0$ such that

$$\|Ty\| \leq M$$

for each $y \in \{y \in X : \|y\| \leq l\}$. Then T maps bounded sets into bounded sets in X.

Step 3. Let $M = \{y \in X : \|y\| \leq r\}$. Prove that both $\{t \to t^{1-\beta}(Tx)(t) : x \in M\}$ and $\{t \to \Phi^{-1}(t^{1-\alpha})\rho(t) D_{0+}^{\beta}(Tx)(t) : x \in M\}$ are equicontinuous on $(0, 1)$.

By $M = \{y \in X : \|y\| \leq r\}$, there exists nonnegative β-well L_α^1 function ϕ_r such that (6) holds. Let $t_1, t_2 \in (0, 1)$ with $t_2 < t_2$. We have

$$\left| t_1^{1-\beta}(Ty)(t_1) - t_2^{1-\beta}(Ty)(t_2) \right|$$
$$= \left| t_1^{1-\beta} \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right.$$
$$\times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu)) d\nu + \Gamma(\alpha) A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds$$
$$-t_2^{1-\beta} \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}$$
$$\left. \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu)) d\nu + \Gamma(\alpha) A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \right|$$

$$= \Bigg| t_1^{1-\beta} \int_{t_2}^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu))d\nu + \Gamma(\alpha)A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds + \int_0^{t_2} \frac{t_1^{1-\beta}(t_1-s)^{\beta-1} - t_2^{1-\beta}(t_2-s)^{\beta-1}}{\Gamma(\beta)} \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu))d\nu + \Gamma(\alpha)A_y s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \Bigg|.$$

Since $t^{1-\beta}(t-s)^{\beta-1}$ is decreasing, we get

$$\begin{aligned} &\left| t_1^{1-\beta}(Ty)(t_1) - t_2^{1-\beta}(Ty)(t_2) \right| \\ &\le t_1^{1-\beta} \int_{t_2}^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \times \frac{\left|\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu))d\nu + \Gamma(\alpha)A_y s^{\alpha-1}\right)\right|}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \\ &\quad + \int_0^{t_2} \frac{t_2^{1-\beta}(t_2-s)^{\beta-1} - t_1^{1-\beta}(t_1-s)^{\beta-1}}{\Gamma(\beta)} \times \frac{\left|\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, y(\nu), D_{0+}^{\beta} y(\nu))d\nu + \Gamma(\alpha)A_y s^{\alpha-1}\right)\right|}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds. \end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned} &\left| t_1^{1-\beta}(Ty)(t_1) - t_2^{1-\beta}(Ty)(t_2) \right| \\ &\le \frac{1}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \left| t_1^{1-\beta} \int_{t_2}^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds + \int_0^{t_2} \frac{t_2^{1-\beta}(t_2-s)^{\beta-1} - t_1^{1-\beta}(t_1-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds \right| \\ &\quad \times \Phi^{-1}\left(2 \max_{s\in[0,1]} s^{1-\alpha} \int_0^s (s-\nu)^{\alpha-1} |f(\nu, y(\nu), D_{0+}^{\beta} y(\nu))| d\nu\right) \\ &\le \frac{1}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \left| t_1^{1-\beta} \int_{t_2}^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds + \int_0^{t_2} \frac{t_2^{1-\beta}(t_2-s)^{\beta-1} - t_1^{1-\beta}(t_1-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds \right| \\ &\quad \times \Phi^{-1}\left(2 \max_{s\in[0,1]} s^{1-\alpha} \int_0^s (s-u)^{\alpha-1} \phi_r(u) du\right) \\ &= \frac{1}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))} \left| t_1^{1-\beta} \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds \right. \end{aligned}$$

$$
\begin{aligned}
&+ t_2^{1-\beta} \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds \\
&- 2t_1^{1-\beta} \int_0^{t_2} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds \Bigg| \, \Phi^{-1} \\
&\times \left(2 \max_{s\in[0,1]} s^{1-\alpha} \int_0^s (s-\nu)^{\alpha-1} \phi_r(\nu) d\nu \right).
\end{aligned}
$$

Since $\frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)} ds$ is a β-well function, then we get

$$
\left| t_1^{1-\beta}(Ty)(t_1) - t_2^{1-\beta}(Ty)(t_2) \right| \to 0 \quad \text{uniformly as } t_1 \to t_2. \tag{14}
$$

Similarly, we get

$$
\begin{aligned}
&\left| \Phi^{-1}(t_1^{1-\alpha})\rho(t_1) D_{0^+}^{\beta}(Ty)(t_1) - \Phi^{-1}(t_2^{1-\alpha})\rho(t_2) D_{0^+}^{\beta}(Ty)(t_2) \right| \\
&= \Bigg| \Phi^{-1}\left(-t_1^{1-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds + A_y \right) \\
&\quad - \Phi^{-1}\left(-t_2^{1-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds + A_y \right) \Bigg|.
\end{aligned} \tag{15}
$$

Then

$$
\begin{aligned}
&\Bigg| \left(-t_1^{1-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds + A_y \right) \\
&\quad - \left(-t_2^{1-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds + A_y \right) \Bigg| \\
&= \Bigg| -t_1^{1-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds \\
&\quad + t_2^{1-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds \Bigg| \\
&\le \Bigg| t_1^{1-\beta} \int_{t_2}^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} |f(s, y(s), D_{0^+}^{\beta} y(s))| ds \\
&\quad + \int_0^{t_2} \frac{t_2^{1-\beta}(t_2-s)^{\beta-1} - t_1^{1-\beta}(t_1-s)^{\beta-1}}{\Gamma(\beta)} |f(s, y(s), D_{0^+}^{\beta} y(s))| ds \Bigg| \\
&\le \Bigg| t_1^{1-\beta} \int_{t_2}^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \phi_r(s) ds \\
&\quad + \int_0^{t_2} \frac{t_2^{1-\beta}(t_2-s)^{\beta-1} - t_1^{1-\beta}(t_1-s)^{\beta-1}}{\Gamma(\beta)} \phi_r(s) ds \Bigg| \\
&= \left| t_1^{1-\beta} I_{0^+}^{\beta} \phi_r(t_1) + t_2^{1-\beta} I_{0^+}^{\beta} \phi_r(t_2) \; -2t_1^{1-\beta} I_{0^+}^{\beta} \phi_r(t_2) \right|.
\end{aligned}
$$

Since ϕ_r is a β-well function, then we get

$$\begin{aligned}&\Big|\Big(-t_1^{1-\alpha}I_{0+}^{\alpha}f(t_1,y(t_1),D_{0+}^{\beta}y(t_1))+A_y\Big)\\&\quad-\Big(-t_2^{1-\alpha}I_{0+}^{\alpha}f(t_2,y(t_2),D_{0+}^{\beta}y(t_2))+A_y\Big)\Big|\to 0\quad\text{uniformly as } t_1\to t_2.\end{aligned}\tag{16}$$

It is easy to see that

$$\Big|\Big(-t^{1-\alpha}I_{0+}^{\alpha}f(t,y(t),D_{0+}^{\beta}y(t))+A_y\Big)\Big|\le 2\Gamma(\alpha)\max_{s\in[0,1]}s^{1-\alpha}I_{0+}^{\alpha}\phi_r(t)=:L,$$

and Φ^{-1} is uniformly continuous on $[-L,L]$, then for each $\epsilon>0$, there exists $\delta_0>0$ such that $|x_1-x_2|<\delta_0$ with $x_1,x_2\in[-L,L]$ implies

$$|\Phi^{-1}(x_1)-\Phi^{-1}(x_2)|<\epsilon.$$

It follows from (16) that there exists $\delta>0$ such that $t_1,t_2\in(0,1)$ and $|t_1-t_2|<\delta$ imply

$$\begin{aligned}&\Big|\Big(-t_1^{1-\alpha}I_{0+}^{\alpha}f(t_1,y(t_1),D_{0+}^{\beta}y(t_1))+A_y\Big)\\&\quad-\Big(-t_2^{1-\alpha}I_{0+}^{\alpha}f(t_2,y(t_2),D_{0+}^{\beta}y(t_2))+A_y\Big)\Big|<\delta_0.\end{aligned}$$

Hence $t_1,t_2\in(0,1)$ and $|t_1-t_2|<\delta$ imply

$$\begin{aligned}&\Big|\Phi^{-1}(t_1^{1-\alpha})\rho(t_1)D_{0+}^{\beta}(Ty)(t_1)-\Phi^{-1}(t_2^{1-\alpha})\rho(t_2)D_{0+}^{\beta}(Ty)(t_2)\Big|\\&\quad=\Big|\Phi^{-1}\Big(-t_1^{1-\alpha}I_{0+}^{\alpha}f(t_1,y(t_1),D_{0+}^{\beta}y(t_1))+A_y\Big)\\&\qquad-\Phi^{-1}\Big(-t_2^{1-\alpha}I_{0+}^{\alpha}f(t_2,y(t_2),D_{0+}^{\beta}y(t_2))+A_y\Big)\Big|<\epsilon.\end{aligned}$$

It follows that

$$\Big|\Phi^{-1}(t_1^{1-\alpha})\rho(t_1)D_{0+}^{\beta}(Ty)(t_1)-\Phi^{-1}(t_2^{1-\alpha})\rho(t_2)D_{0+}^{\beta}(Ty)(t_2)\Big|\to 0\tag{17}$$

uniformly as $t_1\to t_2$.

From (14) and (17), both $\{t\to t^{1-\beta}(Tx)(t): x\in M\}$ and $\{t\to \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}(Tx)(t): x\in M\}$ are equicontinuous on $(0,1)$.

The Arzelà–Ascoli theorem implies that TM is relatively compact. Thus, the operator $T: X\to X$ is completely continuous.

In order to prove the existence of solution of BVP(3), we use the following topological transversality theorem given by Granas [7].

Lemma 2.5. *Let B be a convex subset of a normed linear space X and assume $0\in B$. Let $T: B\to B$ be a completely continuous operator and let $U(T)=\{x: x=\lambda Tx\}$ for some $0<\lambda<1$. Then either $U(T)$ is unbounded or T has a fixed point.*

Theorem 2.1. *Suppose that (H) holds and there exist nonnegative L_α^1 functions A,B and C such that*

$$\left|f\left(t,t^{\beta-1}u,\frac{1}{\Phi^{-1}(t^{1-\alpha})\rho(t)}v\right)\right|\le A(t)\Phi(|u|)+B(t)\Phi(|v|)+C(t)\tag{18}$$

holds for all $t \in (0,1), u, v \in R$. *Then* BVP(3) *has at least one solution if*

$$\begin{aligned}\delta_0 &=: \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} A(t)\right) + \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} B(t)\right)\\ &< \frac{1}{C_q \mu_0} \quad \textit{for } a \neq 0,\\ \delta_1 &=: \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} A(t)\right) + \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} B(t)\right)\\ &< \frac{1}{C_q \mu_1} \quad \textit{for } a = 0,\end{aligned} \tag{19}$$

where

$$\begin{aligned}C_q &= \begin{cases}1, & q \in (1,2],\\ 2^{q-2}, & q > 2.\end{cases} \qquad \frac{1}{p}+\frac{1}{q}=1, p, q \textit{ are defined in Section } 1,\\ \mu_0 &= \frac{2}{\phi^{-1}(\Gamma(\alpha))}\sup_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d} + 1,\\ \mu_1 &= \frac{1}{\phi^{-1}(\Gamma(\alpha))}\sup_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{b}{a} + 1.\end{aligned}$$

Proof. We shall prove that under the assumptions (18) and (19), T has at least one fixed point in X by using Leray–Schauder Alternative principle. Let

$$U(T) = \{x : x = \lambda T x \text{ for some } 0 < \lambda < 1\}.$$

Case 1. $a \neq 0$. Indeed, by the definition of T for $x \in U(T)$, by using (4) and (9), we have the estimates

$$\begin{aligned}&t^{1-\beta}|(Tx)(t)|\\ &\quad= \Bigg|B_x + t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\\ &\qquad\times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, x(\nu), D_{0+}^{\beta}x(\nu))d\nu + \Gamma(\alpha)A_x s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)}ds\Bigg|\\ &\quad\le |B_x| + \Bigg|t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\\ &\qquad\times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, x(\nu), D_{0+}^{\beta}x(\nu))d\nu + \Gamma(\alpha)A_x s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)}ds\Bigg|\\ &\quad\le \left[\frac{2}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d}\right]\Phi^{-1}\\ &\qquad\times\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}|f(t, x(t), D_{0+}^{\beta}x(t))|\right)\end{aligned}$$

$$
\begin{aligned}
&\le \left[\frac{2}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)}+\frac{c}{d}\right]\\
&\quad\times \Phi^{-1}\biggl(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}[A(t)\,\Phi(u^{1-\beta}|x(t)|)\\
&\quad + B(t)\,\Phi(\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}x(t)|)+C(t)]\biggr)\\
&\le \left[\frac{2}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)}+\frac{c}{d}\right]\\
&\quad\times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}[A(t)\,\Phi(\|x\|)+B(t)\,\Phi(\|x\|)+C(t)]\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}(Tx)(t)|\\
&\quad=\left|\Phi^{-1}\left(-t^{1-\alpha}I_{0+}^{\alpha}f(t,x(t),D_{0+}^{\beta}x(t))+A_x\right)\right|\\
&\quad\le \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}[A(t)\,\Phi(\|x\|)+B(t)\,\Phi(\|x\|)+C(t)]\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|Tx\| &= \max\left\{\sup_{t\in(0,1)}|t^{1-\beta}|(Tx)(t)|,\ \sup_{t\in(0,1)}\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}(Tx)(t)|\right\}\\
&\le \left[\frac{2}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)}+\frac{c}{d}+1\right]\\
&\quad\times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}[A(t)\,\Phi(\|x\|)+B(t)\,\Phi(\|x\|)+C(t)]\right).
\end{aligned}
$$

One sees that

$$
\Phi^{-1}(a+b+c)\le C_q[\Phi(a)+\Phi(b)+\Phi(c)],\quad a,b,c\ge 0,\ C_q=\begin{cases}1, & q\in(1,2],\\ 2^{q-2}, & q>2.\end{cases}
$$

Then

$$
\begin{aligned}
\|x\| &= \lambda\|Tx\|\le\|Tx\|\\
&\le C_q\mu_0\,\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha}I_{0+}^{\alpha}A(t)\right)\|x\|\\
&\quad + C_q\mu_0\,\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha}I_{0+}^{\alpha}B(t)\right)\|x\|\\
&\quad + C_q\mu_0\,\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha}I_{0+}^{\alpha}C(t)\right).
\end{aligned}
$$

From (19), we get

$$\|x\| \leq \frac{C_q\mu_0 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} C(t)\right)}{1 - C_q\mu_0\delta}.$$

It follows that $U(T)$ is bounded. Hence Lemma 2.4 implies that T has a fixed point in X. Then BVP(3) has at least one solution.

Case 2. $a = 0$. Indeed, by the definition of T for $x \in X$, by using (4) and (9), we have the estimates

$$\begin{aligned}
& t^{1-\beta}|(Tx)(t)| \\
&= \Bigg| B_x + t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(u, x(\nu), D_{0+}^{\beta} x(\nu))d\nu + \Gamma(\alpha) A_x s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \Bigg| \\
&\leq |B_x| + \Bigg| t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times \frac{\Phi^{-1}\left(-\int_0^s (s-\nu)^{\alpha-1} f(\nu, x(\nu), D_{0+}^{\beta} x(\nu))d\nu + \Gamma(\alpha) A_x s^{\alpha-1}\right)}{\Phi^{-1}(\Gamma(\alpha))\rho(s)} ds \Bigg| \\
&\leq \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta} \frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{b}{a}\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} |f(t, x(t), D_{0+}^{\beta} x(t))|\right) \\
&\leq \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta} \frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{b}{a}\right] \\
&\quad \times \Phi^{-1}\Bigg(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} [A(t)\Phi(t^{1-\beta}|x(t)|) \\
&\quad + B(t)\Phi(\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}x(t)|) + C(t)]\Bigg) \\
&\leq \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta} \frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{b}{a}\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} [A(t)\Phi(\|x\|) + B(t)\Phi(\|x\|) + C(t)]\right)
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}(Tx)(t)| &= \left|\Phi^{-1}\left(-t^{1-\alpha} I_{0+}^{\alpha} f(t, x(t), D_{0+}^{\beta} x(t)) + A_x\right)\right| \\
&\leq \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha} [A(t)\Phi(\|x\|) + B(t)\Phi(\|x\|) + C(t)]\right).
\end{aligned}$$

It follows that

$$\begin{aligned}\|Tx\| &= \max\left\{\sup_{t\in(0,1)}|t^{1-\beta}|(Tx)(t)|, \sup_{t\in(0,1)}\Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}(Tx)(t)|\right\}\\ &\le \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]}t^{1-\beta}I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)}+\frac{b}{a}+1\right]\\ &\quad\times\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}[A(t)\Phi(\|x\|)+B(t)\Phi(\|x\|)+C(t)]\right).\end{aligned}$$

Then

$$\begin{aligned}\|x\| &= \lambda\|Tx\|\le\|Tx\|\le C_q\mu_1\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}A(t)\right)\|x\|\\ &\quad+C_q\mu_1\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}B(t)\right)\|x\|\\ &\quad+C_q\mu_1\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}C(t)\right).\end{aligned}$$

From (19), we get

$$\|x\|\le\frac{C_q\mu_1\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}C(t)\right)}{1-C_q\mu_1\delta}.$$

It follows that $U(T)$ is bounded. Hence Lemma 2.4 implies that T has a fixed point in X. Then BVP(3) has at least one solution. From Cases 1 and 2, the proof is completed.

Theorem 2.2. *Suppose that (H) holds and that there exists L_α^1 function ϕ such that*

$$\left|f\left(t,t^{\beta-1}u,\frac{v}{\Phi^{-1}(t^{1-\alpha})\rho(t)}\right)\right|\le\phi(t)w(|u|+|v|),\quad t\in(0,1],u,v\in R \tag{20}$$

with $w\in C(R,[0,\infty))$ nondecreasing. If

$$\sup_{\mu>0}\frac{\mu}{\mu_0\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}(w(2\mu))}>1\quad\textit{for } a\ne0, \tag{21}$$

$$\sup_{\mu>0}\frac{\mu}{\mu_1\Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]}t^{1-\alpha}I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}(w(2\mu))}>1\quad\textit{for } a=0, \tag{22}$$

then, BVP(3) has at least one solution, where

$$\begin{aligned}\mu_0 &= \frac{2}{\Phi^{-1}(\Gamma(\alpha))}\sup_{t\in[0,1]}t^{1-\beta}I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)}+\frac{c}{d}+1,\\ \mu_1 &= \frac{1}{\Phi^{-1}(\Gamma(\alpha))}\sup_{t\in[0,1]}t^{1-\beta}I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)}+\frac{b}{a}+1.\end{aligned}$$

Proof. It follows from (21) and (22) that there exist constants $\overline{\mu}_0, \overline{\mu}_1 > 0$ such that

$$\frac{\overline{\mu}_0}{\mu_0 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\overline{\mu}_0\right)\right)} > 1 \quad \text{if } a \neq 0, \tag{23}$$

$$\frac{\overline{\mu}_1}{\mu_1 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\overline{\mu}_1\right)\right)} > 1 \quad \text{if } a = 0. \tag{24}$$

Case 1. $a \neq 0$. Let

$$U = \{x \in X : \|x\| \leq \overline{\mu}_0\}.$$

We claim that $x \neq \lambda T x$ for all ∂U and $\lambda \in (0,1)$. In fact, if $x = \lambda T x$ for some $x \in \partial U$ and $\lambda \in (0,1)$. We have

$$\begin{aligned}
\|x\| &= \max\left\{\sup_{t\in(0,1)} \lambda t^{1-\beta}|(Tx)(t)|, \sup_{t\in(0,1)} \Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}\lambda|(Tx)(t)|\right\} \\
&\leq \left[\frac{2}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d} + 1\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}|f(t,x(t),D_{0+}^{\beta}x(t))|\right) \\
&\leq \left[2\max_{t\in[0,1]} t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Phi^{-1}(\Gamma(\alpha))}\frac{\Phi^{-1}(s^{\alpha-1})}{\rho(s)}ds + \frac{c}{d} + 1\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t) w\left(t^{1-\beta}|x(t)| + \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t)|\right)\right) \\
&\leq \left[\frac{2}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d} + 1\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t) w\left(2\|x\|\right)\right) \\
&= \mu_0 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\|x\|\right)\right).
\end{aligned}$$

So

$$\overline{\mu}_0 \leq \mu_0 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\overline{\mu}_0\right)\right).$$

It follows that

$$\frac{\overline{\mu}_0}{\mu_0 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\overline{\mu}_0\right)\right)} \leq 1,$$

which contradicts with (23). From the above discussion, we have $x \neq \lambda T x$ for all ∂U and $\lambda \in (0,1)$. Since T is completely continuous, by Lemma 2.4, we see that BVP(3) has at least one solution x.

Case 2. $a = 0$. Let

$$U = \{x \in X : \|x\| \le \overline{\mu}_1\}.$$

We claim that $x \neq \lambda Tx$ for all ∂U and $\lambda \in (0,1)$. In fact, let $x = \lambda Tx$ for some $x \in \partial U$ and $\lambda \in (0,1)$. We have

$$\begin{aligned}
\|x\| &= \max\left\{\sup_{t\in(0,1)} \lambda t^{1-\beta}|(Tx)(t)|, \sup_{t\in(0,1)} \Phi^{-1}(t^{1-\alpha})\rho(t)|D_{0+}^{\beta}\lambda|(Tx)(t)|\right\} \\
&\le \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d} + 1\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}|f(t,x(t),D_{0+}^{\beta}x(t))|\right) \\
&\le \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d} + 1\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)w\left(t^{1-\beta}|x(t)| + \Phi^{-1}(t^{1-\alpha})\rho(t)D_{0+}^{\beta}x(t)|\right)\right) \\
&\le \left[\frac{1}{\Phi^{-1}(\Gamma(\alpha))}\max_{t\in[0,1]} t^{1-\beta} I_{0+}^{\beta}\frac{\Phi^{-1}(t^{\alpha-1})}{\rho(t)} + \frac{c}{d} + 1\right] \\
&\quad \times \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)w\left(2\|x\|\right)\right) \\
&= \mu_1 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\|x\|\right)\right).
\end{aligned}$$

So

$$\overline{\mu}_1 \le \mu_1 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\overline{\mu}_1\right)\right).$$

It follows that

$$\frac{\overline{\mu}_1}{\mu_1 \Phi^{-1}\left(2\Gamma(\alpha)\max_{t\in[0,1]} t^{1-\alpha} I_{0+}^{\alpha}\phi(t)\right)\Phi^{-1}\left(w\left(2\overline{\mu}_1\right)\right)} \le 1,$$

which contradicts with (24).

From the above discussion, we have $x \neq \lambda Tx$ for all ∂U and $\lambda \in (0,1)$. Since T is completely continuous, by Lemma 2.4, we see that BVP(3) has at least one solution x. The proof is complete.

3. Two examples

In this section, we give two examples to illustrate the main theorems.

Example 3.1. Consider the following BVP

$$\begin{cases} D_{0+}^{\frac{1}{2}}[t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}D_{0+}^{\frac{1}{2}}x(t)] + f(t,x(t),D_{0+}^{\frac{1}{2}}x(t)) = 0, \quad t \in (0,1), \\ \lim\limits_{t\to 0} t^{\frac{1}{2}}x(t) - \lim\limits_{t\to 0} t^{\frac{1}{2}}D_{0+}^{\frac{1}{2}}x(t) = 0, \\ \lim\limits_{t\to 1} t^{\frac{1}{2}}D_{0+}^{\frac{1}{2}}x(t) + \lim\limits_{t\to 1} t^{\frac{1}{2}}x(t) = 0. \end{cases} \tag{25}$$

We consider two cases:

Case 1. $f(t,u(t),D_{0+}^{\frac{1}{2}}x(t)) = 2t^{-\frac{1}{2}} + \lambda\left(t-\frac{1}{2}\right)^4 t^{\frac{1}{2}}x(t) + \mu t(1-t)^{\frac{1}{2}}D_{0+}^{\frac{1}{2}}x(t)$ with $\lambda,\mu > 0$.

Corresponding to BVP(3), we find that $\alpha = \beta = \frac{1}{2}$, $a=b=c=d=1$, $\Phi(x) = x$ with $\Phi^{-1}(x) = x$, $\rho(t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}$ and

$$f(t,u,v) = 2t^{-\frac{1}{2}} + \lambda\left(t-\frac{1}{2}\right)^4 t^{\frac{1}{2}}u + \mu t(1-t)^{\frac{1}{2}}v.$$

Choose $A(t) = \lambda\left(t-\frac{1}{2}\right)^4$, $B(t) = \mu$ and $C(t) = 2t^{-\frac{1}{2}}$. One sees that

$$\left|f(t,t^{-\frac{1}{2}}u,t^{-1}(1-t)^{-\frac{1}{2}}v)\right| \le A(t)|u| + B(t)|v| + C(t), \quad t\in(0,1), u,v\in R.$$

It is easy to see that

$$\begin{aligned}\mu_0 &\le 2\sup_{t\in[0,1]} t^{1/2}\int_0^t \frac{(t-s)^{-1/2}}{[\Gamma(1/2)]^2}\frac{1}{s^{3/4}(t-s)^{1/4}}ds + 2 \\ &= 2 + \frac{2\mathbf{B}(1/4,1/4)}{[\Gamma(1/2)]^2}.\end{aligned}$$

Hence Theorem 2.1 implies that BVP(25) has a solution if

$$\begin{aligned}&2\lambda\max_{t\in[0,1]} t^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}(s-1/2)^4ds + 2\mu\max_{t\in[0,1]} t^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}ds \\ &< \frac{1}{2+\frac{2\mathbf{B}(1/4,1/4)}{[\Gamma(1/2)]^2}}.\end{aligned}$$

So BVP(25) has a solution for sufficiently small λ and μ.

Case 2. $f(t,x(t),D_{0+}^{\frac{1}{2}}x(t)) = t^{-\frac{1}{2}}\left(t^{\frac{1}{2}}x(t) + t(1-t)^{\frac{1}{2}}D_{0+}^{\frac{1}{2}}x(t)\right)^{1/2}$.

Corresponding to BVP(3), we find that $\alpha = \beta = \frac{1}{2}$, $a=b=c=d=1$, $\Phi(x) = x$ with $\Phi^{-1}(x) = x$, $\rho(t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}$ and

$$f(t,t^{-\frac{1}{2}}u,t^{-1}(1-t)^{-\frac{1}{2}}v) = \phi(t)\,(u+v)^{1/2}.$$

Choose $w(x) = x^{1/2}$ and $\phi(t) = t^{-\frac{1}{2}}$. We find that

$$\left|f\left(t,t^{-\frac{1}{2}}u,t^{-1}(1-t)^{-\frac{1}{2}}v\right)\right| \le \phi(t)w(|u|+|v|), \quad t\in(0,1), u,v\in R.$$

One sees that

$$\mu_0 \le 2 \sup_{t\in[0,1]} t^{1/2} \int_0^t \frac{(t-s)^{-1/2}}{[\Gamma(1/2)]^2} \frac{1}{s^{3/4}(t-s)^{1/4}} ds + 2 = 2 + \frac{2\mathbf{B}(1/4,1/4)}{[\Gamma(1/2)]^2}.$$

It is easy to see from Theorem 2.2 that

$$\sup_{\mu>0} \frac{\mu}{\left(2 \max_{t\in[0,1]} t^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds\right) w(2\mu)} > 2 + \frac{2\mathbf{B}(1/4,1/4)}{[\Gamma(1/2)]^2} \tag{26}$$

implies that BVP(25) has at least one solution. It is easy to see that (26) is always correct since $w(x) = x^{1/2}$. Hence BVP(25) has at least one solution.

Example 3.2. Consider the following BVP

$$\begin{cases} D_{0+}^{\frac{1}{2}}[(D_{0+}^{\frac{1}{2}}x(t))^3] + f(t, x(t), D_{0+}^{\frac{1}{2}}x(t)) = 0, \quad t\in(0,1), \\ \lim_{t\to 0} t^{\frac{1}{2}}x(t) - \lim_{t\to 0} t^{\frac{1}{6}} D_{0+}^{\frac{1}{2}}x(t) = 0, \\ \lim_{t\to 1} t^{\frac{1}{6}} D_{0+}^{\beta}x(t) + \lim_{t\to 1} t^{\frac{1}{2}}x(t) = 0. \end{cases} \tag{27}$$

We consider two cases:

Case 1. $f(t, u(t), D_{0+}^{\frac{1}{2}}x(t)) = 2t^{-\frac{1}{2}} + \lambda t^{\frac{1}{2}}x(t) + \mu t^{\frac{1}{6}} D_{0+}^{\frac{1}{2}}x(t)$ with $\lambda, \mu > 0$.

Corresponding to BVP(3), we find that $\alpha = \beta = \frac{1}{2}$, $a = b = c = d = 1$, $\Phi(x) = x^3$ with $\Phi^{-1}(x) = x^{\frac{1}{3}}$, $\rho(t) = 1$ and

$$f(t,u,v) = 2t^{-\frac{1}{2}} + \lambda t^{\frac{1}{2}}u + \mu t^{\frac{1}{6}}v.$$

Choose $A(t) = \lambda$, $B(t) = \mu$ and $C(t) = 2t^{-\frac{1}{2}}$. One sees that

$$\left| f(t, t^{-\frac{1}{2}}u, t^{-\frac{1}{6}}v)\right| \le A(t)|u| + B(t)|v| + C(t), \quad t\in(0,1), u, v\in R.$$

It is easy to see that

$$\mu_0 = 2 + 2 \sup_{t\in[0,1]} t^{\frac{5}{6}} \int_0^1 \frac{(1-u)^{-1/2}}{[\Gamma(1/2)]^{5/3}} u^{-\frac{1}{6}} du \le 2 + \frac{2\mathbf{B}(1/2,5/6)}{[\Gamma(1/2)]^{5/3}}.$$

Hence Theorem 2.1 implies that BVP(27) has a solution if

$$\sqrt[3]{4\lambda} + \sqrt[3]{4\mu} < \frac{[\Gamma(1/2)]^{5/3}}{2[\Gamma(1/2)]^{5/3} + 2\mathbf{B}(1/2,5/6)}.$$

So BVP(27) has a solution for sufficiently small λ and μ.

Case 2. $f(t, x(t), D_{0+}^{\frac{1}{2}}x(t)) = t^{-\frac{1}{2}} w\left(t^{\frac{1}{2}}x(t) + t^{\frac{1}{6}} D_{0+}^{\frac{1}{2}}x(t)\right)$, $w: R\to R$ is a continuous function.

Corresponding to BVP(3), we find that $\alpha = \beta = \frac{1}{2}$, $a = b = c = d = 1$, $\Phi(x) = x^3$ with $\Phi^{-1}(x) = x^{\frac{1}{3}}$, $\rho(t) = 1$ and

$$f(t, t^{-\frac{1}{2}}u, t^{-\frac{1}{6}}v) = \phi(t) w(u+v).$$

Choose $w(x) = x^{1/2}$ and $\phi(t) = t^{-\frac{1}{2}}$. We find that

$$\left| f\left(t, t^{-\frac{1}{2}}u, t^{-\frac{1}{6}}v\right)\right| \le \phi(t)w(|u| + |v|), \quad t \in (0,1), u, v \in R.$$

One sees that

$$\mu_0 \le 2 \sup_{t\in[0,1]} t^{1/2} \int_0^t \frac{(t-s)^{-1/2}}{[\Gamma(1/2)]^{5/3}} s^{-\frac{1}{6}} ds + 2 = 2 + \frac{2\mathbf{B}(1/2, 5/6)}{[\Gamma(1/2)]^{5/3}}.$$

It is easy to see from Theorem 2.2 that

$$\sup_{u>0} \frac{u}{\sqrt[3]{2\mathbf{B}(1/2,1/2)}\sqrt[3]{w(2u)}} > 2 + \frac{2\mathbf{B}(1/2, 5/6)}{[\Gamma(1/2)]^{5/3}}. \tag{28}$$

implies that BVP(27) has at least one solution.

4. Conclusions

The existence of solutions of a class of boundary value problems for nonlinear fractional differential equations involving Riemann–Liouville fractional derivatives is studied. The fractional differential equation concerned in (3) is a composition of two left-sided Riemann–Liouville fractional derivatives. The investigation shows that these results and methods are helpful for study in the nonlinear area.

As is well known for $\alpha \in [n-1, n)$ with $n \in \{1, 2, 3, \ldots)$ that

$$D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial^n t} \int_a^t (t-s)^{n-\alpha-1} g(s) ds, \quad \text{and}$$
$$D_{b-}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial^n t} \int_t^b (s-t)^{n-\alpha-1} g(s) ds$$

are called left-sided Riemann–Liouville fractional derivative and right-sided Riemann–Liouville fractional derivative respectively and

$${}^cD_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \frac{\partial^n}{\partial^n s} g(s) ds, \quad \text{and}$$
$${}^cD_{b-}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \frac{\partial^n}{\partial^n s} g(s) ds$$

are called left-sided Caputo fractional derivative and right-sided Caputo fractional derivative respectively.

Further studies are also located on seeking solutions of such kind of BVPs in which the fractional differential equations are concerned with the composition of left- and right-sided Riemann–Liouville fractional derivatives or the composition of left- and right-sided Caputo fractional derivatives, or the composition of right- and left-sided Riemann–Liouville fractional derivatives, etc. are involved.

Another important part is to demonstrate the application of powerful mathematical tools (fixed point theorems in Banach spaces) for solving nonlinear fractional differential equations.

Acknowledgments

The authors thank anonymous referees and editors for valuable remarks and suggestions and some useful comments on improving the presentation of this paper.

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