Arab Journal of Mathematical Sciences

Arab J Math Sci 22 (2016) 138–149

Solutions to Kirchhoff equations with critical exponent

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Received 2 February 2015; received in revised form 30 March 2015; accepted 14 April 2015
Available online 28 April 2015

Abstract. In this paper, we study the following problems

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2^* = \frac{2N}{N-4}$ is the critical exponent. Under some conditions on M and f, we prove the existence of nontrivial solutions by using variational methods.

Keywords: Fourth-order elliptic equations; Nonlocal problem; Critical exponent; Lions principle

2010 Mathematics Subject Classification: 34B10; 35B33

1. Introduction and main results

In this paper we are concerned with the existence of nontrivial solutions for the following nonlocal elliptic problems

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.04.001

1319-5166

where Ω is a bounded smooth domain of $\mathbb{R}^N, N \geq 5,\ 2^* = \frac{2N}{N-4}$ is the critical Sobolev exponent, and $M : \mathbb{R}^+ \to \mathbb{R}^+,\ f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy some conditions which will be stated later on.

Problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff [22]. More precisely, the authors in [8,22] introduced a model given by the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Later (1.2) was developed to form

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x,u) \quad \text{in } \Omega. \tag{1.3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-M\left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Problems like (1.4) can be used for modeling several physical and biological systems where u describes a process which depends on the average of itself, such as the population density, see [4]. Many interesting results for problems of Kirchhoff type were obtained see, for example, [1,13,20,21].

The investigation of fourth order boundary value problems has drawn the attention of many authors, because the static form change of beam or the sport of rigid body can be described by a fourth order equation, and specially a model to study traveling waves in suspension bridges can be furnished by the fourth order equation of nonlinearity. Several results are known concerning the existence and multiplicity of solutions for fourth order boundary value problems, see [10,11,18] and the references therein. In [26], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = \lambda f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Also, in [24] employing a smooth version of Ricceri's variational principle [25], the authors ensured the existence of infinitely many solutions for the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - M\left[\int_\Omega |\nabla u|^p dx \right]^{p-1} \Delta_p u + \rho |u|^{p-2} u = \lambda f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Much interest has grown on problems involving critical exponents, starting from the celebrated paper by Brezis and Nirenberg [12]. This pioneering work has stimulated a vast amount of research on this class of problems. We refer the reader to [2,7,3,5,9,14–17,19,28] and reference therein for the study of problems with critical exponent.

Before stating our main result, we need the following hypotheses on the function $M : \mathbb{R}^+ \to \mathbb{R}^+$:

(m_1) There is a positive constant m_0 such that

$$M(t) \geq m_0 \quad \text{for all } t \geq 0.$$

(m_2) There exists $\sigma > 2/2^*$ such that

$$\widehat{M}(t) \geq \sigma M(t)t \quad \text{for all } t \geq 0,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$.

A typical example of a function satisfying the conditions (m_1)–(m_2) is given by

$$M(t) = m_0 + bt$$

with $b \geq 0$ and for all $t \geq 0$.

The hypotheses on function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are the following:

(f_1)

$$f(x,t) = o(|t|) \quad \text{as } t \to 0 \text{ uniformly in } x \in \Omega.$$

(f_2) There exists $q \in (2, 2^*)$ such that

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{q-2}t} = 0 \quad \text{uniformly in } x \in \Omega.$$

(f_3) There exists $\theta \in (\max(2, 2/\sigma), 2^*)$ such that

$$0 < \theta F(x,t) = \theta \int_0^t f(x,s)ds \leq tf(x,t) \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R} \setminus \{0\},$$

where σ is given in (m_2).

A typical example of a function satisfying the conditions (f_1)–(f_3) is given by

$$f(x,t) = \sum_{i=1}^{k} a_i(x)|t|^{q_i - 2}t,$$

where $k \geq 1,\ 2 < q_i < 2^*$, $a_i \in \mathcal{C}(\overline{\Omega})$ with $a_i \geq 0$ for all $x \in \Omega$.

Now, we formulate our main result as follows.

Theorem 1.1. *Suppose that* (m_1), (m_2) *and* (f_1)–(f_3) *hold. Then, there exists* $\lambda_* > 0$, *such that problem* (1.1) *has a nontrivial solution for all* $\lambda \geq \lambda_*$.

2. Preliminary results

We denote by $H = H^2(\Omega) \cap H_0^1(\Omega)$ the Hilbert space equipped with the inner product

$$(u,v) = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v)dx,$$

and the deduced norm

$$\|u\|^2 = \int_\Omega (|\Delta u|^2 + |\nabla u|^2)dx.$$

We consider the following energy functional $I_\lambda : H \to \mathbb{R}$, defined by

$$\begin{aligned} I_\lambda(u) = {} & \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2 dx\right) \\ & - \lambda \int_\Omega F(x,u)dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx. \end{aligned} \tag{2.1}$$

It is well known that a critical point of I_λ is a weak solution of problem (1.1).

In the sequel, we show that the functional I_λ has the mountain pass geometry.

Lemma 2.1. *Suppose that* (m_1), (m_2) *and* (f_1)–(f_3) *hold, then we have*

(i) *There exist* $r, \rho > 0$ *such that* $\inf_{\|u\|=r} I_\lambda(u) \geq \rho > 0$.
(ii) *There exists a nonnegative function* $e \in H$ *such that* $\|e\| > r$ *and* $I_\lambda(e) < 0$.

Proof. (i) It follows from (f_1) and (f_2) that for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$F(x,t) \leq \frac{1}{2}\varepsilon t^2 + C(\varepsilon)|t|^q. \tag{2.2}$$

Together with (m_1) and Sobolev's inequalities, we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{m_0}{2}\int_\Omega |\nabla u|^2 dx - \lambda C_1 \varepsilon \|u\|^2 - \lambda C_2(\varepsilon)\|u\|^q - C_3\|u\|^{2^*} \\ &\geq \left(\frac{\min(1,m_0)}{2} - \lambda C_1 \varepsilon\right)\|u\|^2 - \lambda C_2(\varepsilon)\|u\|^q - C_3\|u\|^{2^*}. \end{aligned} \tag{2.3}$$

We take $\varepsilon < \frac{\min(1,m_0)}{2\lambda C_1}$, since $2 < q < 2^*$, choosing $\|u\| = r$ small enough, we can obtain a positive constant ρ such that $I_\lambda(u) \geq \rho$ as $\|u\| = r$.

(ii) Choose a nonnegative function $\phi_1 \in C_0^\infty(\Omega)$ with $\|\phi_1\| = 1$.

By integrating (m_2), we get

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{\sigma}}} t^{\frac{1}{\sigma}} = C_0 t^{\frac{1}{\sigma}} \quad \text{for all } t \geq t_0 > 0. \tag{2.4}$$

Moreover, from (f_3), one has $\int_\Omega F(x, t\phi_1) > 0$. Hence for $t \geq t_0$, we obtain

$$\begin{aligned} I_\lambda(t\phi_1) &\leq \frac{t^2}{2}\int_\Omega |\Delta\phi_1|^2 dx + \frac{1}{2}\widehat{M}\left(t^2\int_\Omega |\nabla\phi_1|^2 dx\right) - \frac{t^{2^*}}{2^*}\int_\Omega \phi_1^{2^*} dx \\ &\leq \frac{t^2}{2}\int_\Omega |\Delta\phi_1|^2 dx + \frac{C_0 t^{\frac{2}{\sigma}}}{2}\left(\int_\Omega |\nabla\phi_1|^2 dx\right)^{\frac{1}{\sigma}} - \frac{t^{2^*}}{2^*}\int_\Omega \phi_1^{2^*} dx \\ &\leq \frac{t^2}{2} + \frac{C_0 t^{\frac{2}{\sigma}}}{2} - \frac{t^{2^*}}{2^*}\int_\Omega \phi_1^{2^*} dx. \end{aligned} \tag{2.5}$$

The fact that $\max(2, 2/\sigma) < 2^*$, the assertion (ii) is proved by choosing $e = t_*\phi_1$ with $t_* > 0$ large enough. □

From Lemma 2.1, using a version of the Mountain Pass theorem due to Ambrosetti and Rabinowitz [6], without (PS) condition (see [27]), there exists a sequence $(u_n) \subset H$ such that

$$I_\lambda(u_n) \to c_* \quad \text{and} \quad I'_\lambda(u_n) \to 0,$$

where

$$c_* = \inf_{\gamma \in \Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0, \tag{2.6}$$

with

$$\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\}.$$

Let S_* be the best positive constant of the Sobolev embedding $H \hookrightarrow L^{2^*}(\Omega)$ given by

$$S_* = \inf\left\{\|u\|^2 : u \in H,\ \int_\Omega |u|^{2^*} dx = 1\right\}. \tag{2.7}$$

Lemma 2.2. *Suppose that* (m_1), (m_2) *and* (f_1)–(f_3) *hold. Then there exists* $\lambda_* > 0$ *such that* $c_* \in \left(0, \left(\frac{1}{\theta} - \frac{1}{2^*}\right) S_*^{\frac{N}{4}}\right)$ *for all* $\lambda \geq \lambda_*$.

Proof. For the nonnegative function e given in (ii) of Lemma 2.1, we have $\lim_{t\to+\infty} I_\lambda(te) = -\infty$, then there exists $t_\lambda > 0$ such that

$$I_\lambda(t_\lambda e) = \max_{t\geq 0} I_\lambda(te).$$

Therefore

$$\begin{aligned} &t_\lambda \int_\Omega |\Delta e|^2 dx + M\left(t_\lambda^2 \int_\Omega |\nabla e|^2 dx\right) \int_\Omega t_\lambda |\nabla e|^2 dx \\ &\quad = \lambda \int_\Omega f(x, t_\lambda e) e dx + t_\lambda^{2^*-1} \int_\Omega |e|^{2^*} dx. \end{aligned}$$

By (m_2) and (f_3) it follows that

$$\begin{aligned} t_\lambda^{2^*} \int_\Omega e^{2^*} dx &\leq \lambda t_\lambda \int_\Omega f(x, t_\lambda e) e dx + t_\lambda^{2^*} \int_\Omega e^{2^*} dx \\ &= t_\lambda^2 \int_\Omega |\Delta e|^2 dx + M\left(t_\lambda^2 \int_\Omega |\nabla e|^2 dx\right) \int_\Omega t_\lambda^2 |\nabla e|^2 dx \\ &\leq t_\lambda^2 \int_\Omega |\Delta e|^2 + \frac{1}{\sigma} \widehat{M}\left(\int_\Omega t_\lambda^2 |\nabla e|^2 dx\right). \end{aligned} \tag{2.8}$$

Hence, from (2.4), we obtain

$$t_\lambda^{2^*} \int_\Omega e^{2^*} dx \leq t_\lambda^2 \int_\Omega |\Delta e|^2 + \frac{C_0}{\sigma} t_\lambda^{2/\sigma} \left(\int_\Omega |\nabla e|^2 dx\right)^{1/\sigma}, \quad \text{with } t_\lambda > t_0.$$

Since $2^* > \max(2, 2/\sigma)$, (t_λ) is bounded. So, there exists a sequence $\lambda_n \to +\infty$ and $s_0 \geq 0$ such that $t_{\lambda_n} \to s_0$ as $n \to \infty$. Hence, there exists $C > 0$ such that

$$t_{\lambda_n}^2 \int_\Omega |\Delta e|^2 + \frac{C_0}{\sigma} t_{\lambda_n}^{2/\sigma} \left(\int_\Omega |\nabla e|^2 dx \right)^{1/\sigma} \leq C \quad \text{for all } n,$$

that is,

$$\lambda_n t_{\lambda_n} \int_\Omega f(x, t_{\lambda_n} e) e dx + t_{\lambda_n}^{2^*} \int_\Omega e^{2^*} dx \leq C \quad \text{for all } n.$$

If $s_0 > 0$, the last inequality implies that

$$\lambda_n t_{\lambda_n} \int_\Omega f(x, t_{\lambda_n} e) e \, dx + t_{\lambda_n}^{2^*} \int_\Omega e^{2^*} dx \to +\infty \leq C, \quad \text{as } n \to \infty,$$

which is impossible, and consequently, $s_0 = 0$. Let $\gamma_*(t) = te$ for $t \in [0, 1]$. Clearly $\gamma_* \in \Gamma$, thus

$$0 < c_* \leq \max_{t \geq 0} I_\lambda(\gamma_*(t)) = I_\lambda(t_\lambda e) \leq \frac{t_\lambda^2}{2} \int_\Omega |\Delta e|^2 + \frac{1}{2} \widehat{M} \left(t_\lambda^2 \int_\Omega |\nabla e|^2 dx \right).$$

Since $t_{\lambda_n} \to 0$ and $\left(\frac{1}{\theta} - \frac{1}{2^*}\right) (m_0 S_*)^{\frac{N}{4}} > 0$, for $\lambda > 0$ sufficiently large, we have

$$\frac{t_\lambda^2}{2} \int_\Omega |\Delta e|^2 + \frac{1}{2} \widehat{M} \left(t_\lambda^2 \int_\Omega |\nabla e|^2 dx \right) < \left(\frac{1}{\theta} - \frac{1}{2^*} \right) S_*^{\frac{N}{4}},$$

and hence

$$0 < c_* < \left(\frac{1}{\theta} - \frac{1}{2^*} \right) S_*^{\frac{N}{4}}. \quad \square$$

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, there exists a sequence $(u_n) \subset H$ such that

$$I_\lambda(u_n) \to c_* \quad \text{and} \quad I'_\lambda(u_n) \to 0, \tag{2.9}$$

with $c_* \in \left(0, \left(\frac{1}{\theta} - \frac{1}{2^*}\right) S_*^{\frac{N}{4}}\right)$ for $\lambda \geq \lambda_*$. Then, there exists $C > 0$ such that $|I_\lambda(u_n)| \leq C$ and by (f_3) for n large enough, it follows from (m_1) and (m_2) that

$$\begin{aligned} C + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \int_\Omega |\Delta u_n|^2 dx + \left(\frac{\sigma}{2} - \frac{1}{\theta} \right) m_0 \int_\Omega |\nabla u_n|^2 dx \\ &\geq \min \left[\left(\frac{1}{2} - \frac{1}{\theta} \right), \left(\frac{\sigma}{2} - \frac{1}{\theta} \right) m_0 \right] \|u_n\|^2. \end{aligned} \tag{2.10}$$

Since $\theta > \max(2, 2/\sigma)$, (u_n) is bounded. Hence, up to a subsequence, we may assume that

$$u_n \rightharpoonup u \quad \text{weakly in } H^2(\Omega) \cap H_0^1(\Omega),$$

$$\begin{aligned}
&u_n \to u \quad \text{a.e. in } \Omega,\\
&u_n \to u \quad \text{in } L^s(\Omega),\ 1 \le s < 2^*,\\
&|\Delta u_n|^2 \rightharpoonup \mu \text{ (weak}^*\text{—sense of measures)}\\
&|u_n|^{2^*} \rightharpoonup \nu \text{ (weak}^*\text{—sense of measures)},
\end{aligned} \tag{2.11}$$

where μ and ν are nonnegative bounded measures on $\overline{\Omega}$. Then, by concentration-compactness principle due to Lions [23], there exists some at most countable index set J such that

$$\begin{cases} \nu = |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}\,, & \nu_j > 0,\\ \mu \ge |\Delta u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, & \mu_j > 0,\\ S_* \nu_j^{2/2^*} \le \mu_j, \end{cases} \tag{2.12}$$

where δ_{x_j} is the Dirac measure mass at $x_j \in \overline{\Omega}$.

For $\varepsilon > 0$ and $j \in J$, define a function $\psi_\varepsilon^j(x) \in C_0^\infty$ such that $0 \le \psi_\varepsilon^j \le 1$,

$$\psi_\varepsilon^j(x) = \begin{cases} 1 & \text{if} \quad |x - x_j| < \varepsilon\\ 0 & \text{if} \quad |x - x_j| \ge 2\varepsilon, \end{cases} \tag{2.13}$$

$|\nabla \psi_\varepsilon^j|_\infty \le 2/\varepsilon$ and $|\Delta \psi_\varepsilon^j|_\infty \le 2/\varepsilon^2$.

Since $I'_\lambda(u_n) \to 0$ and $(\psi_\varepsilon^j u_n)$ is bounded, $\langle I'_\lambda(u_n), \psi_\varepsilon^j u_n\rangle \to 0$ as $n \to \infty$, that is

$$\begin{aligned}
&\int_\Omega |\Delta u_n|^2 \psi_\varepsilon^j dx + \int_\Omega \Delta u_n \left(2\nabla u_n \nabla \psi_\varepsilon^j dx + u_n \Delta \psi_\varepsilon^j\right) dx\\
&\qquad + M\left(\int_\Omega |\nabla u_n|^2 dx\right)\left(\int_\Omega u_n \nabla u_n \nabla \psi_\varepsilon^j dx + \int_\Omega \psi_\varepsilon^j |\nabla u_n|^2 dx\right)\\
&\quad = \lambda \int_\Omega f(x, u_n) u_n \psi_\varepsilon^j dx + \int_\Omega |u_n|^{2^*} \psi_\varepsilon^j dx + o_n(1).
\end{aligned} \tag{2.14}$$

Note that

$$\|\nabla(u_n - u)\|_{L^2(\Omega)}^2 = -\int_\Omega (u_n - u)\Delta(u_n - u)dx \le \|u_n - u\|.\|u_n - u\|_{L^2(\Omega)},$$

then, (2.11) implies

$$\nabla u_n \to \nabla u \quad \text{in } L^2(\Omega). \tag{2.15}$$

Now, by Vitali's theorem we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega |u_n \nabla \psi_\varepsilon^j|^2 dx = \int_\Omega |u \nabla \psi_\varepsilon^j|^2 dx \quad \text{and}\\
&\lim_{n\to\infty} \int_\Omega |u_n \Delta \psi_\varepsilon^j|^2 dx = \int_\Omega |u \Delta \psi_\varepsilon^j|^2 dx.
\end{aligned}$$

In what follows, the letter C will be indiscriminately used to denote various constants. By Hölder's inequality, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_\Omega u_n\nabla u_n\nabla\psi_\varepsilon^j dx\right| &\le \limsup_{n\to\infty}\left(\int_\Omega|\nabla u_n|^2\right)^{\frac12}\left(\int_\Omega|u_n|^2|\nabla\psi_\varepsilon^j|^2dx\right)^{\frac12}\\ &\le C\left(\int_\Omega|u|^2|\nabla\psi_\varepsilon^j|^2dx\right)^{\frac12}\\ &\le C\left(\int_{B(x_j,\varepsilon)}|\nabla\psi_\varepsilon^j|^N dx\right)^{\frac1N}\left(\int_{B(x_j,\varepsilon)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\\ &\le C\left(\int_{B(x_j,\varepsilon)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\underset{\varepsilon\to0}{\longrightarrow}0,\end{aligned}$$

and similarly, we have

$$\begin{aligned}&\limsup_{n\to\infty}\left|\int_\Omega \Delta u_n\nabla u_n\nabla\psi_\varepsilon^j dx\right| \le \limsup_{n\to\infty}\left(\int_\Omega|\Delta u_n|^2\right)^{\frac12}\left(\int_\Omega|\nabla u_n|^2|\nabla\psi_\varepsilon^j|^2dx\right)^{\frac12}\\ &\le C\left(\int_\Omega|\nabla u|^2|\nabla\psi_\varepsilon^j|^2dx\right)^{\frac12}\\ &\le C\left(\int_{B(x_j,\varepsilon)}|\nabla\psi_\varepsilon^j|^N dx\right)^{\frac1N}\left(\int_{B(x_j,\varepsilon)}|\nabla u|^{2^*}dx\right)^{\frac{1}{2^*}}\\ &\le C\left(\int_{B(x_j,\varepsilon)}|\nabla u|^{2^*}dx\right)^{\frac{1}{2^*}}\underset{\varepsilon\to0}{\longrightarrow}0,\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_\Omega u_n\Delta u_n\Delta\psi_\varepsilon^j dx\right| &\le \limsup_{n\to\infty}\left(\int_\Omega|\Delta u_n|^2\right)^{\frac12}\left(\int_\Omega|u_n|^2|\Delta\psi_\varepsilon^j|^2dx\right)^{\frac12}\\ &\le C\left(\int_\Omega|u|^2|\Delta\psi_\varepsilon^j|^2dx\right)^{\frac12}\\ &\le C\left(\int_{B(x_j,\varepsilon)}|\Delta\psi_\varepsilon^j|^{\frac N2} dx\right)^{\frac2N}\left(\int_{B(x_j,\varepsilon)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\\ &\le C\left(\int_{B(x_j,\varepsilon)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\underset{\varepsilon\to0}{\longrightarrow}0.\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to0}\left[\lim_{n\to\infty}\int_\Omega\left(\Delta u_n(2\nabla u_n\nabla\psi_\varepsilon^j+u_n\Delta\psi_\varepsilon^j)+u_n\nabla u_n\nabla\psi_\varepsilon^j+\psi_\varepsilon^j|\nabla u_n|^2\right)dx\right] = 0. \tag{2.16}$$

On the other hand, from (2.11) we have

$$f(x,u_n)u_n \to f(x,u)u \quad \text{a.e. in } \Omega,$$

$u_n \to u$ strongly in $L^2(\Omega)$ and in $L^q(\Omega)$. By (m_1) and (m_2), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t| + C_\varepsilon |t|^{q-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \tag{2.17}$$

thus

$$|f(x,u_n)u_n| \leq \varepsilon|u_n|^2 + C_\varepsilon|u_n|^q.$$

This is what we need to apply Vitali's theorem, which yields

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)u_n dx = \int_\Omega f(x,u)u dx.$$

Since ψ_ε^j has compact support, from (2.11), (2.14) and (2.16) we deduce

$$\begin{aligned}\int_\Omega \psi_\varepsilon^j d\mu \leq C\left(\int_{B(x_j,\varepsilon)} |u|^{2^*} dx\right)^{\frac{1}{2^*}} + C\left(\int_{B(x_j,\varepsilon)} |\nabla u|^{2^*} dx\right)^{\frac{1}{2^*}} \\ + \int_{B(x_j,\varepsilon)} |\nabla u|^2 dx + \lambda \int_{B(x_j,\varepsilon)} f(x,u)u dx + \int_\Omega \psi_\varepsilon^j d\nu,\end{aligned}$$

letting $\varepsilon \to 0$, we get

$$\mu_j \leq \nu_j.$$

It follows from (2.12) that

$$S_*^{\frac{N}{4}} \leq \nu_j. \tag{2.18}$$

Now, we shall prove that the above expression cannot occur, and therefore the set J is empty. Indeed, arguing by contradiction, let us suppose that $S_*^{\frac{N}{4}} \leq \nu_{j_0}$ for some $j_0 \in J$. Then, from the fact that

$$c_* = I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle + o_n(1),$$

we obtain

$$\begin{aligned} c_* &\geq \left(\frac{1}{\theta} - \frac{1}{2^*}\right)\int_\Omega |u_n|^{2^*} dx + o_n(1) \\ &\geq \left(\frac{1}{\theta} - \frac{1}{2^*}\right)\int_\Omega \psi_\varepsilon^j |u_n|^{2^*} dx + o_n(1).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} c_* &\geq \left(\frac{1}{\theta} - \frac{1}{2^*}\right) \sum_{j \in J} \psi_\varepsilon^j(x_j)\nu_j \\ &= \left(\frac{1}{\theta} - \frac{1}{2^*}\right) \sum_{j \in J} \nu_j \geq \left(\frac{1}{\theta} - \frac{1}{2^*}\right) S_*^{\frac{N}{4}} \end{aligned}$$

which contradicts Lemma 2.2. This implies that $J = \emptyset$ and it follows that $u_n \to u$ in $L^{2^*}(\Omega)$. The relation (2.17) implies that

$$\begin{aligned} \int_\Omega |f(x,u_n)(u_n - u)|dx &\leq \int_\Omega \left(\varepsilon|u_n| + C_\varepsilon|u_n|^{q-1}\right)|u_n - u|dx \\ &\leq \varepsilon \left(\int_\Omega |u_n|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_n - u|^2 dx\right)^{\frac{1}{2}} \\ &\quad + C_\varepsilon \left(\int_\Omega |u_n|^q dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |u_n - u|^q dx\right)^{\frac{1}{q}}, \end{aligned}$$

and using again (2.11), we get

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)(u_n - u)dx = 0. \tag{2.19}$$

Since $u_n \to u$ in $L^{2^*}(\Omega)$, we see that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{2^*-2} u_n(u_n - u)dx = 0. \tag{2.20}$$

From $\langle I'_\lambda(u_n), u_n - u\rangle = o_n(1)$, we deduce that

$$\begin{aligned} \langle I'_\lambda(u_n), u_n - u\rangle &= \int_\Omega \Delta u_n \Delta(u_n - u)dx + M\left(\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega \nabla u_n \nabla(u_n - u) \\ &\quad - \lambda \int_\Omega f(x,u_n)(u_n - u)dx - \int_\Omega |u_n|^{2^*-2}u_n(u_n - u)dx \\ &= o_n(1). \end{aligned}$$

By continuity of M, (2.15), (2.19) and (2.20) we have

$$\lim_{n\to\infty} \int_\Omega \Delta u_n \Delta(u_n - u)dx = 0.$$

In the same way, we obtain

$$\lim_{n\to\infty} \int_\Omega \Delta u \Delta(u_n - u)dx = 0.$$

Taking into account (2.15), we conclude that $\|u_n\| \to \|u\|$. By the uniform convexity of H, it follows that $u_n \to u$ strongly in H, and hence

$$I'_\lambda(u) = 0 \quad \text{and} \quad I_\lambda(u) = c_* \neq 0.$$

The proof of Theorem 1.1 is complete. □

Acknowledgment

The authors would like to thank the anonymous referee for his/her valuable suggestions and comments, which greatly improve the manuscript.

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