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# Slant Riemannian submersions from Sasakian manifolds

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**Abstract.** We introduce and characterize slant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We survey main results of slant Riemannian submersions defined on Sasakian manifolds. We give a sufficient condition for a slant Riemannian submersion from Sasakian manifolds onto Riemannian manifolds to be harmonic. We also give an example of such slant submersions. Moreover, we find a sharp inequality between the scalar curvature and norm squared mean curvature of fibres.

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## 1. Introduction

Let $F$ be a $C^\infty$-submersion from a Riemannian manifold $(M, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then according to the conditions on the map $F : (M, g_M) \to (N, g_N)$, $F$ can be any one of the following types: semi-Riemannian submersion and Lorentzian submersion [11], Riemannian submersion [22,12], slant submersion [9,27], almost Hermitian submersion [29], contact-complex submersion [13], quaternionic submersion [14], almost $h$-slant submersion and $h$-slant submersion [24], semi-invariant submersion [28], $h$-semi-invariant submersion [25], etc.

As we know, Riemannian submersions are related to physics and have their applications in the Yang–Mills theory [6,30], Kaluza–Klein theory [7,15], supergravity and superstring theories [16,21]. In [26], Şahin introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. He gave a generalization of Hermitian

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submersions and anti-invariant submersions by defining and studying slant submersions from almost Hermitian manifolds onto Riemannian manifolds [27].

The present work is another step in this direction, more precisely from the point of view of slant Riemannian submersions from Sasakian manifolds. We also want to carry anti-invariant submanifolds of Sasakian manifolds to anti-invariant Riemannian submersion theory and to prove dual results for submersions. For instance, a slant submanifold of a $K$-contact manifold is an anti invariant submanifold if and only if $\nabla Q = 0$ (see Proposition 4.1 of [8]). We get a result similar to Proposition 4. Although slant submanifolds of contact metric manifolds were studied by several different authors and are considered a well-established topic in contact Riemannian geometry, only little about slant submersions are known. So, we study slant Riemannian submersions from almost contact metric manifolds onto Riemannian manifolds. Recently, the authors in [17,20] and [18] studied anti-invariant Riemannian submersions from almost contact manifolds independently of each other.

The paper is organized as follows: In Section 2, we present the basic information about Riemannian submersions needed throughout this paper. In Section 3, we mention about Sasakian manifolds. In Section 4, we give the definition of slant Riemannian submersions and introduce slant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We survey main results on slant submersions defined on Sasakian manifolds. We give a sufficient condition for a slant Riemannian submersion from Sasakian manifolds onto Riemannian manifolds to be harmonic. Moreover, we investigate the geometry of leaves of $(\ker F_*)$ and $(\ker F_*)^\perp$. We give an example of slant submersions such that the characteristic vector field $\xi$ is vertical. Moreover, we find a sharp inequality between the scalar curvature and squared mean curvature of fibres.

## 2. Riemannian submersions

In this section we recall several notions and results which will be needed throughout the paper.

Let $(M, g_M)$ be an $m$-dimensional Riemannian manifold and let $(N, g_N)$ be an $n$-dimensional Riemannian manifold. A Riemannian submersion is a smooth map $F : M \to N$ which is onto and satisfies the following axioms:

$S1$. $F$ has maximal rank.

$S2$. The differential $F_*$ preserves the lengths of horizontal vectors.

The fundamental tensors of a submersion were defined by O'Neill [22], [23]. They are $(1,2)$-tensors on $M$, given by the following formulas:

$$\mathcal{T}(E,F) = \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.1}$$

$$\mathcal{A}(E,F) = \mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.2}$$

for any vector fields $E$ and $F$ on $M$. Here $\nabla$ denotes the Levi-Civita connection of $(M, g_M)$. These tensors are called integrability tensors for the Riemannian submersions. Here we denote the projection morphism on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. The following lemmas are well known [22,23]:

**Lemma 1.** *For any $U, W$ vertical and $X, Y$ horizontal vector fields, the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy*

$$\text{(i)}\mathcal{T}_U W = \mathcal{T}_W U, \tag{2.3}$$

$$\text{(ii)}\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}\left[X, Y\right]. \tag{2.4}$$

It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$, $\mathcal{A}$ is horizontal and $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$.

For each $q \in N$, $F^{-1}(q)$ is an $(m-n)$-dimensional submanifold of $M$. The submanifolds $F^{-1}(q)$, $q \in N$, are called fibres. A vector field on $M$ is called vertical if it is always tangent to fibres. A vector field on $M$ is called horizontal if it is always orthogonal to fibres. A vector field $X$ on $M$ is called basic if $X$ is horizontal and $F$-related to a vector field $X$ on $N$, i. e., $F_* X_p = X_{*F(p)}$ for all $p \in M$.

**Lemma 2.** *Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian submersion. If $X$, $Y$ are basic vector fields on $M$, then*

(i) $g_M(X, Y) = g_N(X_*, Y_*) \circ F$,

(ii) $\mathcal{H}[X, Y]$ is basic and $F$-related to $[X_*, Y_*]$,

(iii) $\mathcal{H}(\nabla_X Y)$ is a basic vector field corresponding to $\overset{*}{\nabla}_{X_*} Y_*$ where $\nabla^*$ is the connection on $N$,

(iv) for any vertical vector field $V$, $[X, V]$ is vertical.

Moreover, if $X$ is basic and $U$ is vertical, then $\mathcal{H}(\nabla_U X) = \mathcal{H}(\nabla_X U) = \mathcal{A}_X U$. On the other hand, from (2.1) and (2.2) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. On any fibre $F^{-1}(q)$, $q \in N, \hat{\nabla}$ coincides with the Levi-Civita connection with respect to the metric induced by $g_M$. This induced metric on fibre $F^{-1}(q)$ is denoted by $\hat{g}$.

Notice that $\mathcal{T}$ acts on the fibres as the second fundamental form of the submersion and restricted to vertical vector fields and it can be easily seen that $\mathcal{T} = 0$ is equivalent to the condition that the fibres are totally geodesic. A Riemannian submersion is called a Riemannian submersion with totally geodesic fibres if $\mathcal{T}$ vanishes identically. Let $U_1, \ldots, U_{m-n}$ be an orthonormal frame of $\Gamma(\ker F_*)$. Then the horizontal vector field $H = \frac{1}{m-n}\sum_{j=1}^{m-n} \mathcal{T}_{U_j} U_j$ is called the mean curvature vector field of the fibre. If $H = 0$, then the Riemannian submersion is said to be minimal. A Riemannian submersion is called a Riemannian submersion with totally umbilical fibres if

$$\mathcal{T}_U W = g_M(U, W) H \tag{2.9}$$

for $U, W \in \Gamma(\ker F_*)$. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g_M)$ reversing the horizontal and the vertical distributions. By Lemma 1, the horizontal distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A}$ =0. For any $D, E, G \in \Gamma(TM)$, one has

$$g(\mathcal{T}_D E, G) + g(\mathcal{T}_D G, E) = 0 \tag{2.10}$$

and

$$g(\mathcal{A}_D E, G) + g(\mathcal{A}_D G, E) = 0. \tag{2.11}$$

The tensor fields $\mathcal{A}$, $\mathcal{T}$ and their covariant derivatives play a fundamental role in expressing the Riemannian curvature $R$ of $(M, g)$. By (2.5) and (2.6), we have

$$R(U,V,S,W) = g(R(S,W)V,U) \\ = \hat{R}(U,V,S,W) + g(\mathcal{T}_U W, \mathcal{T}_V S) - g(\mathcal{T}_V W, \mathcal{T}_U S), \tag{2.12}$$

where $\hat{R}$ is a Riemannian curvature tensor of any fibre $(F^{-1}(q), \hat{g}_q)$. Precisely, if $\{U, V\}$ is an orthonormal basis of the vertical 2-plane, then Eq. (2.12) implies that

$$K(U \wedge V) = \hat{K}(U \wedge V) + \| \mathcal{T}_U V \|^2 - g(\mathcal{T}_U U, \mathcal{T}_V V), \tag{2.13}$$

where $K$ and $\hat{K}$ denote the sectional curvature of $M$ and fibre $F^{-1}(q)$, respectively.

We recall the notion of harmonic maps between Riemannian manifolds. Let $(M, g_M)$ and $(N, g_N)$ be two Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth map between them. Then the differential $\varphi_*$ of $\varphi$ can be viewed as a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection $\nabla$ induced from the Levi-Civita connection $\nabla^M$ and the pullback connection. Then the second fundamental form of $\varphi$ is given by

$$(\nabla \varphi_*)(X,Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.14}$$

for $X, Y \in \Gamma(TM)$, where $\nabla^{\varphi}$ is the pullback connection. It is known that the second fundamental form is symmetric. If $\varphi$ is a Riemannian submersion, it can be easily proved that

$$(\nabla \varphi_*)(X,Y) = 0 \tag{2.15}$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla \varphi_*) = 0$. On the other hand, the tension field of $\varphi$ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div \varphi_* = \sum_{i=1}^{m} (\nabla \varphi_*)(e_i, e_i), \tag{2.16}$$

where $\{e_1, \dots, e_m\}$ is the orthonormal frame on $M$. Then it follows that $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$; for details, see [2].

## 3. Sasakian manifolds

An $n$-dimensional differentiable manifold $M$ is said to have an almost contact structure $(\phi, \xi, \eta)$ if it carries a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ on $M$ respectively such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1, \tag{3.1}$$

where $I$ denotes the identity tensor.

The almost contact structure is said to be normal if $N + d\eta \otimes \xi = 0$, where $N$ is the Nijenhuis tensor of $\phi$. Suppose that a Riemannian metric tensor $g$ is given in $M$ and satisfies the condition

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi). \tag{3.2}$$

Then the $(\phi, \xi, \eta, g)$-structure is called an almost contact metric structure. Define a tensor field $\Phi$ of type $(0,2)$ by $\Phi(X,Y) = g(\phi X, Y)$. If $d\eta = \Phi$ then an almost contact metric structure is said to be normal contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{3.3}$$

where $\nabla$ denotes the Levi-Civita connection of $g$. For a Sasakian manifold $M = M^{2n+1}$, it is known that

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{3.4}$$

$$S(X,\xi) = 2n\eta(X) \tag{3.5}$$

and

$$\nabla_X \xi = -\phi X. \tag{3.6}$$

[5].

The curvature tensor $R$ of a Sasakian space form $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) \\ & + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & - g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & - 2g(\phi X,Y)\phi Z), \end{aligned} \tag{3.7}$$

in [4] for any tangent vector fields $X, Y, Z$ to $M(c)$.

Now we will introduce a well known Sasakian manifold example on $\mathbb{R}^{2n+1}$.

**Example 1** (*[4]*)**.** We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z)(i = 1,\dots,n)$ and its usual contact form

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y_i dx_i\right).$$

The characteristic vector field $\xi$ is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric $g$ and tensor field $\phi$ are given by

$$g = \frac{1}{4}\eta\otimes\eta + \sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2), \qquad \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix}, \quad i = 1,\dots,n.$$

This gives a contact metric structure on $\mathbb{R}^{2n+1}$. The vector fields $E_i = 2\frac{\partial}{\partial y_i}$, $E_{n+i} = 2\left(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}\right)$, $\xi$ form a $\phi$-basis for the contact metric structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi, \xi, \eta, g)$ is a Sasakian manifold.

## 4. Slant Riemannian submersions

**Definition 1.** Let $M(\phi, \xi, \eta, g_M)$ be a Sasakian manifold and $(N, g_N)$ be a Riemannian manifold. A Riemannian submersion $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ is said to be slant if for any nonzero vector $X \in \Gamma(\ker F_*) - \{\xi\}$, the angle $\theta(X)$ between $\phi X$ and the space $\ker F_*$ is a constant (which is independent of the choice of $p \in M$ and of $X \in \Gamma(\ker F_*) - \{\xi\}$). The angle $\theta$ is called the slant angle of the slant submersion. Invariant and anti-invariant submersions are slant submersions with $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submersion which is not invariant nor anti-invariant is called proper submersion.

Now we will give an example.

**Example 2.** $\mathbb{R}^5$ has got a Sasakian structure as in Example 1. Let $F : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $F(x_1, x_2, y_1, y_2, z) = (x_1 - 2\sqrt{2}x_2 + y_1, 2x_1 - 2\sqrt{2}x_2 + y_1)$. Then, a simple calculation gives

$$\ker F_* = span\left\{V_1 = 2E_1 + \frac{1}{\sqrt{2}}E_4, V_2 = E_2, V_3 = \xi = E_5\right\}$$

and

$$(\ker F_*)^{\perp} = span\left\{H_1 = 2E_1 - \frac{1}{\sqrt{2}}E_4,\ H_2 = E_3\right\}.$$

Then it is easy to see that $F$ is a Riemannian submersion. Moreover, $\phi V_1 = 2E_3 - \frac{1}{\sqrt{2}}E_2$ and $\phi V_2 = E_4$ imply that $|g(\phi V_1, V_2)| = \frac{1}{\sqrt{2}}$. So $F$ is a slant submersion with slant angle $\theta = \frac{\pi}{4}$.

In Example 2, we note that the characteristic vector field $\xi$ is a vertical vector field. If $\xi$ is orthogonal to $\ker F_*$, we will then give the following theorem.

**Theorem 1.** *Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. If $\xi$ is orthogonal to $\ker F_*$, then $F$ is anti-invariant.*

**Proof.** By (2.3), (2.6), (2.10) and (3.6), we have

$$\begin{aligned} g(\phi U, V) &= -g(\nabla_U \xi, V) = -g(T_U \xi, V) = g(T_U V, \xi) \\ &= g(T_V U, \xi) = g(U, \phi V) \end{aligned}$$

for any $U, V \in \Gamma(\ker F_*)$. Using the skew symmetry property of $\phi$ in the last relation, we complete the proof of the theorem. □

**Remark 1.** Lotta [19] proved that if $M_1$ is a submanifold of a contact metric manifold of $\tilde{M}_1$ and $\xi$ is orthogonal to $M_1$, then $M_1$ is an anti-invariant submanifold. So, our result can be seen as a submersion version of Lotta's result.

Now, let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then for any $U, V \in \Gamma(\ker F_*)$, we put

$$\phi U = \psi U + \omega U, \tag{4.1}$$

where $\psi U$ and $\omega U$ are vertical and horizontal components of $\phi U$, respectively. Similarly, for any $X \in \Gamma(\ker F_*)^{\perp}$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{4.2}$$

where $\mathcal{B}X$ (resp. $\mathcal{C}X$) is the vertical part (resp. horizontal part) of $\phi X$.

From (3.2), (4.1) and (4.2), we obtain

$$g_M(\psi U, V) = -g_M(U, \psi V) \tag{4.3}$$

and

$$g_M(\omega U, Y) = -g_M(U, \mathcal{B}Y), \tag{4.4}$$

for any $U, V \in \Gamma(\ker F_*)$ and $Y \in \Gamma((\ker F_*)^{\perp})$.

Using (2.5), (3.6) and (4.1), we obtain

$$\mathcal{T}_U \xi = -\omega U, \qquad \hat{\nabla}_U \xi = -\psi U, \tag{4.5}$$

for any $U \in \Gamma(\ker F_*)$.

Now we will give the following proposition for a Riemannian submersion with two dimensional fibres in a similar way to Proposition 3.2 of [1].

**Proposition 1.** *Let $F$ be a Riemannian submersion from an almost contact manifold onto a Riemannian manifold. If $\dim(\ker F_*) = 2$ and $\xi$ is a vertical vector field, then the fibres are anti-invariant.*

As the proof of the following proposition is similar to slant submanifolds (see [8]), we omit its proof.

**Proposition 2.** *Let $F$ be a Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $\xi \in \Gamma(\ker F_*)$. Then $F$ is an anti-invariant submersion if and only if $D$ is integrable, where $D = \ker F_* - \{\xi\}$.*

**Theorem 2.** *Let $M(\phi, \xi, \eta, g_M)$ be a Sasakian manifold of dimension $2m+1$ and $(N, g_N)$ is a Riemannian manifold of dimension $n$. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be a slant Riemannian submersion. Then the fibres are not totally umbilical.*

**Proof.** Using (2.5) and (3.6), we obtain

$$\mathcal{T}_U \xi = -\omega U, \tag{4.6}$$

for any $U \in \Gamma(\ker F_*)$. If the fibres are totally umbilical, then we have $\mathcal{T}_U V = g_M(U, V)H$ for any vertical vector fields $U, V$ where $H$ is the mean curvature vector field of any fibre. Since $\mathcal{T}_\xi \xi = 0$, we have $H = 0$, which shows that fibres are minimal. Hence the fibres are totally geodesic, which is a contradiction to the fact that $\mathcal{T}_U \xi = -\omega U \neq 0$. □

By (2.5), (2.6), (4.1) and (4.2), we have

$$(\nabla_U \omega)V = \mathcal{C}\mathcal{T}_U V - \mathcal{T}_U \psi V, \tag{4.7}$$

$$(\nabla_U \psi)V = \mathcal{B}\mathcal{T}_U V - \mathcal{T}_U \omega V + R(\xi, U)V, \tag{4.8}$$

where

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega \hat{\nabla}_U V \tag{4.9}$$

$$(\nabla_U \psi)V = \hat{\nabla}_U \psi V - \psi \hat{\nabla}_U V, \tag{4.10}$$

for $U, V \in \Gamma(\ker F_*)$. Now we will characterize slant submersions in the following theorem.

**Theorem 3.** *Let $F$ be a Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $\xi \in \Gamma(\ker F_*)$. Then, $F$ is a slant Riemannian submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\psi^2 = -\lambda(I - \eta \otimes \xi). \tag{4.11}$$

*Furthermore, in such a case, if $\theta$ is the slant angle of $F$, it satisfies that $\lambda = \cos^2 \theta$.*

**Proof.** Firstly we suppose that $F$ is not an anti-invariant Riemannian submersion. Then, for $U \in \Gamma(\ker F_*)$,

$$\cos\theta = \frac{g_M(\phi U, \psi U)}{|\phi U|\,|\psi U|} = \frac{|\psi U|^2}{|\phi U|\,|\psi U|} = \frac{|\psi U|}{|\phi U|}. \tag{4.12}$$

Since $\phi U \perp \xi$, we have $g(\psi U, \xi) = 0$. Now, substituting $U$ by $\psi U$ in (4.12) and using (3.2) we obtain

$$\cos\theta = \frac{|\psi^2 U|}{|\phi\psi U|} = \frac{|\psi^2 U|}{|\psi U|}. \tag{4.13}$$

From (4.12) and (4.13), we have

$$|\psi U|^2 = |\psi^2 U|\,|\phi U|\,. \tag{4.14}$$

On the other hand, one can get the following

$$\begin{aligned} g_M(\psi^2 U, U) &= g_M(\phi\psi U, U) = -g_M(\psi U, \phi U) \\ &= -g_M(\psi U, \psi U) = -|\psi U|^2\,. \end{aligned} \tag{4.15}$$

Using (4.14) and (4.15), we get

$$\begin{aligned} g_M(\psi^2 U, U) &= -|\psi^2 U|\,|\phi U| \\ &= -|\psi^2 U|\,|\phi^2 U|\,. \end{aligned} \tag{4.16}$$

Also, one can easily get

$$g_M(\psi^2 U, \phi^2 U) = -g_M(\psi^2 U, U). \tag{4.17}$$

So, by means of (4.16) and (4.17), we obtain $g_M(\psi^2 U, \phi^2 U) = |\psi^2 U||\phi^2 U|$ and it follows that $\psi^2 U$ and $\phi^2 U$ are collinear, that is $\psi^2 U = \lambda\phi^2 U = -\lambda(I - \eta \otimes \xi)$. Using the last relation together with (4.12) and (4.13) we obtain that $\cos\theta = \sqrt{\lambda}$ is constant and so $F$ is a slant Riemannian submersion.

If $F$ is an anti-invariant Riemannian submersion then $\phi U$ is normal, $\psi U = 0$ and it is equivalent to $\psi^2 U = 0$. In this case $\theta = \frac{\pi}{2}$ and so Eq. (4.12) is again satisfied. □

By using (3.2), (4.1), (4.3) and (4.11), we have the following lemma.

**Lemma 3.** *Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ with slant angle $\theta$. Then the following relations are valid for any $U, V \in \Gamma(\ker F_*)$:*

$$g_M(\psi U, \psi V) = \cos^2\theta(g_M(U, V) - \eta(U)\eta(V)), \tag{4.18}$$

$$g_M(\omega U, \omega V) = \sin^2\theta(g_M(U, V) - \eta(U)\eta(V)). \tag{4.19}$$

We denote the complementary orthogonal distribution to $\omega(\ker F_*)$ in $(\ker F_*)^\perp$ by $\mu$. Then we have

$$(\ker F_*)^\perp = \omega(\ker F_*) \oplus \mu. \tag{4.20}$$

**Lemma 4.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then $\mu$ is an invariant distribution of $(\ker F_*)^\perp$ under the endomorphism $\phi$.*

**Proof.** For $X \in \Gamma(\mu)$ and $V \in \Gamma(\ker F_*)$, from (3.2) and (4.1), we obtain

$$\begin{aligned} g_M(\phi X, \omega V) &= g_M(\phi X, \phi V) - g_M(\phi X, \psi V) \\ &= g_M(X, V) - \eta(X)\eta(V) - g_M(\phi X, \psi V) \\ &= -g_M(X, \phi\psi V) \\ &= 0. \end{aligned}$$

In a similar way, we have $g_M(\phi X, U) = -g_M(X, \phi U) = 0$ due to $\phi U \in \Gamma((\ker F_*) \oplus \omega(\ker F_*))$ for $X \in \Gamma(\mu)$ and $U \in \Gamma(\ker F_*)$. Thus the proof of the lemma is completed. □

By means of (4.19), we can give the following result:

**Corollary 1.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian manifold $M^{2m+1}(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N^n, g_N)$. Let*

$$\{e_1, e_2, \ldots, e_{2m-n}, \xi\}$$

*be a local orthonormal frame of $(\ker F_*)$, then $\{\csc\theta\omega e_1, \csc\theta\omega e_2, \ldots, \csc\theta\omega e_{2m-n}\}$ is a local orthonormal frame of $\omega(\ker F_*)$.*

By using (4.20) and Corollary 1, one can easily prove the following proposition:

**Proposition 3.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian manifold $M^{2m+1}(\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N^n,g_N)$. Then* $\dim(\mu)=2(n-m)$. *If* $\mu=\{0\}$*, then* $n=m$.

By (4.3) and (4.18), we have

**Lemma 5.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian manifold $M^{2m+1}(\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N^n,g_N)$. If $e_1,e_2,\dots,e_k,\xi$ are orthogonal unit vector fields in* $(\ker F_*)$*, then*

$$\{e_1,\sec\theta\psi e_1,e_2,\sec\theta\psi e_2,\dots,e_k,\sec\theta\psi e_k,\xi\}$$

*is a local orthonormal frame of* $(\ker F_*)$. *Moreover* $\dim(\ker F_*)=2m-n+1=2k+1$ *and* $\dim N=n=2(m-k)$.

**Lemma 6.** *Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. If $\omega$ is parallel, then we have*

$$\mathcal{T}_{\psi U}\psi U=-\cos^2\theta(\mathcal{T}_U U+\eta(U)\omega U). \tag{4.21}$$

**Proof.** If $\omega$ is parallel, from (4.7), we obtain $\mathcal{C}T_U V=\mathcal{T}_U\psi V$ for $U,V\in\Gamma(\ker F_*)$. Antisymmetrizing with respect to $U$, $V$ and using (2.3), we get

$$\mathcal{T}_U\psi V=\mathcal{T}_V\psi U.$$

Substituting $V$ by $\psi U$ in the above equation and using Theorem 3, we get the required formula. □

We give a sufficient condition for a slant Riemannian submersion to be harmonic as an analogue of a slant Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold in [27].

**Theorem 4.** *Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. If $\omega$ is parallel, then $F$ is a harmonic map.*

**Proof.** From [10] we know that $F$ is harmonic if and only if $F$ has minimal fibres. Thus $F$ is harmonic if and only if $\sum_{i=1}^{n_1}\mathcal{T}_{e_i}e_i=0$. Hence using the adapted frame for slant Riemannian submersion and by the help of (2.16) and Lemma 5, we can write

$$\tau=-\sum_{i=1}^{m-\frac{n}{2}}F_*(\mathcal{T}_{e_i}e_i+\mathcal{T}_{\sec\theta\psi e_i}\sec\theta\psi e_i)-F_*(\mathcal{T}_\xi\xi).$$

Regarding $\mathcal{T}_\xi\xi=0$, we have

$$\tau=-\sum_{i=1}^{m-\frac{n}{2}}F_*(\mathcal{T}_{e_i}e_i+\sec^2\theta\mathcal{T}_{\psi e_i}\psi e_i).$$

Using (4.21) in the above equation, we obtain

$$\begin{aligned}\tau &= -\sum_{i=1}^{m-\frac{n}{2}} F_*(\mathcal{T}_{e_i}e_i + \sec^2\theta(-\cos^2\theta(\mathcal{T}_{e_i}e_i + \eta(e_i)\omega e_i))) \\ &= -\sum_{i=1}^{m-\frac{n}{2}} F_*(\mathcal{T}_{e_i}e_i - \mathcal{T}_{e_i}e_i) = 0.\end{aligned}$$

So we prove that $F$ is harmonic. □

Now setting $Q = \psi^2$, we define $\nabla Q$ by

$$(\nabla_U Q)V = \mathcal{V}\nabla_U QV - Q\hat{\nabla}_U V$$

for any $U, V \in \Gamma(\ker F_*)$.We give a characterization for a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ by using the value of $\nabla Q$.

**Proposition 4.** *Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then, $\nabla Q = 0$ if and only if $F$ is an anti-invariant submersion.*

**Proof.** By using (4.11),

$$Q\hat{\nabla}_U V = -\cos^2\theta(\hat{\nabla}_U V - \eta(\hat{\nabla}_U V)\xi) \tag{4.22}$$

for each $U, V \in \Gamma(\ker F_*)$, where $\theta$ is the slant angle.

On the other hand, it follows that

$$\mathcal{V}(\nabla_U QV) = -\cos^2\theta(\hat{\nabla}_U V - \eta(\hat{\nabla}_U V)\xi + g(V, \psi U)\xi + \eta(V)\psi U). \tag{4.23}$$

So, from (4.22) and $\nabla Q = 0$ if and only if $\cos^2\theta(g(V, \psi U)\xi + \eta(V)\psi U) = 0$ which implies that $\psi U = 0$ or $\theta = \frac{\pi}{2}$. Both the cases verify that $F$ is an anti-invariant submersion. □

We now investigate the geometry of leaves of $(\ker F_*)^\perp$ and $\ker F_*$.

**Theorem 5.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on $M$ if and only if*

$$\begin{aligned}g_M(\mathcal{H}\nabla_X Y, \omega\psi U) - \sin^2\theta g_M(Y, \phi X)\eta(U) &= g_M(\mathcal{A}_X \mathcal{B}Y, \omega U) \\ &\quad + g_M(\mathcal{H}\nabla_X \mathcal{C}Y, \omega U)\end{aligned}$$

*for any $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.*

**Proof.** From (3.3) and (4.1), we have

$$\begin{aligned}g_M(\nabla_X Y, U) &= -g_M(\phi\nabla_X \phi Y, U) + g_M(Y, \phi X)\eta(U) \\ &= g_M(\nabla_X \phi Y, \phi U) + g_M(Y, \phi X)\eta(U) \\ &= g_M(\nabla_X \phi Y, \psi U) + g_M(\nabla_X \phi Y, \omega U) + g_M(Y, \phi X)\eta(U)\end{aligned} \tag{4.24}$$

for any $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Using (3.3) and (4.1) in (4.24), we obtain

$$g_M(\nabla_X Y, U) = -g_M(\nabla_X Y, \psi^2 U) - g_M(\nabla_X Y, \omega\psi U) + g_M(Y, \phi X)\eta(U) + g_M(\nabla_X \phi Y, \omega U). \tag{4.25}$$

By (4.2) and (4.11), we have

$$g_M(\nabla_X Y, U) = \cos^2\theta g_M(\nabla_X Y, U) - \cos^2\theta \eta(U)\eta(\nabla_X Y) - g_M(\nabla_X Y, \omega\psi U) + g_M(Y, \phi X)\eta(U) + g_M(\nabla_X \mathcal{B}Y, \omega U) + g_M(\nabla_X \mathcal{C}Y, \omega U). \tag{4.26}$$

Using (2.7), (2.8) and (3.6) in the last equation, we obtain

$$\sin^2\theta g_M(\nabla_X Y, U) = \sin^2\theta g_M(Y, \phi X)\eta(U) - g_M(\mathcal{H}\nabla_X Y, \omega\psi U) + g_M(\mathcal{A}_X \mathcal{B}Y, \omega U) + g_M(\mathcal{H}\nabla_X \mathcal{C}Y, \omega U)$$

which proves the theorem. □

**Proposition 5.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. If the distribution $\ker F_*$ defines a totally geodesic foliation on $M$, then $F$ is an invariant submersion.*

**Proof.** By (4.5), if the distribution $\ker F_*$ defines a totally geodesic foliation on $M$, then we conclude that $\omega U = 0$ for any $U \in \Gamma(\ker F_*)$ which shows that $F$ is an invariant submersion. □

Now we establish a sharp inequality between norm squared mean curvature $\|H\|^2$ and the scalar curvature $\hat{\tau}$ of fibre through $p \in M^5(c)$.

**Theorem 6.** *Let $F$ be a proper slant Riemannian submersion from a Sasakian space form $M^5(c)$ onto a Riemannian manifold $(N^2, g_N)$. Then, we have*

$$\|H\|^2 \geq \frac{8}{9}\hat{\tau} - \frac{2}{9}[c + 3 + (3c + 5)\cos^2\theta] \tag{4.27}$$

*where $H$ denotes the mean curvature of fibres. Moreover, the equality sign of (4.27) holds at a point $p$ of a fibre if and only if with respect to some suitable slant orthonormal frame $\{e_1, e_2 = \sec\theta\psi e_1, e_3 = \xi, e_4 = \csc\theta\omega e_1, e_5 = \csc\theta\omega e_2\}$ at p, we have*

$$T_{11}^4 = 3T_{22}^4, \qquad T_{12}^4 = 0 \quad \text{and} \quad T_{11}^5 = 0,$$

*where $T_{ij}^\alpha = g(\mathcal{T}(e_i, e_j), e_\alpha)$ for $1 \leq i, j \leq 3$ and $\alpha = 4, 5$.*

**Proof.** By Corollary 1, Lemma 5 and Proposition 3 we construct a slant orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ defined by

$$e_1, e_2 = \sec\theta\psi e_1, \qquad e_3 = \xi, \qquad e_4 = \csc\theta\omega e_1, \qquad e_5 = \csc\theta\omega e_2, \tag{4.28}$$

where $e_1, e_2, e_3 = \xi \in \Gamma(\ker F_*)$ and $e_4, e_5 \in \Gamma((\ker F_*)^\perp)$. Let $\hat{\tau}$ be scalar curvature of fibre $F^{-1}(q)$. We choose an arbitrary point $p$ of the fibre $F^{-1}(q)$. We obtain

$$\hat{\tau}(p) = \hat{K}(e_1 \wedge e_2) + \hat{K}(e_1 \wedge e_3) + \hat{K}(e_2 \wedge e_3). \tag{4.29}$$

By (2.12), (2.13) and (3.7), we get

$$\hat{K}(e_1 \wedge e_2) = \frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + T^4_{11}T^4_{22} + T^5_{11}T^5_{22} - (T^4_{12})^2 - (T^5_{12})^2, \tag{4.30}$$

where $T^\alpha_{ij} = g(\mathcal{T}(e_i, e_j), e_\alpha)$ for $1 \leq i, j \leq 3$ and $\alpha = 4, 5$. Using Theorem 3 and the relation (4.19), one has

$$\psi e_2 = -\cos\theta e_1 \quad \text{and} \quad \omega e_2 = \sin\theta e_5. \tag{4.31}$$

From (4.8), we have

$$g((\hat{\nabla}_{e_2}\psi)e_2, e_1) = g(\mathcal{B}\mathcal{T}_{e_2}e_2, e_1) - g(\mathcal{T}_{e_2}\omega e_2, e_1).$$

Using (4.1), (4.2), (4.10), (4.28) and (4.31) in the last relation, we obtain

$$\sin\theta[g(\mathcal{T}_{e_2}e_2, e_4) - g(\mathcal{T}_{e_2}e_1, e_5)] = 0. \tag{4.32}$$

Since the submersion is proper, Eq. (4.32) implies that

$$T^4_{22} = T^5_{12}.$$

Now we choose the unit normal vector $e_4 \in \Gamma(\ker(F_*))^\perp$ parallel to the mean curvature vector $H(p)$ of fibre. Then we have

$$\|H(p)\|^2 = \frac{1}{9}(T^4_{11} + T^4_{22})^2, \qquad T^5_{11} + T^5_{22} = 0.$$

So the relation (4.30) becomes

$$\hat{K}(e_1 \wedge e_2) = \frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + T^4_{11}T^4_{22} - (T^5_{11})^2 - (T^4_{12})^2 - (T^4_{22})^2. \tag{4.33}$$

From the trivial inequality $(\mu - 3\lambda)^2 \geq 0$, one has $(\mu + \lambda)^2 \geq 8(\lambda\mu - \lambda^2)$. Putting $\mu = T^4_{11}$ and $\lambda = T^4_{22}$ in the last inequality, we find

$$\|H\|^2 \geq \frac{8}{9}\left[\hat{K}(e_1 \wedge e_2) - \frac{c+3}{4} - \frac{3}{4}(c-1)\cos^2\theta\right]. \tag{4.34}$$

Using (2.13), we get

$$\hat{K}(e_1 \wedge e_3) = \hat{K}(e_2 \wedge e_3) = \cos^2\theta.$$

By (2.13), (4.29) and the last relation, we get the required inequality. Moreover, the equality sign of (4.27) holds at a point $p$ of a fibre if and only if $T^4_{11} = 3T^4_{22}, T^4_{12} = 0$ and $T^5_{11} = 0$. □

**Open Problem:**

Let $F$ be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. In [3], Barrera et al. defined and studied the Maslov form of non-invariant slant submanifolds of $S$-space form $\tilde{M}(c)$. They find conditions for it to be closed. By a similar discussion in [3], we can define Maslov form $\varOmega H$ of $M$ as the dual form of the vector field $\mathcal{B}H$, that is,

$$\varOmega H(U) = g_M(U, \mathcal{B}H)$$

for any $U \in \Gamma(\ker F_*)$. So it will be interesting to give a characterization with respect to $\varOmega H$ for slant submersions, where $H = \sum_{i=1}^{m-\frac{n}{2}} \mathcal{T}_{e_i} e_i + \mathcal{T}_{\sec\theta\psi e_i} \sec\theta\psi e_i$ and $\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, \ldots, e_k, \sec\theta\psi e_k, \xi\}$ is a local orthonormal frame of $(\ker F_*)$.

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