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Skew generalized power series Hopfian modules

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Abstract. In this paper we study the transfer of the property of Hopfian modules between the right R-module M_R and some of its extension classes. Namely, under certain conditions, we show that: M_R is a Hopfian right R-module if and only if the skew generalized power series module $[[M^{S,\leq}]]$ is a Hopfian right $[[R^{S,\leq},\omega]]$-module.

Keywords: Hopfian module; Hopfian ring; Skew generalized power series module

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1. INTRODUCTION

Throughout this paper R denotes an associative ring not necessarily commutative with the identity and M_R a unitary right R-module. As it has been noted by Hiremath [3], the concept of Hopfian groups was introduced by Baumslag [1]. In fact, the study of endomorphism rings of various rings and modules has been a topic of keen interest since the end of the nineteen sixties when injectivity and its variants began to flourish. In 1986, Hiremath introduced the concept of the Hopfian module as follows: A right R-module M_R is called *Hopfian* if any surjective endomorphism of M_R is an isomorphism. The term "Hopfian" is said to be in honor of Heinz Hopf and his use of the concept of the Hopfian group in his work on fundamental groups of surfaces. Any noetherian module is Hopfian and if R is a right noetherian ring, then

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every finitely generated R-module is a Hopfian module. Also, a simple ring is Hopfian, since the kernel of any endomorphism is an ideal, which is necessarily zero in a simple ring.

The module R_R is Hopfian if and only if R is a directly finite ring. Symmetrically, these two are also equivalent to the left R-module ${}_RR$ being Hopfian. The full linear ring $\mathrm{End}_D(V)$ of a countable dimensional vector space is a Hopfian ring which is not Hopfian as a module, since it only has three ideals, but it is not directly finite.

Varadarajan [5,6] showed that the right R-module M_R is Hopfian if and only if the right $R[x]$-module $M[x]$ is Hopfian.

The motivation of this paper is to investigate how the property of Hopfian modules behaves under passage to the skew generalized power series modules.

2. Hopfian modules over skew generalized power series rings

In this section we extend the results of [9] to the skew generalized power series modules.

Let $(S, \leq)$ be an ordered commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the identity element by 0. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. The following construction is due to Zhongkui [10]:

Let $(S, \leq)$ be a strictly ordered monoid (that is, if $s, s^{'}, t \in S$ and $s < s^{'}$, then $s + t < s^{'} + t$), R a ring and $\omega : S \longrightarrow \mathrm{End}\,(R)$ a monoid homomorphism. Consider the set $A = \left[\left[R^{S,\leq}, \omega\right]\right]$ of all maps $f : S \longrightarrow R$ whose support ($\mathrm{supp}\,(f) = \{s \in S |\ f\,(s) \neq 0\}$) is artinian and narrow.

For every $s \in S$ and $f, g \in A$, let

$$X_s\,(f, g) = \{(u, v) \in S \times S |\ u + v = s; f\,(u) \neq 0, g\,(v) \neq 0\}\,.$$

It follows from ([4], 4.1) that $X_s\,(f, g)$ is a finite set.

This fact allows us to define the operation of multiplication (*convolution*) as follows:

$$(fg)\,(s) = \sum_{(u,v)\in X_s(f,g)} f\,(u)\,\omega_u\,(g\,(v))\,,$$

and $(fg)\,(s) = 0$ if $X_s\,(f, g) = \phi$. With this operation and pointwise addition $A = \left[\left[R^{S,\leq}, \omega\right]\right]$ becomes a ring, which is called the *ring of skew generalized power series* with coefficients in R and exponents in S.

In [8], Zhao and Jiao generalized this construction to obtain the skew generalized power series modules over skew generalized power series rings, as follows:

Let M_R be a right R-module, let B be the set of all maps $\varphi : S \longrightarrow M$ such that supp(φ) $= \{s \in S\ |\varphi\,(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $B = \left[\left[M^{S,\leq}\right]\right]$ is an abelian additive group. For each $f \in A = \left[\left[R^{S,\leq}, \omega\right]\right]$ and $\varphi \in B$, the set

$$X_s\,(\varphi, f) = \{(u, v) \in S \times S |\ u + v = s; \varphi\,(u) \neq 0, f\,(v) \neq 0\}$$

is finite (see [9], Lemma 1). This allows us to define the scalar multiplication of the elements of B by scalars from A as follows:

$$(\varphi f)\,(s) = \sum_{(u,v)\in X_s(\varphi,f)} \varphi\,(u)\,\omega_u\,(f\,(v))\,,$$

and $(\varphi f)(s) = 0$ if $X_s(\varphi, f) = \phi$. With this operation and pointwise addition, one can easily show that B is a right A-module, which is called the *module of skew generalized power series* with coefficients in M and exponents in S.

For every $s \in S$ if we set $\omega(s) = \mathrm{Id}_R \in \mathrm{Aut}(R) \subset \mathrm{End}(R)$, the identity map of R, then $A = \left[\left[R^{S,\leq}, \omega\right]\right] = \left[\left[R^{S,\leq}\right]\right]$ is the ring of generalized power series in the sense of Ribenboim [4] and $B = \left[\left[M^{S,\leq}\right]\right]$ is the untwisted module of generalized power series in the sense of [7].

For any $r \in R$ we associated the map $c_r \in A$ defined by:

$$c_r(x) = \begin{cases} r, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

For any $m \in M$ and $s \in S$, we define a map $d_m^s \in B$ by:

$$d_m^s(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases}$$

If $(S, \leq)$ is a strictly totally ordered monoid, then $\mathrm{supp}(f)$ is a nonempty well-ordered subset of S, for every $0 \neq f \in A$, and we denote by $\pi(f)$ the smallest element of $\mathrm{supp}(f)$. Also, $\mathrm{supp}(\varphi)$ is a nonempty well-ordered subset of S, for every $0 \neq \varphi \in B$, and we denote by $\pi(\varphi)$ the smallest element of $\mathrm{supp}(\varphi)$.

The following are required in the sequel.

Definition 1 (*[2]*)**.** A monoid S is called finitely generated if there exists a finite subset $\{s_1, \ldots, s_n\}$ of S such that $S = \{\sum_{i=1}^{n} k_i s_i \,|k_i \geq 0\}$.

Lemma 1 (*[2]*)**.** *If S is a finitely generated monoid, then every ideal of S is finitely generated, so every strictly increasing sequence of ideals is finite.*

The following lemma is crucial in developing the proof of the main result.

Lemma 2. *Let $(S, \leq)$ be a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0 \leq s$ for every $s \in S$. Assume that $\omega_s(1) = 1$ for each $s \in S$. Then $\pi(\alpha(\varphi)) \geq \pi(\varphi)$ where $\alpha \in \mathrm{End}_A(B)$ and $0 \neq \varphi \in B$.*

Proof. For each element $x \in S$, we denote by x^+ the subset $x^+ = \{x + y \mid y \in S\}$. Set $s = \pi(\alpha(\varphi))$ and $t = \pi(\varphi)$. Suppose that $s < t$. Define $\psi_1 \in B$ via

$$\psi_1(x) = \varphi(x + t).$$

Set $\varphi_1 = \varphi - \psi_1 e_t$. If $\varphi_1 = 0$, then $\varphi = \psi_1 e_t$. Thus

$$\alpha(\varphi) = \alpha(\psi_1 e_t) = \alpha(\psi_1) e_t.$$

For any $x \in S$ with $x < t$, we have

$$(\alpha(\psi_1)e_t)(x) = \sum_{(u,v) \in X_x(\alpha(\psi_1), e_t)} \alpha(\psi_1)(u)\,\omega_u(e_t(v)).$$

For each pair $(u,v) \in X_x(\alpha(\psi_1), e_t)$, we have $x = u + v < t$. Then $u < t$ and $v < t$. Thus $e_t(v) = 0$ and $(\alpha(\psi_1)e_t)(x) = 0$. Hence $\pi(\alpha(\psi_1)e_t) \geq t$. It follows that $s \geq t$, a contradiction. So $\varphi_1 \neq 0$. Since $\varphi = \varphi_1 + \psi_1 e_t$, we have

$$\alpha(\varphi) = \alpha(\varphi_1 + \psi_1 e_t) = \alpha(\varphi_1) + \alpha(\psi_1)e_t.$$

It follows that

$$\mathrm{supp}\,(\alpha(\varphi)) \subseteq \mathrm{supp}\,(\alpha(\varphi_1)) \cup \mathrm{supp}\,(\alpha(\psi_1)e_t)\,.$$

If $s \in \mathrm{supp}\,(\alpha(\psi_1)e_t)$, then $s \geq \pi(\alpha(\psi_1)e_t) \geq t$, a contradiction with the assumption that $s < t$. Thus $s \in \mathrm{supp}\,(\alpha(\varphi_1))$.

Denote $s_1 = \pi(\alpha(\varphi_1))$ and $t_1 = \pi(\varphi_1)$. Then $s \geq s_1$. Since

$$\begin{aligned}\varphi_1(t) &= (\varphi - \psi_1 e_t)(t)\\ &= \varphi(t) - (\psi_1 e_t)(t)\\ &= \varphi(t) - \psi_1(0)\omega_0(e_t(t))\\ &= \varphi(t) - \psi_1(0)\omega_0(1) = \varphi(t) - \psi_1(0),\end{aligned}$$

we have

$$\varphi_1(t) = \varphi(t) - \varphi(0+t) = \varphi(t) - \varphi(t) = 0.$$

It is clear that $t_1 > t$. If $t_1 \in t^+ = \{t + u \,|u \in S\}$, then there exists $u \in S$ such that $t_1 = t + u$. Thus

$$\begin{aligned}0 \neq \varphi_1(t_1) &= (\varphi - \psi_1 e_t)(t_1)\\ &= \varphi(t_1) - (\psi_1 e_t)(t_1)\\ &= \varphi(t_1) - \sum_{(y,z)\in X_{t_1}(\psi_1, e_t)} \psi_1(y)\omega_y(e_t(z))\\ &= \varphi(t_1) - \psi_1(u)\omega_u(e_t(t))\\ &= \varphi(t_1) - \psi_1(u)\omega_u(1) = \varphi(t_1) - \psi_1(u)\\ &= \varphi(t_1) - \varphi(u+t) = \varphi(t_1) - \varphi(t_1) = 0,\end{aligned}$$

a contradiction. Hence $t_1 \notin t^+$ and $t_1 + S \nsubseteq t + S$. Then $t + S \subsetneqq (t+S) \cup (t_1 + S)$. Suppose that for a positive integer $n \geq 2$, we have found $\varphi_1, \psi_1, \ldots, \varphi_{n-1}, \psi_{n-1} \in B$ such that

$$\psi_i(x) = \varphi_{i-1}(x + t_{i-1}) \quad \text{and} \quad \varphi_i = \varphi_{i-1} - \psi_i e_{t_{i-1}},$$

where

$$\begin{aligned}&t_i = \pi(\varphi_i), t < t_1 < \cdots < t_{i-1} < t_i \quad \text{and}\\ &t_i \notin t^+ \cup t_1^+ \cup \cdots \cup t_{i-1}^+ \quad \text{for each } i = 1, \ldots, n-1.\end{aligned}$$

We set $t_0 = t$ and $\varphi_0 = \varphi$. Define $\psi_n \in B$ via

$$\psi_n(x) = \varphi_{n-1}(x + t_{n-1}).$$

Let $\varphi_n = \varphi_{n-1} - \psi_n e_{t_{n-1}}$. Hence

$$\varphi = \psi_1 e_t + \psi_2 e_{t_1} + \cdots + \psi_n e_{t_{n-1}} + \varphi_n.$$

If $\varphi_n = 0$, then

$$\alpha(\varphi) = \alpha(\psi_1)e_t + \alpha(\psi_2)e_{t_1} + \cdots + \alpha(\psi_n)e_{t_{n-1}}.$$

Thus

$$\operatorname{supp}(\alpha(\varphi)) \subseteq \bigcup_{i=0}^{n-1} \operatorname{supp}(\alpha(\psi_{i+1})e_{t_i}),$$

which implies that there exists i such that $s = \pi(\alpha(\varphi)) \in \operatorname{supp}(\alpha(\psi_{i+1})e_{t_i})$. Thus $s \geq \pi(\alpha(\psi_{i+1})e_{t_i}) \geq t_i \geq t$, a contradiction. Now, suppose that $\varphi_n \neq 0$. Denote $t_n = \pi(\varphi_n)$. Since $\varphi_n = \varphi_{n-1} - \psi_n e_{t_{n-1}}$, we have

$$\begin{aligned}
\varphi_n(t_{n-1}) &= \varphi_{n-1}(t_{n-1}) - \left(\psi_n e_{t_{n-1}}\right)(t_{n-1}) \\
&= \varphi_{n-1}(t_{n-1}) - \sum_{(y,z)\in X_{t_{n-1}}\left(\psi_n, e_{t_{n-1}}\right)} \psi_n(y)\omega_y\left(e_{t_{n-1}}(z)\right) \\
&= \varphi_{n-1}(t_{n-1}) - \psi_n(0)\,\omega_0\left(e_{t_{n-1}}(t_{n-1})\right) \\
&= \varphi_{n-1}(t_{n-1}) - \psi_n(0)\,\omega_0(1) = \varphi_{n-1}(t_{n-1}) - \psi_n(0) \\
&= \varphi_{n-1}(t_{n-1}) - \varphi_{n-1}(0 + t_{n-1}) = 0.
\end{aligned}$$

For any $x \in S$ with $x < t_{n-1}$ and for every $(y, z) \in X_x\left(\psi_n, e_{t_{n-1}}\right)$, we have $x = y + z < t_{n-1}$. Then $y < t_{n-1}$ and $z < t_{n-1}$, and hence $\omega_y\left(e_{t_{n-1}}(z)\right) = \omega_y(0) = 0$. It follows that

$$\begin{aligned}
\varphi_n(x) &= \varphi_{n-1}(x) - \left(\psi_n e_{t_{n-1}}\right)(x) \\
&= 0 - \sum_{(y,z)\in X_x\left(\psi_n, e_{t_{n-1}}\right)} \psi_n(y)\omega_y\left(e_{t_{n-1}}(z)\right) = 0.
\end{aligned}$$

So $\pi(\varphi_n) = t_n > t_{n-1}$. Thus

$$t < t_1 < \cdots < t_{n-1} < t_n.$$

Suppose that $t_n \in t^+ \cup t_1^+ \cup \cdots \cup t_{n-1}^+$. Then there exists i such that

$$t_n \notin t_{n-1}^+, \ldots, t_n \notin t_{i+1}^+, \quad \text{but } t_n \in t_i^+.$$

Let $t_n = t_i + v$ for some $v \in S$. Then

$$\begin{aligned}
\varphi_i(t_n) &= \left(\psi_{i+1}e_{t_i} + \psi_{i+2}e_{t_{i+1}} + \cdots + \psi_n e_{t_{n-1}} + \varphi_n\right)(t_n) \\
&= (\psi_{i+1}e_{t_i})(t_n) + \left(\psi_{i+2}e_{t_{i+1}}\right)(t_n) + \cdots + (\psi_n e_{t_{n-1}})(t_n) + \varphi_n(t_n).
\end{aligned}$$

Note that, since $t_n \in t_i^+$ and $t_n \notin t_{i+1}^+$, we see that

$$(\psi_{i+1}e_{t_i})(t_n) = \psi_{i+1}(v)\omega_v(e_{t_i}(t_i)) = \psi_{i+1}(v)\,\omega_v(1) = \psi_{i+1}(v),$$

and

$$\left(\psi_{i+2}e_{t_{i+1}}\right)(t_n) = \sum_{(y,z)\in X_{t_n}\left(\psi_{i+2},e_{t_{i+1}}\right)} \psi_{i+2}(y)\omega_y\left(e_{t_{i+1}}(z)\right) = 0.$$

Thus

$$\varphi_i\left(t_n\right) = \psi_{i+1}(v) + \varphi_n(t_n) = \varphi_i(v+t_i) + \varphi_n\left(t_n\right) = \varphi_i\left(t_n\right) + \varphi_n(t_n),$$

which implies that $\varphi_n(t_n) = 0$, a contradiction. Hence $t_n \notin t^+ \cup t_1^+ \cup \cdots \cup t_{n-1}^+$. Now, we have the infinite strictly increasing sequence of ideals of S

$$t+S \subsetneqq (t+S)\cup(t_1+S) \subsetneqq (t+S)\cup(t_1+S)\cup(t_2+S) \subsetneqq \cdots$$

a contradiction with Lemma 1. Therefore $s \geq t$. ■

Now, we are able to prove the main result of this paper.

Theorem 3. *Suppose that $(S,\leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0 \leq s$ for every $s \in S$. Assume that $\omega_s(1) = 1$ for each $s \in S$. Then M_R is a Hopfian right R-module if and only if B_A is a Hopfian right A-module.*

Proof. Suppose that M_R is a Hopfian right R-module. Let $\alpha : B_A \longrightarrow B_A$ be any surjective A-homomorphism. We want to prove that α is injective to be an isomorphism. Define $f : B \longrightarrow M$ via $f(\varphi) = \varphi(0)$. Now, define $h : M_R \longrightarrow M_R$ via $h(m) = f\alpha(d_m^0)$.

(1) h is an R-homomorphism: For any $m \in M$ and $r \in R$, we have

$$\begin{aligned} h(mr) &= f\alpha\left(d_{mr}^0\right) = f\alpha\left(d_m^0 c_r\right) = f\left(\alpha\left(d_m^0 c_r\right)\right) \\ &= f\left(\alpha\left(d_m^0\right)c_r\right) = \left(\alpha\left(d_m^0\right)c_r\right)(0). \end{aligned}$$

Since $0 \leq s$ for every $s \in S$, we get $X_0\left(\alpha(d_m^0), c_r\right) = \{(0,0)\}$ and so

$$\begin{aligned} h(mr) &= (\alpha(d_m^0))(0)\omega_0\left(c_r(0)\right) = (\alpha(d_m^0))(0)\omega_0(r) \\ &= (\alpha(d_m^0))(0)r = f\alpha(d_m^0)r \\ &= h(m)r. \end{aligned}$$

(2) h is a surjective map: For any $m \in M$, there exists $\beta \in B$ such that $\alpha(\beta) = d_m^0$ since α is surjective. Let $\psi = \beta - d_{\beta(0)}^0$. Then

$$\psi(0) = (\beta - d_{\beta(0)}^0)(0) = \beta(0) - d_{\beta(0)}^0(0) = \beta(0) - \beta(0) = 0.$$

So $\pi(\psi) > 0$ and using Lemma 2, $\pi(\alpha(\psi)) > 0$. Hence

$$\begin{aligned} m &= d_m^0(0) = \alpha(\beta)(0) = \alpha\left(\psi + d_{\beta(0)}^0\right)(0) = \left(\alpha(\psi) + \alpha\left(d_{\beta(0)}^0\right)\right)(0) \\ &= \alpha(\psi)(0) + \alpha\left(d_{\beta(0)}^0\right)(0) = 0 + \alpha\left(d_{\beta(0)}^0\right)(0) = \alpha\left(d_{\beta(0)}^0\right)(0) \\ &= f\alpha\left(d_{\beta(0)}^0\right) = h(\beta(0)). \end{aligned}$$

Hence h is a surjective R-homomorphism which must be an isomorphism, since M_R is a Hopfian right R-module. To prove that α is injective, let $\varphi \in B_A$ be such that $\alpha(\varphi) = 0$.

It follows that $h(\varphi(0)) = \alpha(\varphi)(0) = 0$. Then $\varphi(0) = 0$, since h is an R-isomorphism. Suppose that $u \in S$ and for any $v \in S$ with $v < u, \varphi(v) = 0$. We show that $\varphi(u) = 0$. Define $\psi \in B_A$ via

$$\psi(s) = \begin{cases} \varphi(u), & \text{if } s = u, \\ 0, & \text{if } s \neq u. \end{cases}$$

Thus $(\varphi - \psi)(s) = 0$, for any $s \leq u$, and it follows that $\pi(\varphi - \psi) > u$. Using Lemma 2, we have $\pi(\alpha(\varphi - \psi)) > u$. Hence

$$\begin{aligned} \alpha(\psi)(u) &= \alpha(\psi)(u) + 0 = \alpha(\psi)(u) + \alpha(\varphi - \psi)(u) \\ &= \alpha(\psi + \varphi - \psi)(u) = \alpha(\varphi)(u) = 0. \end{aligned}$$

Consider the following computation, for any $s \in S$

$$\begin{aligned} \left(d^0_{\varphi(u)} e_u\right)(s) &= \sum_{(x,y) \in X_s\left(d^0_{\varphi(u)}, e_u\right)} d^0_{\varphi(u)}(x)\, \omega_x(e_u(y)) \\ &= d^0_{\varphi(u)}(0)\, \omega_0(e_u(s)) \\ &= \varphi(u)\, e_u(s) \\ &= \begin{cases} \varphi(u), & \text{if } s = u, \\ 0, & \text{if } s \neq u. \end{cases} \\ &= \psi(s). \end{aligned}$$

It follows that $\psi = d^0_{\varphi(u)} e_u$. Thus

$$\begin{aligned} h(\varphi(u)) &= f\alpha\left(d^0_{\varphi(u)}\right) = \alpha\left(d^0_{\varphi(u)}\right)(0) = \left(\alpha\left(d^0_{\varphi(u)}\right) e_u\right)(u) \\ &= \alpha\left(d^0_{\varphi(u)} e_u\right)(u) = \alpha(\psi)(u) = 0, \end{aligned}$$

which implies that $\varphi(u) = 0$, since h is an R-isomorphism. Hence $\varphi(s) = 0$ for any $s \in S$, and so $\varphi = 0$. Therefore α is an A-isomorphism and B_A is a Hopfian right A-module.

Conversely, suppose that B_A is a Hopfian right A-module. Let $h : M_R \longrightarrow M_R$ be any surjective R-homomorphism. We want to prove that h is injective.

Define $\alpha : B_A \longrightarrow B_A$ via $\alpha(\varphi)(s) = h(\varphi(s))$ for any $\varphi \in B_A$ and $s \in S$. We show that α is an A-isomorphism.

(1) α is an A-homomorphism: For any $\varphi \in B_A$, $f \in A$ and $s \in S$, we set

$$\begin{aligned} &X_1 = \{(u, v) \in X_s(\varphi, f)\, |\, h(\varphi(u)) = \alpha(\varphi)(u) = 0\} \quad \text{and} \\ &X_2 = \{(u, v) \in X_s(\varphi, f)\, |\, h(\varphi(u)) = \alpha(\varphi)(u) \neq 0\}. \end{aligned}$$

Then clearly $X_2 = X_s(\alpha(\varphi), f)$. Consider the following computation, for any $s \in S$

$$\begin{aligned} \alpha(\varphi f)(s) &= h((\varphi f)(s)) = h\left(\sum_{(u,v) \in X_s(\varphi, f)} \varphi(u)\, \omega_u(f(v))\right) \\ &= \sum_{(u,v) \in X_s(\varphi, f)} h(\varphi(u))\, \omega_u(f(v)) \end{aligned}$$

$$= \sum_{(u,v)\in X_1} h(\varphi(u))\,\omega_u(f(v)) + \sum_{(u,v)\in X_2} h(\varphi(u))\,\omega_u(f(v))$$
$$= 0 + \sum_{(u,v)\in X_2} h(\varphi(u))\,\omega_u(f(v))$$
$$= \sum_{(u,v)\in X_s(\alpha(\varphi),f)} \alpha(\varphi)(u)\,\omega_u(f(v))$$
$$= (\alpha(\varphi)f)(s).$$

Thus $\alpha(\varphi f) = \alpha(\varphi) f$.

(2) α is a surjective map: For any $\psi \in B_A$ and any $s \in \operatorname{supp}(\psi)$, there exists an element $m_s \in M$ such that $h(m_s) = \psi(s)$, since h is a surjective map.

Define $\beta : S \longrightarrow M$ via

$$\beta(s) = \begin{cases} m_s, & \text{if } s \in \operatorname{supp}(\psi), \\ 0, & \text{if } s \notin \operatorname{supp}(\psi). \end{cases}$$

Clearly, $\operatorname{supp}(\beta) \subseteq \operatorname{supp}(\psi)$, which implies that $\operatorname{supp}(\beta)$ is an artinian and narrow subset of S and thus $\beta \in B_A$.

If $s \in \operatorname{supp}(\psi)$, then

$$\alpha(\beta)(s) = h(\beta(s)) = h(m_s) = \psi(s).$$

If $s \notin \operatorname{supp}(\psi)$, then

$$\alpha(\beta)(s) = h(\beta(s)) = h(0) = 0.$$

Thus $\alpha(\beta) = \psi$. Hence α is a surjective A-homomorphism which must be an isomorphism, since B_A is a Hopfian right A-module. To prove that h is injective, let $m \in M$ be such that $h(m) = 0$. Then for any $s \in S$,

$$\alpha\left(d_m^0\right)(s) = h\left(d_m^0(s)\right) = \begin{cases} h\left(d_m^0(0)\right) = h(m) = 0, & \text{if } s = 0, \\ h(0) = 0, & \text{if } s \neq 0. \end{cases}$$

Thus $\alpha\left(d_m^0\right) = 0$ and so $m = 0$, since α is an A-isomorphism. Therefore h is an R-isomorphism and M_R is a Hopfian right R-module. ■

If we set $\omega(s) = \mathrm{Id}_R$, for every $s \in S$, we get the following result as a corollary.

Corollary 4 ([9]). *Suppose that $(S, \leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0 \leq s$ for every $s \in S$. Then M_R is a Hopfian right R-module if and only if $\left[\left[M^{S,\leq}\right]\right]_{[[R^{S,\leq}]]}$ is a Hopfian right $\left[\left[R^{S,\leq}\right]\right]$-module.*

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