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Semipotency and the total of rings and modules

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Abstract. Let M and N be two modules over a ring R. The object of this paper is the study of substructures of $\mathrm{Hom}_R(M,N)$ such as, radical, the singular, and co-singular ideal and the total. New results obtained include necessary and sufficient conditions for the total to equal the radical, $\mathrm{Hom}_R(M,J(N))$, a description of (Δ-, $-, *I*-) semipotency rings and the endomorphism ring of locally projective module. New structure theorems are obtained by studying the relationship between two concepts of the total and (Δ-, $-, *I*-) semi-potentness. In addition, locally injective and locally projective modules are characterized in new ways.

Mathematics subject classification: Primary 16E50; 16E60; 16D70

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1. Introduction

In this paper rings R are associative with identity unless otherwise indicated. Modules over a ring R are unitary right modules. A submodule N of a module M is said to be small in M if $N + K \neq M$ for any proper submodule K of M, [8]. A submodule N of a module M is said to be large (essential) in M if $N \cap K \neq 0$ for any nonzero submodule K of M, [8]. If M is an R-module, the radical of M denoted by $J(M)$, is defined to be the intersection of all maximal submodules of M. Also, $J(M)$ coincides with the sum of all small submodules of M. It my happen that M has no maximal submodules in which case $J(M) = M$, [11]. Thus, for a ring R, $J(R)$ is the Jacobson radical of R. For a submodule N of a module M, we use $N \subseteq^{\oplus} M$ to mean that N is a direct summand

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of M, and we write $N \leqslant_e M$ and N M to indicate that N is a large, respectively small, submodule of M. If M_R is a module, we use the notation $E_M = \text{End}_R(M)$ is the ring of endomorphisms of M and we write $\Delta E_M = \{\alpha{:}\alpha \in E_M;\ \text{Ker}(\alpha) \leqslant_e M\}$, $\$E_M = \{\alpha{:}\alpha \in E_M; \text{Im}(\alpha)$ $M\}$ and $I(E_M) = \{\alpha{:}\alpha \in E_M; \text{Im}(\alpha) \subseteq J(M)\}$. It is well-known that ΔE_M, $\$E_M$ and $I(E_M)$ are ideals in E_M, [8]. It is easy to see that $\$E_M \subseteq I(E_M)$. If M_R and N_R are modules, we use $[M,N] = \text{Hom}_R(M,N)$. Thus $[M,N]$ is an (E_N,E_M)-bi-module. Our main concern is about the substructures of $\text{Hom}_R(M,N)$ and the (Δ-, \$-, I-) semi-potency of $\text{Hom}_R(M,N)$ (see [13]).

The total is a concept that was first introduced by Kasch in 1982 [8], and Zhou [13] in 2009. In the study of the total, one of the interesting questions is when the total equals the Jacobson radical, the singular ideal and the co-singular ideal. In Section 2 it is proved that $\text{Tot}(R) = I$ if and only if, R is an I- semi-potent ring and the ideal I contains no nonzero idempotents. In Section 3 it is proved that a quasi-projective module P is semi-potent if and only if E_P is an I- semi-potent ring. Interesting corollaries are obtained in this section. In particular, $\text{Tot}[M,N] = \{\alpha{:}\alpha \in [M,N]; \beta\alpha \in \text{Tot}(E_M)$ for all $\beta \in [N,M]\}$. In Section 5 it is proved that $[M,N]$ is Δ- semi-potent if and only if $\text{Tot}[M,N] = \Delta[M,N]$. Also, in this section we characterize the modules V and W for which $\text{Tot}[V,N] = \Delta[V,N]$ and $\text{Tot}[M,W] = \Delta[M,W]$ for all $N,M \in \text{mod} - R$. The main result states that E_V is Δ- semi-potent if and only if $\text{Tot}[V,N] = \Delta[V,N]$ for all $N \in \text{mod} - R$. Also, in this section it is proved that $[M,N]$ is \$- semi-potent if and only if $\text{Tot}[M,N] = \$[M,N]$. Also, in this section, we characterize the modules V and W for which $\text{Tot}[V,N] = \$[V,N]$ and $\text{Tot}[M,W] = \$[M,W]$ for all $M,N \in \text{mod} - R$. The main result states that E_V is \$- semi-potent if and only if $\text{Tot}[V,N] = \$[V,N]$ for all $N \in \text{mod} - R$ if and only if $\text{Tot}[M,V] = \$\ [M,V]$ for all $M \in \text{mod} - R$. In Section 6 it is proved that, a module Q_R is a locally injective if and only if $\text{Tot}[N,Q] = \Delta[N,Q]$ for all $N \in \text{mod} - R$. Also, a module P_R is locally projective if and only if $\text{Tot}[P,M] = \$\ [P,M]$ for all $M \in \text{mod} - R$. Interesting corollaries are obtained in this section.

2. (I-) Semipotent rings

Recall that a ring R is a semi-potent ring, also called I_0-ring by Nicholson [4], Hamza [3], if every principal right ideal not contained in $J(R)$ contains a nonzero idempotent. Examples of such rings include: (a) Exchange ring (see [6, Proposition 1.9], a ring R is an exchange ring, if for every $a \in R$, there exists an idempotent $e \in aR$ such that $a - e \in (a^2 - a)R$). (b) Endomorphism rings of injective modules (see [4, Proposition 1.4]). (c) Endomorphism ring of regular modules in the sense Zelmanowitz [14], (see [3, Corollary 4]). Let N and L are submodules of a module M_R. N is called a supplement of L in M if $N + L = M$ and $N \cap L$ is small in N. N is said to be a supplement submodule of M if N is a supplement of some submodule of M.

Theorem 2.1. *For any ring R the following conditions are equivalent:*

(1) *R is a semi-potent ring.*
(2) *For any $a \in R$ there exists $0 \neq x \in R$ such that R/axR has a projective cover (as a right R-module).*

(3) *For any* $a \in R$ *there exists* $0 \neq x \in R$ *such that* axR *has a supplement in* R_R *(as a right* R*-module) which also has a supplement.*

Proof. (1) ⇒ (2). Let $a \in R$, if $a \in J(R)$ then for any $x \in R$ the natural epimorphism $R \to R/axR$ is a projective cover of R/axR. Suppose that $a \notin J(R)$ then there is $e = ax$, where $e \neq 0$ is an idempotent in R and $axR = eR$. Since $(1 - e)R \cong R/axR$ we have R/axR has a projective cover. (2) ⇒ (3) follows by [2, Proposition 1.4]. (3) ⇒ (1). Let $a \in R$, $a \notin J(R)$. Then there exists $y \in R$ such that ayR has a supplement L which has also a supplement. By [2, Proposition 1.4], ayR has a supplement K which is a direct summand of R. Thus $R = ayR + K$ and by [2, Proposition 1.2] there exists a direct summand eR of R, $eR \subseteq ayR \subseteq aR$, where e is a non-zero idempotent of R. Thus R is a semi-potent ring. □

If T is a left ideal or right ideal of R, we say that idempotents lift modulo T if, whenever $a^2 - a \in T$, $a \in R$, there exists $e^2 = e \in R$ such that $e - a \in T$. Nicholson in [7] gave an example of a commutative semi-potent ring where idempotents do not lift modulo $J(R)$ (see [7, Example 25]). Therefore, we extend this notion as follows:

Lemma 2.2. *Let* T *be an ideal of* R *and* $a \in R$, $a \notin T$. *The following equivalent:*

(1) *If* $a^2 - a \in T$ *there exists* $e^2 = e \in aR$, $e \notin T$.
(2) *If* $a^2 - a \in T$ *there exists* $e^2 = e \in Ra$, $e \notin T$.

Proof. Suppose (1) holds. Then $e^2 = e = ax$ for some $x \in R$ and $e \notin T$. We put $y = xax$ then $f = ya$ is an idempotent of R and $f \in Ra$ and $f \notin T$. (2) ⇒ (1) is analogous. □

We say that an ideal T of R is weakly lifting, or that idempotents lift weakly modulo T, if for any $a \in R$, $a^2 - a \in T$, $a \notin T$, there exists an idempotent $e = ax \in aR$ such that $e \notin T$.

Proposition 2.3. *For any ring* R *the following conditions are equivalent:*

(1) R *is a semi-potent ring.*
(2) $\overline{R} = R/J(R)$ *is semi-potent and* $J(R)$ *is weakly lifting.*

Proof. (1) ⇒ (2). Suppose R is semi-potent. Obviously $\overline{R}$ is semi-potent. Let $a^2 - a \in J(R)$ such that $a \notin J(R)$. Then there exists a non-zero idempotent $e = ax \in aR$. Clearly $e \notin J(R)$. Hence $J(R)$ is weakly lifting. (2) ⇒ (1). Let $a \in R$ such that $a \notin J(R)$. As $\overline{R}$ is semi-potent, there exists a non-zero idempotent $\overline{f} \in \overline{a}\overline{R}$. Now $f = ar + x$ for some $r \in R$ and $x \in J(R)$. As $f^2 - f \in J(R)$, there exists a non-zero idempotent $e = fy = ary + xy \in fR$. As $xy \in J(R)$, there exists $b \in R$ such that $(1 - xy)b = 1 = b(1 - xy)$. So, $xyb = b - 1$. We can take y such that $ye = y$. Now $eb = aryb + xyb = aryb + b - 1$, $ebe = arybe + be - e$, $e = arybe + (1 - e)be$. Unless $(1 - e)be = 0$, we cannot conclude that $e = arybe$. However by multiplication by e on the left, we conclude that $e = earybe$. Let $g = arybe$. Then $g^2 = arybearybe = arybe = g$ and $g \in aR$. Hence R is semi-potent. □

The semi-potent rings generalize as follows:

Lemma 2.4 [7, Lemma 19]. *The following conditions are equivalent for an ideal I of a ring R:*

(1) *If $T \subsetneq I$ is a right (resp. left) ideal there exists $e^2 = e \in T \backslash I$.*
(2) *If $a \notin I$ there exists $e^2 = e \in aR \backslash I$ (resp. $e \in Ra \backslash I$).*
(3) *If $a \notin I$ there exists $x \in R$ such that $x = xax \notin I$.* □

Let R be a ring and I is an ideal of R, recall R is I- semi-potent [7], if the conditions in Lemma 2.4, are satisfied.

Corollary 2.5. *Let I be an ideal of a ring R. If R is I- semi-potent then $J(R) \subseteq I$.*

Proof. Suppose $J(R) \subsetneq I$ there exists $a \in J(R)$, $a \notin I$, so $x = xax \notin I$ for some $x \in R$. Since $x \neq 0$ then $0 \neq (ax)^2 = ax \in J(R)$ this is a contradiction. □

Proposition 2.6. *Let I be an ideal of a ring R. The following are equivalent:*

(1) *R is an I- semi-potent ring.*
(2) *R/I is a semi-potent ring with $J(R/I) = \bar{0}$ and I is weakly lifting.*

Proof. Suppose (1) holds. First we prove that $J(R/I) = \bar{0}$. Assume $J(R/I) \neq \bar{0}$ then there exists $\bar{0} \neq \bar{a} \in J(R/I)$. So $a \in R$, $a \notin I$ therefore $x = xax \notin I$ for some $x \in R$. Thus, $\bar{0} \neq (\bar{a}\bar{x})^2 = \bar{a}\bar{x} \in J(R/I)$, a contradiction, so $J(R/I) = \bar{0}$. It is clear that R/I is semi-potent. Finally, we prove that I is a weakly lifting. Let $a^2 - a \in I$ and $a \in R \backslash I$. Since R is I- semi-potent there exists $y \in R$, $y = yay \notin I$, so $0 \neq (ay)^2 = ay \in aR$, and $ay \notin I$. (2) $\Rightarrow$ (1). Let $a \in R \backslash I$ then $\bar{0} \neq \bar{a} \in R/I$. Since R/I is semi-potent and $J(R/I) = \bar{0}$ then $\bar{x} = \bar{x}\bar{a}\bar{x}$ for some $\bar{0} \neq \bar{x} \in R/I$. Since $(ax)^2 - ax \in I$ and I is a weakly lifting there exists $0 \neq e^2 = e \in axR \subseteq aR$ and $e \notin I$, so R is I- semi-potent. □

Following [13], the total of a ring R is

$$\mathrm{Tot}(R) = \{a : a \in R;\ aR \text{ contains no idempotents}\}$$
$$\mathrm{Tot}(R) = \{a : a \in R;\ Ra \text{ contains no idempotents}\}$$

Y.Zhou, proved that, for a ring R; $\mathrm{Tot}(R) = J(R)$ if and only if R is a semi-potent, [13, Theorem 2.2]. For an I- semi-potent ring we have:

Theorem 2.7. *Let I be an ideal of a ring R. The following are equivalent:*

(1) *$Tot(R) = I$.*

(2) *R is an I- semi-potent ring and I contains no nonzero idempotents.*
(3) *R/I is a semi-potent and $J(R/I) = \bar{0}$ with I contains no nonzero idempotents and I is weakly lifting.*

Proof. (1) ⇒ (2). It is clear that I contains no nonzero idempotents. Let $a \in R\backslash I$. Then aR contains a nonzero idempotent. This shows that R is an I- semi-potent ring. (2) ⇒ (1). Suppose that $\mathrm{Tot}(R) \neq I$, Since $I \subseteq \mathrm{Tot}(R)$ there exists $a \in \mathrm{Tot}(R)$ such that $a \notin I$. So, for some $x \in R$, $x = xax \notin I$ and $0 \neq (ax)^2 = ax \in aR$, a contradiction. (2) $\iff$ (3). By Proposition 2.6. □

3. Semipotent modules

Let M_R be a module and $K \subseteq^{\oplus} M$. Then $K \subseteq J(M)$ if and only if $K = J(K)$. Put $\Gamma(M) = \{K{:}K \subseteq^{\oplus} M;\ K \subseteq J(K)\}$. Note that for any projective module P, $\Gamma(P) = \{0\}$. In addition to, if $J(M)$ M (or M finitely generated) for some $M \in \mathrm{mod} - R$ then $\Gamma(M) = \{0\}$. Let M_R be a module, letting $I = I(E_M) = \{\alpha{:}\alpha \in E_M;\mathrm{Im}(\alpha) \subseteq J(M)\}$ It is clear that $I = I(E_M)$ is an ideal in E_M.

Recall a module M_R is a semi-potent module also, called I_0- module [3] and weakly regular module [1] if each submodule of M not contained in $J(M)$ contains a direct summand N of M, $N \notin \Gamma(M) = \{T \subseteq^{\oplus} M{:}J(T) = T\}$.

Lemma 3.1. *Let M be a semi-potent module. The following holds:*

(1) *Every submodule N of M such that $J(N) = N \cap J(M)$ is semi-potent.*
(2) *Every direct summand of M is semi-potent.*
(3) *Every supplement submodule of M is semi-potent.*

Proof. (1). It is clear. □

A module P_R is called a quasi-projective module [12] if given an epimorphism $\beta \in [P,M]$ and any morphism $\alpha \in [P,M]$ there exists $\lambda \in E_P$ such that $\beta\lambda = \alpha$. For a quasi-projective module we have the following:

Theorem 3.2. *For any quasi-projective module P the following are equivalent:*

(1) *P is a semi-potent module.*
(2) *or any $\alpha \in E_P$, $\alpha \notin I(E_P)$, there exists a direct summand N of P contained in Im(α) such that $N \notin \Gamma(P)$.*
(3) *E_P is an $I = I(E_P)$- semi-potent ring.*

Proof. (1) ⇒ (2). It is clear. (2) ⇒ (3). Let $\alpha \in E_P\backslash I(E_P)$, $\mathrm{Im}(\alpha) \subsetneq J(P)$ and there exists $N \subseteq^{\oplus} P$, $N \subseteq \mathrm{Im}(\alpha)$ and $N \notin \Gamma(P)$. Let γ be the projection of P on to N. Then $\mathrm{Im}(\gamma\alpha) = N$, so there exists $\beta \in E_P$ such that $\gamma\alpha\beta = \gamma$. We put $\mu = \beta\gamma$, then

$\mu\alpha\mu = \mu \notin I(E_P)$. Because if $\mu \in I(E_P)$ then $\gamma = \gamma\gamma = \gamma\alpha\beta\gamma \in I(E_P)$ that is $N \in \Gamma(P)$ a contradiction. So, E_P is I- semi-potent. (3) $\Rightarrow$ (1). Let E_P be I- semi-potent, where $I = I(E_P)$. Let K be a submodule of P, $K \subsetneq J(P)$. Then there exists a maximal submodule D of P such that $K \subsetneq D$. Thus $K + D = P$. By [10, Lemma 1.1] there are $f,g \in E_P$ such that $1 = f + g$ and $\mathrm{Im}(f) \subseteq A$, $\mathrm{Im}(g) \subseteq D$. It is clear that $f \notin I(E_P)$. By assumption there exists $\varphi \in E_P$ such that $\varphi = \varphi f \varphi \notin I(E_P)$. Since $(f\varphi)^2 = f\varphi$ then $\mathrm{Im}(f\varphi) \subseteq^{\oplus} P$, $\mathrm{Im}(f\varphi) \subseteq A$ and $\mathrm{Im}(f\varphi) \notin \Gamma(P)$. So P is semi-potent. □

Corollary 3.3. *For any quasi-projective module P the following are equivalent:*

(1) *P is a semi-potent module and $\Gamma(P) = \{0\}$.*
(2) *E_P is an I- semi-potent ring and $\Gamma(P) = \{0\}$.*
(3) *$Tot(E_P) = I(E_P)$.*

Proof. (1) $\Longleftrightarrow$ (2). By Theorem 3.2. (2) $\Longleftrightarrow$ (3). By Theorem 2.7 because $\Gamma(P) = \{0\}$ if and only if $I(E_P)$ contain no nonzero idempotents. □

Corollary 3.4. *Let P be a quasi-projective module with $J(P)$ P. Then following are equivalent:*

(1) *P is a semi-potent module.*
(2) *For any $\alpha \in E_P$, $\alpha \notin J(E_P)$, there exists $0 \neq N \subseteq^{\oplus} P$, $N \subseteq \mathrm{Im}(\alpha)$.*
(3) *E_P is a semi-potent ring.*
(4) *$Tot(E_P) = J(E_P) = \$E_P = I(E_P)$.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) As in Theorem 3.2, because for a quasi-projective module with $J(P)$ P, $J(E_P) = \$E_P = I(E_P)$ by [11, Lemma 2]. (3) $\Longleftrightarrow$ (4) By [13, Theorem 2.2]. □

A module P_R is called a direct-projective module [5], if given any direct summand N of P with projection $\pi{:}P \to N$ and any epimorphism $\alpha{:}P \to N$ there exists $\beta \in E_P$ such that $\alpha\beta = \pi$. If P is a direct-projective module then $\$E_P \subseteq J(E_P)$, (see [5, Theorem 3.1]). For a direct projective modules we have the following:

Proposition 3.5. *Let P_R be a direct-projective module. If P is semi-potent then:*

(1) *E_P is an I- semi-potent ring.*
(2) *$J(E_P) \subseteq I(E_P)$.*

Proof. (1). Let $\alpha \in E_P$, $\alpha \notin I(E_P)$ there exists $N \subseteq^{\oplus} P$, $N \notin \Gamma(P)$ and $N \subseteq \mathrm{Im}(\alpha)$. Let γ be the projection of P on to N. Then $N = \mathrm{Im}(\gamma) = Im(\gamma\alpha)$. Since P is a direct-projective there exists $\beta \in E_P$ such that $\gamma\alpha\beta = \gamma$. Putting $\mu = \beta\gamma$ then $0 \neq \mu \in E_P$, $\mu\alpha\mu = \mu$ and $\mu \notin I(E_P)$, because, if $\mu \in I(E_P)$, so $\gamma = \gamma\alpha\beta\gamma \in I(E_P)$ thus $N = \mathrm{Im}(\gamma) \subseteq J(P)$ this means that $N \in \Gamma(P)$, a contradiction. This shows that E_P is I- semi-potent. (2). By Corollary 2.5. □

Corollary 3.6. *Let P_R be a direct-projective module. If P is semi-potent and $J(P)\ \ P$ then E_P is a semi-potent ring.*

Proof. We have by [5, Theorem 3.1], $\$E_P \subseteq J(E_P)$ and by Proposition 3.5, $J(E_P) \subseteq I(E_P)$. Since $J(P)\ \ P$ then $I(E_P) = \$E_P$ thus $J(E_P) = \$E_P = I(E_P)$, so E_P is a semi-potent ring. □

A module P_R is called π- projective [10] if, for any two submodules U,V of P with $P = U + V$; $E_P = [P,U] + [P,V]$. For a π- projective modules we have the following:

Proposition 3.7. *Let P_R be a π-projective module. The following are equivalent:*

(1) *P is a semi-potent module.*
(2) *For any $\alpha \in E_P$, $\alpha \notin I(E_P)$ there exists $N \subseteq^{\oplus} P$, $N \notin \Gamma(P)$* and $N \subseteq \mathrm{Im}(\alpha)$.

Proof. (1) $\Rightarrow$ (2). It is clear. (2) $\Rightarrow$ (1). Let A be a submodule of P, $A \subsetneq J(P)$. Then there exists a maximal submodule M of P, $A \subsetneq M$ therefore $P = A + M$. Since P is a π- projective there are $\alpha,\beta \in E_P$ such that $1 = \alpha + \beta$ and $\mathrm{Im}(\alpha) \subseteq A$, $\mathrm{Im}(\beta) \subseteq M$. It is clear that $\mathrm{Im}(\alpha) \subsetneq J(P)$, because if $\mathrm{Im}(\alpha) \subseteq J(P)$ we have $P = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta) \subseteq J(P) + M \subseteq M \subseteq P$ thus $P = M$, a contradiction. By assumption $\mathrm{Im}(\alpha) \subseteq A$ contains a direct summand N of P, $N \notin \Gamma(P)$. So P is a semi-potent module. □

Corollary 3.8. *Let P_R be a π-projective module. If E_P is an I- semi-potent ring the following hold:*

(1) *For any $\alpha \in E_P$, $\alpha \notin I(E_P)$ there exists $N \subseteq^{\oplus} P$, $N \notin \Gamma(P)$* and $N \subseteq \mathrm{Im}(\alpha)$.
(2) *P is a semi-potent module.*

Proof. (1). Let $\alpha \in E_P \backslash I(E_P)$, so there exists $\gamma \in E_P \backslash I(E_P)$ such that $\gamma = \gamma\alpha\,\gamma$. Since $0 \neq (\alpha\gamma)^2 = \alpha\gamma \in E_P$ then $\mathrm{Im}(\alpha\gamma) \subseteq^{\oplus} P$, $\mathrm{Im}(\alpha\,\gamma) \notin \Gamma(P)$ and $\mathrm{Im}(\alpha\gamma) \subseteq \mathrm{Im}(\alpha)$. (2). By (1) and Proposition 3.7. □

Proposition 3.9. *For any projective module P_R the following are equivalent:*

(1) *P is a semi-potent module and $J(P)\ \ P$.*
(2) *E_P is a semi-potent ring.*
(3) *For any $\alpha \in E_P$, $P/\mathrm{Im}(\alpha\beta)$ has a projective cover for some $0 \neq \beta \in E_P$.*
(4) *For any $\alpha \in E_P$, $\mathrm{Im}(\alpha\beta)$ has a supplement which also has a supplement for some $0 \neq \beta \in E_P$.*

Proof. (1) $\iff$ (2). By [3, Theorem 3.5]. (2) $\Rightarrow$ (3). Suppose that E_P is a semi-potent ring, by Theorem 2.1, for any $\alpha \in E_P$ there exists $0 \neq \beta \in E_P$ such that $E_P/(\alpha\beta)E_P$ has a projective cover, by [2, Proposition 2.9] $P/\mathrm{Im}(\alpha\beta)$ has a projective cover. (3) $\Rightarrow$ (2) follows immediately from [2, Proposition 2.9] and Theorem 2.1. (3) $\iff$ (4). By [2, Proposition 1.4]. □

Lemma 3.10. *Let M_R be a module with E_M is a semi-potent ring. Then:*

(1) $\$E_M \subseteq J(E_M)$ *and* $\Delta\ E_M \subseteq J(E_M)$.
(2)] *If* $\Gamma(M) = \{0\}$ *then* $I(E_M) \subseteq J(E_M)$.
(3) *If* $J(M)\ \ M$ *then* $\$E_M = I(E_M) \subseteq\ J(E_M)$.

Proof.

(1) Let $\alpha \in \$E_M$, so $\mathrm{Im}(\alpha)\ \ M$. Suppose that $\alpha \notin J(E_M)$ then $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in E_M$. Let $\gamma = \alpha\beta$. Then $0 \neq \gamma^2 = \gamma \in E_M$ and $\mathrm{Im}(\gamma) \subseteq \mathrm{Im}(\alpha)\ M$. Hence $\mathrm{Im}(\gamma)\ M$ and $\mathrm{Im}(\gamma) = \mathrm{Im}(\gamma) \oplus \mathrm{Ker}(\gamma)$. Thus $\mathrm{Ker}(\gamma) = M$, and $\gamma = 0$ which is a contradiction, hence $\alpha \in J(E_M)$ and $\$\ E_M \subseteq J(E_M)$.If $g \in \Delta E_M$ then $\mathrm{Ker}(g)$ is large in M. Suppose that $g \notin J(E_M)$ then $\mu = \mu g\mu$ for some $0 \neq \mu \in E_M$. Let $t = \mu g$, so $0 \neq t^2 = t\ \in E_M$ and $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(t)$, therefore $\mathrm{Ker}(g) \cap \mathrm{Im}(t) = 0$ thus $\mathrm{Im}(t) = 0$, hence $\mathrm{Ker}(g)$ is large in M and $t = 0$ this is a contradiction, hence $g \in J(E_M)$ and $\Delta E_M \subseteq J(E_M)$.
(2) Suppose that $\Gamma(M) = \{0\}$. If $\alpha \in I(E_M)$ then $\mathrm{Im}(\alpha) \subseteq J(M)$. Suppose that $\alpha \notin J(E_M)$ then $\gamma = \gamma\alpha\gamma$ for some $0 \neq \gamma\ \in E_M$. We put $g = \alpha\gamma$ then $0 \neq g^2 = g \in E_M$, $\mathrm{Im}(g) \subseteq^{\oplus} M$ and $\mathrm{Im}(g) \subseteq J(M)$, therefore $\mathrm{Im}(g) \in \Gamma(M) = \{0\}$, so $\mathrm{Im}(g) = 0$ and $g = 0$ this is a contradiction. Thus $\alpha \in J(E_M)$. (3). It is clear. □

It is known that for any module M_R, $\$E_M \subseteq I(E_M)$. So, if E_M is an I- semi-potent ring we have the following:

Lemma 3.11. *Let M_R be a module and assume that E_M is an I- semi-potent ring. Then:* $J(E_M) \subseteq I(E_M)$ and $\Delta E_M \subseteq I(E_M)$.

Proof. By Corollary 2.5 we have $J(E_M) \subseteq I(E_M)$. If $\alpha \in \Delta E_M$, then $\mathrm{Ker}(\alpha) \leqslant_e M$. Suppose $\alpha\ \notin I(E_M)$ then $\gamma = \gamma\alpha\gamma$ for some $\gamma \in E_M$, $\gamma \notin I(E_M)$. If $t = \gamma\alpha$, then $0 \neq t^2 = t \in E_M$ and $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(t)$, therefore $\mathrm{Ker}(\alpha) \cap \mathrm{Im}(t) = 0$ thus $\mathrm{Im}(t) = 0$, so $t = 0$ this is a contradiction, hence $\alpha \in\ I(E_M)$. □

4. Semipotent [M,N]

Following [13], let M_R, N_R be two modules. Then $[M,N] = \mathrm{Hom}_R(M,N)$ is an (E_N,E_M)-bi-module. The following are defined:

- Radical:

$$J[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in J(E_M)\ \text{ for all } \beta \in [N,M]\}$$
$$J[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in J(E_N)\ \text{for all } \beta \in [N,M]\}$$

Thus $J[M,M] = J(E_M)$. In particular $J[R,R] = J(R)$.

- Singular ideal:

$$\Delta[M,N] = \{\alpha : \alpha \in [M,N];\ \mathrm{Ker}(\alpha) \leqslant_e M\}$$

- Co-singular ideal:

$$\nabla[M,N] = \{\alpha : \alpha \in [M,N];\ \mathrm{Im}(\alpha) \ll N\}$$

- Total:

$$\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N];\ [N,M]\alpha \ \text{ contains no nonzero idempotents}\}$$
$$\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha[N,M] \ \text{ contains no nonzero idempotents}\}$$

Lemma 4.1. *Let M_R, N_R be modules then:*

(1) *$Tot[M,N] = \{\alpha{:}\alpha \in [M,N];\ \beta\alpha \in Tot(E_M)$ for all $\beta \in [N,M]\}$.*
(2) *$Tot[M,N]$* $= \{\alpha{:}\alpha \in [M,N];\ \alpha\beta \in \mathrm{Tot}(E_N)$ for all $\beta \in [N,M]\}$.

Proof. (1) Let $\alpha \in \mathrm{Tot}[M,N]$. If $\beta\alpha \notin \mathrm{Tot}(E_M)$ for some $\beta \in [N,M]$ there exists $\gamma \in E_M$ such that $0 \neq \gamma(\beta\alpha) = [\gamma(\beta\alpha)]^2 \in E_M$. Since $\gamma\beta \in [N,M]$ then $0 \neq (\gamma\beta)\alpha = [(\gamma\varpi \beta)\alpha]^2 \in [N,M]\alpha$, a contradiction. Let $\alpha \in [M,N]$ such that $\beta\alpha \in \mathrm{Tot}(E_M)$ for all $\beta \in [N,M]$. If $\alpha \notin \mathrm{Tot}[M,N]$, then $[N,M]\alpha$ contains a nonzero idempotent. So there exists $\gamma \in [N,M]$ such that $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ and $\gamma\alpha \in (\gamma\alpha)E_M$, so $\gamma\alpha \notin \mathrm{Tot}(E_M)$, a contradiction. Similarly (2) holds. □

Lemma 4.2. [13, Lemma 2.1]*Let M_R, N_R be modules. The following are equivalent:*

(1) *If $\alpha \in [M,N]\backslash J[M,N]$, there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$.*
(2) *If $\alpha \in [M,N]\backslash J[M,N]$, there exists $\beta \in [N,M]$*such that $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.
(3) *If $\alpha \in [M,N]\backslash J[M,N]$, there exists $\gamma \in [N,M]$*such that $\gamma\alpha\gamma = \gamma \notin J[N,M]$. □

Following [13], we say that $[M,N]$ is semi-potent if the conditions in Lemma 4.2, are satisfied.

Lemma 4.3. *Let M_R, N_R be modules and $[M,N]$ is semi-potent. Then:*

(1) $\Delta[M,N] \subseteq J[M,N]$ *and* $\$[M,N] \subseteq J[M,N]$.
(2) *If $\Gamma(M) = \{0\}$ then $I[M,N] \subseteq J[M,N]$.*
(3) *If $\Gamma(N) = \{0\}$ then $I[M,N] \subseteq J[M,N]$.*

Proof. Suppose that $[M,N]$ is semi-potent.

(1) Let $\alpha \in \Delta[M,N]$, so $\mathrm{Ker}(\alpha) \leqslant_e M$. Suppose that $\alpha \notin J[M,N]$ then there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. Since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta\alpha)$ then $\mathrm{Ker}(\alpha) \cap \mathrm{Im}(\beta\alpha) \subseteq \mathrm{Ker}(\beta\alpha) \cap \mathrm{Im}(\beta\alpha) = 0$. Thus, $\mathrm{Im}(\beta\alpha) = 0$ and $\beta\alpha = 0$ this is a contradiction. Hence $\alpha \in J[M,N]$. Let $\alpha \in \$[M,N]$ then $\mathrm{Im}(\alpha)\ N$. Suppose that $\alpha \notin J[M,N]$ then there exists $\beta \in [N,M]$ such that $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$. Since $\mathrm{Im}(\alpha\beta) \subseteq \mathrm{Im}(\alpha)$ then $\mathrm{Im}(\alpha\beta)\ N$ and $N = \mathrm{Ker}(\alpha\beta)$. So, $\mathrm{Ker}(\alpha\beta) \cap \mathrm{Im}(\alpha\beta) = \mathrm{Im}(\alpha\beta) = 0$. Thus, $\beta\alpha = 0$ this is a contradiction. Hence $\alpha \in J[M,N]$. (2). Suppose that $\Gamma(M) = \{0\}$. Let $\alpha \in I[M,N]$ then $\mathrm{Im}(\alpha) \subseteq J(N)$. Assume that $\alpha \notin J[M,N]$ then there exists $\beta \in [N,M]$ such that

$0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. So $\text{Im}(\beta\alpha) \subseteq^{\oplus} M$ and $\text{Im}(\beta\alpha) \subseteq J(M)$ thus $\text{Im}(\beta\alpha) \in \Gamma(M) = \{0\}$, so $\beta\alpha = 0$ a contradiction. Thus $\alpha \in J[M,N]$. Similarly, (3) holds. □

Proposition 4.4. *Let M_R, N_R be modules, the following hold:*

(1) *If $Tot(E_M) = J(E_M)$ then $Tot[M,N] = J[M,N]$.*
(2) *If $Tot(E_N) = J(E_N)$ then $Tot[M,N] = J[M,N]$.*
(3) *If E_M is a semi-potent ring then $[M,N]$ is semi-potent.*
(4) *If E_N is a semi-potent ring then $[M,N]$ is semi-potent.*

Proof. (1) Suppose that $\text{Tot}(E_M) = J(E_M)$. It is clear that $J[M,N] \subseteq \text{Tot}[M,N]$. Let $\alpha \in \text{Tot}[M,N]$ then by Lemma 4.1 for any $\beta \in [N,M]$; $\beta\alpha \in \text{Tot}(E_M) = J(E_M)$ so $\alpha \in J[M,N]$. The proof of (2) is analogous. (3) Suppose that E_M is a semi-potent ring then by [13, Theorem 2.2] $\text{Tot}(E_M) = J(E_M)$ and by (1) $\text{Tot}[M,N] = J[M,N]$, again by [13, Theorem 2.2], $[M,N]$ is semi-potent. The proof of (4) is analogous. □

Remark. Zhou [13], gave an example of two modules M_R, N_R such that $[M,N]$ is semi-potent, but neither E_M, nor E_N is semi-potent, (see [13, Example 4.9]). So in general, if $\text{Tot}[M,N] = J[M,N]$ then $\text{Tot}(E_M) \neq J(E_M)$ and $\text{Tot}(E_N) \neq J(E_N)$. Hence it is possible that $\text{Tot}[M,N] = J[M,N]$ while $\text{Tot}(E_M) \neq J(E_M)$ and $\text{Tot}(E_N) \neq J(E_N)$.

Following [13], let

$$\Phi(R) = \{M \in \text{mod} - R : \ \text{Tot}[M,N] = J[M,N] \ \forall \ N \in \text{mod} - R\}$$
$$\Gamma(R) = \{N \in \text{mod} - R : \ \text{Tot}[M,N] = J[M,N] \ \forall \ M \in \text{mod} - R\}$$

Corollary 4.5. [13, Theorem 4.5]. *The following holds:*

(1) $\Phi(R) = \{M \in \text{mod} - R$: E_M is a semi-potent ring$\}$.
(2) $\Gamma(R) = \{N \in \text{mod} - R$: E_N is a semi-potent ring$\}$.
(3) $\Phi(R) = \Gamma(R)$.

Proof. (1) ($\Rightarrow$). If $M \in \Phi(R)$, then $\text{Tot}(E_M) = J(E_M)$, so E_M is semi-potent by [13, Theorem 2.2]. ($\Leftarrow$). Let $M \in \text{mod} - R$ with E_M is semi-potent, then for any $N \in \text{mod} - R$; $[M,N]$ is semi-potent by Proposition 4.4, so $M \in \Phi(R)$. Similarly (2) holds. (3) By (1) and (2). □

5. (D-, \$-, *I*-) Semipotent [*M*,*N*]

Proposition 5.1. *Let M_R, N_R be modules.*

(a) *The following hold:*
  (1) $\Delta[M,N] \subseteq \{\alpha{:}\alpha \in [M,N];\ \beta\alpha \in \Delta E_M$ for all $\beta \in [N,M]\}$.
  (2) $\Delta[M,N] \subseteq \{\alpha{:}\alpha \in [M,N];\ \alpha\beta \in \Delta E_N$ for all $\beta \in [N,M]\}$.

(b) *If* $Tot[M,N] = \Delta[M,N]$ *then*
  (1) $\Delta[M,N] = \{\alpha:\alpha \in [M,N];\ \beta\alpha \in \Delta E_M$ for all $\beta \in [N,M]\}$.
  (2) $\Delta[M,N] = \{\alpha:\alpha \in [M,N];\ \alpha\beta \in \Delta E_N$ for all $\beta \in [N,M]\}$.

Proof.

(a) (1) Let $\alpha \in \Delta[M,N]$, so $\mathrm{Ker}(\alpha) \leqslant_e M$, and for any $\beta \in [N,M]$, $\mathrm{Ker}(\beta\alpha) \leqslant_e M$ hence $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta\alpha)$, this follows that $\beta\alpha \in \Delta E_M$. (2) If $\alpha \in \Delta[M,N]$, then $\mathrm{Ker}(\alpha) \leqslant_e M$. Let $\beta \in [N,M]$ and K be a submodule of N such that $\mathrm{Ker}(\alpha\beta) \cap K = 0$. Hence $\mathrm{Ker}(\beta) \subseteq \mathrm{Ker}(\alpha\beta)$ then $\mathrm{Ker}(\beta) \cap K = 0$. Let $y \in \mathrm{Ker}(\alpha) \cap \beta(K)$ then $y \in \mathrm{Ker}(\alpha)$, so $\alpha(y) = 0$ and $y \in \beta(K)$ therefore $y = \beta(x)$ for some $x \in K$. So $0 = \alpha(y) = \alpha\beta(x)$ thus $x \in \mathrm{Ker}(\alpha\beta)$, $x \in K$, so $x \in \mathrm{Ker}(\alpha\beta) \cap K = 0$ thus, $x = 0$, so $y = \beta(x) = 0$ thus, $\mathrm{Ker}(\alpha) \cap \beta(K) = 0$. Since $\mathrm{Ker}(\alpha) \leqslant_e M$ follows that $\beta(K) = 0$ so $K \subseteq \mathrm{Ker}(\beta)$ thus $K = \mathrm{Ker}(\beta) \cap K = 0$ so $\mathrm{Ker}(\alpha\beta) \leqslant_e N$, thus $\alpha\beta \in \Delta E_N$.

(b) Suppose that $\mathrm{Tot}[M,N] = \Delta[M,N]$. (1) We have by (a) $\Delta[M,N] \subseteq \{\alpha:\alpha \in [M,N];\ \beta\alpha \in \Delta E_M$ for all $\beta \in [N,M]\}$. Let $\alpha \in [M,N]$ such that $\beta\alpha \in \Delta E_M$ for all $\beta \in [N,M]$. Suppose $\alpha \notin \Delta[M,N]$. Then $\alpha \notin \mathrm{Tot}[M,N]$, so there exists $\gamma \in [N,M]$ such that $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$. Therefore $M = \mathrm{Im}(\gamma\alpha) \oplus \mathrm{Ker}(\gamma\alpha)$. Since $\mathrm{Ker}(\gamma\alpha) \cap \mathrm{Im}(\gamma\alpha) = 0$ and $\mathrm{Ker}(\gamma\alpha) \leqslant_e M$, it follows that $\mathrm{Im}(\gamma\alpha) = 0$, so $\gamma\alpha = 0$, a contradiction. Thus, $\alpha \in \Delta[M,N]$. Similarly (2) holds. □

Lemma 5.2. *Let* M_R, N_R *be modules. The following are equivalent:*

(1) *If* $\alpha \in [M,N]\backslash\Delta[M,N]$, *there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$.
(2) *If* $\alpha \in [M,N]\backslash\Delta[M,N]$, *there exists* $\beta \in [N,M]$ *such that* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.
(3) *If* $\alpha \in [M,N]\backslash\Delta[M,N]$, *there exists* $\gamma \in [N,M]$ *such that* $\gamma = \gamma\alpha\gamma \notin \Delta\ [N,M]$.

Proof. Suppose (1) holds. Then $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ for some $\beta \in [N,M]$. Let $\gamma = \beta\alpha\beta \in [N,M]$ we have $\gamma\alpha\gamma = \gamma \neq 0$ and $\gamma \notin \Delta\ [N,M]$ because $0 \neq \alpha\gamma = (\alpha\gamma)^2$, giving (3). Suppose (3) holds, then $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ for some $\gamma \in [N,M]\backslash\Delta[N,M]$, gives (1). Similarly, the equivalence (2) $\Longleftrightarrow$ (3) holds. □

We say that $[M,N]$ is Δ- semi-potent if the conditions in Lemma 5.2 are satisfied.

Theorem 5.3. *Let* M_R, N_R *be modules.* $[M,N]$ *is* Δ- *semi-potent if and only if,* $Tot[M,N] = \Delta[M,N]$. *In particular,* E_M *is a* Δ*-semi-potent if and only if,* $Tot\,(E_M) = \Delta E_M$.

Proof.

($\Rightarrow$) Suppose that $\mathrm{Tot}[M,N] \neq \Delta\ [M,N]$, Since $\Delta[M,N] \subseteq \mathrm{Tot}[M,N]$, there exists $\alpha \in \mathrm{Tot}[M,N]$ such that $\alpha \notin \Delta[M,N]$. So, for any $\beta \in [N,M]$, either $\alpha\beta = 0$ or $\alpha\beta \neq (\alpha\beta)^2$. Hence $[M,N]$ is not Δ- semi-potent.

($\Leftarrow$) If $\alpha \in [M,N]\backslash\Delta[M,N]$, then $\alpha \notin \mathrm{Tot}[M,N]$. So $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$ for some $\beta \in [N,M]$. This shows that $[M,N]$ is Δ- semi-potent. □

Let

$$\Delta\Phi(R) = \{M \in \mathrm{mod} - R: \ \mathrm{Tot}[M,N] = \Delta[M,N] \ \ \forall \ \ N \in \mathrm{mod} - R\}$$
$$\Delta\Gamma(R) = \{N \in \mathrm{mod} - R: \ \mathrm{Tot}[M,N] = \Delta[M,N] \ \ \forall \ \ M \in \mathrm{mod} - R\}$$

We define the following two sets:

(a) $\Delta S\Phi(R)$ the set of all modules $M \in$ mod $- R$ which have the following two properties:
 (1) E_M is a Δ- semi-potent ring.
 (2) For any $N \in \mathrm{mod} - R$;

$$\Delta[M,N] = \{\alpha : \alpha \in [M,N]; \ \beta\alpha \in \Delta E_M; \ \text{for all } \beta \in [N,M]\}.$$

(b) $\Delta S\Gamma(R)$ the set of all modules $N \in \mathrm{mod} - R$ which satisfy the following two properties:
 (1) E_N is a Δ- semi-potent ring.
 (2) For any $M \in \mathrm{mod} - R$;

$$\Delta[M,N] = \{\alpha : \alpha \in [M,N]; \ \alpha\beta \in \Delta E_N; \ \text{for all } \beta \in [N,M]\}.$$

Theorem 5.4. *The following are holds:*

(1) $\Delta\Phi(R) = \Delta S\Phi(R)$.
(2) $\Delta\Gamma(R) = \Delta S\Gamma(R)$.
(3) $\Delta\Phi(R) = \Delta\Gamma(R)$.

Proof. (1) ($\Rightarrow$). Let $M \in \Delta\Phi(R)$, $\mathrm{Tot}[M,N] = \Delta[M,N]$ for any $N \in \mathrm{mod} - R$; by Proposition 5.1(b) we have $\Delta[M,N] = \{\alpha{:}\alpha \in [M,N];\ \beta\alpha \in \Delta E_M;$ for all $\beta \in [N,M]\}$. It is clear that E_M is a Δ- semi-potent ring, so $M \in \Delta S\ \Phi(R)$.

($\Leftarrow$). Let $M \in \Delta S\Phi(R)$, for any $N \in \mathrm{mod} - R$ we have $\Delta[M,N] \subseteq \mathrm{Tot}[M,N]$. Let $\alpha \in \mathrm{Tot}[M,N]$, by Lemma 4.1, for any $\beta \in [N,M]$; $\beta\alpha \in \mathrm{Tot}(E_M)$. Since E_M is Δ- semi-potent, by Theorem 5.3 $\mathrm{Tot}(E_M) = \Delta E_M$, so $\beta\alpha \in \Delta E_M$ for all $\beta \in [M,N]$ thus, $M \in \Delta\Phi(R)$.

(2) ($\Rightarrow$). Let $N \in \Delta\Gamma(R)$, so for any $M \in \mathrm{mod} - R$; $\mathrm{Tot}[M,N] = \Delta[M,N]$ by proposition 5.1(b) we have $\Delta[M,N] = \{\alpha{:}\alpha \in [M,N];\ \alpha\beta \in \Delta\ E_N;$ for all $\beta \in [N,M]\}$ and E_N is a Δ- semi-potent ring, so $N \in \Delta S\Gamma(R)$.

($\Leftarrow$). Let $N \in \Delta S\Gamma(R)$, so for any $M \in \mathrm{mod} - R$ we have $\Delta[M,N] \subseteq \mathrm{Tot}[M,N]$. Let $\alpha \in \mathrm{Tot}[M,N]$ by Lemma 4.1 for any $\beta \in [N,M]$; $\alpha\beta \in \mathrm{Tot}(E_N)$. Since E_N is a Δ- semi-potent ring by Theorem 5.3, $\mathrm{Tot}(E_N) = \Delta E_N$ so $\alpha\beta \in \Delta E_N$ for all $\beta \in [N,M]$ by assumption $\alpha \in \Delta[M,N]$. Thus, $N \in \Delta\Gamma(R)$.

(3) By (1) and (2). □

Proposition 5.5. *Let M_R, N_R be modules.*

(a) *The following hold:*
  (1) $\$[M,N] \subseteq \{\alpha : \alpha \in [M,N];\ \beta\alpha \in \$E_M$ for all $\beta \in [N,M]\}$.
  (2) $\$[M,N] \subseteq \{\alpha : \alpha \in [M,N];\ \alpha\beta \in \$E_N$ for all $\beta \in [N,M]\}$.

(b) *If $Tot[M,N] = \$[M,N]$ then*
  (1) $\$[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in \ \$E_M$ for all $\beta \in [N,M]\}$.
  (2) $\$[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in \$E_N$ for all $\beta \in [N,M]\}$.

Proof. (a) (1) Let $\alpha \in \$[M,N]$. So for any $\beta \in [N,M]$, $\mathrm{Im}(\beta\alpha)\ \ M$ thus $\beta\alpha \in \$E_M$. (2) If $\alpha \in \$[M,N]$ then $\mathrm{Im}(\alpha)\ \ N$, since for any $\beta \in [N,M]$, $\mathrm{Im}(\alpha\beta) \subseteq \mathrm{Im}(\alpha)$ then $\mathrm{Im}(\alpha\beta)\ \ N$ so $\alpha\beta \in \$E_N$.

(b) Suppose that $\mathrm{Tot}[M,N] = \$[M,N]$. (1) We have by (a) $\$[M,N] \subseteq \{\alpha : \alpha \in [M,N];$ $\beta\alpha \in \$\ E_M$ for all $\beta \in [N,M]\}$. Let $\alpha \in [M,N]$ such that $\beta\alpha \in \$E_M$ for all $\beta \in [N,M]$, suppose $\alpha \notin \$[M,N]$, so there exists $\gamma \in [N,M]$ such that $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ therefore $0 \neq \mathrm{Im}(\gamma\alpha) \subseteq^{\oplus} M$. Since $\gamma\alpha \in \$E_M$, $\mathrm{Im}(\gamma\alpha)\ \ M$ so $\mathrm{Im}(\gamma\alpha) = 0$, a contradiction. Thus, $\alpha \in \$[M,N]$. Similarly (2) holds. □

Lemma 5.6. *Let M_R, N_R be modules. The following are equivalent:*

(1) *If $\alpha \in [M,N] \backslash \$[M,N]$, there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$.*
(2) *If $\alpha \in [M,N] \backslash \$[M,N]$, there exists $\beta \in [N,M]$ such that $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.*
(3) *If $\alpha \in [M,N] \backslash \$[M,N]$, there exists $\gamma \in [N,M]$ such that $\gamma = \gamma\alpha\gamma \notin \$\ [N,M]$.*

Proof. (1) $\Rightarrow$ (3). Let $\alpha \in [M,N] \backslash \$[M,N]$. Then $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ for some $\beta \in [N,M]$. Let $\gamma = \beta\alpha\beta$. Then $\gamma\alpha\gamma = \gamma \notin \$[N,M]$ because $\beta\alpha \notin \$E_M$. Suppose (3) holds, if $\alpha \in [M,N] \backslash\ \$[M,N]$ then $\gamma = \gamma\alpha\gamma$ for some $\gamma \in [N,M] \backslash \$[N,M]$, so $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$, gives (1). Similarly, the equivalence (2) $\Longleftrightarrow$ (3) holds. □

We say that $[M,N]$ is \$- semi-potent if the conditions in Lemma 5.6 are satisfied.

Theorem 5.7. *Let M_R, N_R be modules. $[M,N]$ is \$- semi-potent if and only if, $Tot[M,N] = \$[M,N]$. In particular, E_M is a \$- semi-potent if and only if, $Tot\,(E_M) = \$E_M$.*

Proof. ($\Rightarrow$). Suppose that $\mathrm{Tot}[M,N] \neq \$[M,N]$, Since $\$[M,N] \subseteq \mathrm{Tot}[M,N]$, there exists $\alpha \in \mathrm{Tot}[M,N]$ such that $\alpha \notin \$[M,N]$. So, for any $\beta \in [N,M]$, either $\alpha\beta = 0$ or $\alpha\beta \neq (\alpha\beta)^2$. Hence $[M,N]$ is not \$- semi-potent.

($\Leftarrow$). If $\alpha \in [M,N] \backslash \$[M,N]$, then $\alpha \notin \mathrm{Tot}[M,N]$. So $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$ for some $\beta \in [N,M]$. This shows that $[M,N]$ is \$- semi-potent. □

Let

$$\nabla\Phi(R) = \{M \in \mathrm{mod} - R:\ \mathrm{Tot}[M,N] = \nabla[M,N] \ \ \forall \ \ N \in \mathrm{mod} - R\}$$
$$\nabla\Gamma(R) = \{N \in \mathrm{mod} - R:\ \mathrm{Tot}[M,N] = \nabla[M,N] \ \ \forall \ \ M \in \mathrm{mod} - R\}$$

We define the following two sets:

(a) $\$S\Phi(R)$ the set of all modules $M \in \text{mod} - R$ which have the following two properties:
  (1) E_M is a \$- semi-potent ring.
  (2) For any $N \in \text{mod} - R$;

$$\nabla[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in \nabla E_M;\ \text{for all } \beta \in [N,M]\}.$$

(b) $\$S\Gamma(R)$ the set of all modules $N \in \text{mod} - R$ which satisfy the following two properties:
  (1) E_N is a \$- semi-potent ring.
  (2) For any $M \in \text{mod} - R$;

$$\nabla[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in \nabla E_N;\ \text{for all } \beta \in [N,M]\}.$$

Theorem 5.8. *The following hold:*

(1) $\$\Phi(R) = \$S\Phi(R)$.
(2) $\$\Gamma(R) = \$S\Gamma(R)$.
(3) $\$\Phi(R) = \$\Gamma(R)$.

Proof. (1) ($\Rightarrow$). If $M \in \$\Phi(R)$. Then for any $N \in \text{mod} - R$; $\text{Tot}[M,N] = \$[M,N]$ by Proposition 5.5(b), $\$[M,N] = \{\alpha{:}\alpha \in [M,N];\ \beta\alpha \in \$E_M;$ for all $\beta \in [N,M]\}$ in addition, E_M is a \$- semi-potent ring, so $M \in \$S\Phi(R)$.

($\Leftarrow$). Let $M \in \$S\Phi(R)$. So, for any $N \in \text{mod} - R$, $\$[M,N] \subseteq \text{Tot}[M,N]$. Let $\alpha \in \text{Tot}[M,N]$ then by Lemma 4.1 for any $\beta \in [N,M]$; $\beta\alpha \in \text{Tot}(E_M)$. Since E_M is \$-semi-potent then by Theorem 5.7 $\text{Tot}(E_M) = \$E_M$, so $\beta\alpha \in \$E_M$ for all $\beta \in [M,N]$ thus, $M \in \$\Phi(R)$.

(2) ($\Rightarrow$). If $N \in \$\Gamma(R)$, for any $M \in \text{mod} - R$; $\text{Tot}[M,N] = \$[M,N]$ by proposition 5.5(b) we have $\$[M,N] = \{\alpha{:}\alpha \in [M,N];\ \alpha\beta \in \$E_N;$ for all $\beta \in [N,M]\}$. In addition, E_N is a \$- semi-potent ring, so $N \in \$S\Gamma(R)$.

($\Leftarrow$). Let $N \in \$S\Gamma(R)$, for any $M \in \text{mod} - R$ we have $\$[M,N] \subseteq \text{Tot}[M,N]$. Let $\alpha \in \text{Tot}[M,N]$ by Lemma 4.1 for any $\beta \in [N,M]$; $\alpha\beta \in \text{Tot}(E_N)$. Since E_N is a \$- semi-potent ring then by Theorem 5.7, $\text{Tot}(E_N) = \$E_N$ so $\alpha\beta \in \$E_N$ for all $\beta \in [N,M]$ by assumption $\alpha \in \$[M,N]$. Thus, $N \in \$\Phi(R)$.

(3) By (1) and (2). $\square$

Let M_R, N_R be modules. We put

$$I[M,N] = \{\alpha : \alpha \in [M,N]; \text{Im}(\alpha) \subseteq J(N)\}$$

Since any small submodule of N contained in $J(N)$ then $\$\,[M,N] \subseteq I[M,N]$. If $J(N)\ N$ then $I[M,N] = \$[M,N]$. Thus $I = I(E_M) = I[M,M] = \{\alpha{:}\alpha \in E_M;\ \text{Im}(\alpha) \subseteq J(M)\}$. In particular for a ring R, $I(R) = I[R,R] = J[R,R] = J(R)$. Recall that for a module M_R we defined $\Gamma(M) = \{K{:}K \subseteq^{\oplus} M \text{ and } K \subseteq J(M)\}$.

Proposition 5.9. *Let M_R, N_R be modules.*

(a) *The following hold:*
  (1) $I[M,N] \subseteq \{\alpha{:}\alpha \in [M,N];\ \beta\alpha \in I(E_M)$ *for all* $\beta \in [N,M]\}$.
  (2) $I[M,N] \subseteq \{\alpha{:}\alpha \in [M,N];\ \alpha\beta \in I(E_N)$ *for all* $\beta \in [N,M]\}$.

(b) *If* $Tot[M,N] = I[M,N]$ *and* $\Gamma(M) = \{0\}$, *then*

$$I[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in I(E_M) \text{ for all } \beta \in [N,M]\}.$$

(c) *If* $Tot[M,N] = I[M,N]$ *and* $\Gamma(N) = \{0\}$, *then*

$$I[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in I(E_N) \text{ for all } \beta \in [N,M]\}.$$

Proof.

(a) (1) If $\alpha \in I[M,N]$, then $\mathrm{Im}(\alpha) \subseteq J(N)$, so for any $\beta \in [N,M]$; $\beta\alpha \in E_M$ and $\mathrm{Im}(\beta\alpha) \subseteq J(M)$. Thus, $\beta\alpha \in I(E_M)$. (2). If $\alpha \in I[M,N]$, then $\mathrm{Im}(\alpha) \subseteq J(N)$, so for any $\beta \in [N,M]$; $\alpha\beta \in E_N$ and $\mathrm{Im}(\alpha\beta) \subseteq \mathrm{Im}(\alpha) \subseteq J(N)$. Thus, $\alpha\beta \in I(E_N)$.
(b) Suppose that $\mathrm{Tot}[M,N] = I[M,N]$ and $\Gamma(M) = \{0\}$. Let $\alpha \in [M,N]$ such that $\beta\alpha \in I(E_M)$ for all $\beta \in [N,M]$. Suppose $\alpha \notin I[M,N]$, so there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$, since $\mathrm{Im}(\beta\alpha) \subseteq J(M)$ and $\mathrm{Im}(\beta\alpha) \subseteq^{\oplus} M$, then $\mathrm{Im}(\beta\alpha) \in \Gamma(M) = \{0\}$, a contradiction.
(c) Suppose that $\mathrm{Tot}[M,N] = I[M,N]$ and $\Gamma(N) = \{0\}$. Let $\alpha \in [M,N]$ such that $\alpha\beta \in I(E_N)$ for all $\beta \in [N,M]$. Suppose $\alpha \notin I[M,N]$, so there exists $\gamma \in [N,M]$ such that $0 \neq \alpha\gamma = (\alpha\gamma)^2 \in E_N$. Since $\mathrm{Im}(\alpha\gamma) \subseteq J(N)$ and $\mathrm{Im}(\alpha\gamma) \subseteq^{\oplus} N$ then $\mathrm{Im}(\alpha\gamma) \in \Gamma(N) = \{0\}$, a contradiction. Thus $\alpha \in I[M,N]$. □

Lemma 5.10. *Let M_R, N_R be modules. The following are equivalent:*

(1) *If* $\alpha \in [M,N]\backslash I[M,N]$, *there exists* $\beta \in [N,M]$; $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$, $\beta\alpha \notin I(E_M)$.
(2) *If* $\alpha \in [M,N]\backslash I[M,N]$, *there exists* $\beta \in [N,M]$; $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$, $\alpha\beta \notin I(E_N)$.
(3) *If* $\alpha \in [M,N]\backslash I[M,N]$, *there exists* $\gamma \in [N,M]$; $\gamma\alpha\gamma = \gamma \notin I[N,M]$.

Proof. Suppose (1) holds. Then $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ and $\beta\alpha \notin I(E_M)$ for some $\beta \in [N,M]$. By letting $\gamma = \beta\alpha\beta \in [N,M]$ we have $\gamma\alpha\gamma = \gamma \neq 0$ and $\gamma \notin I[N,M]$ because $\beta\alpha \notin I(E_M)$, giving (3). Suppose (3) holds. Then $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ and $\gamma\alpha \notin I(E_M)$ because $\gamma \notin I[N,M]$ gives (1). Similarly, the equivalence (2) $\Longleftrightarrow$ (3) holds. □

We say that $[M,N]$ is I- semi-potent if the conditions in lemma 5.10 are satisfied.

Theorem 5.11. *Let M_R, N_R be modules. Then the following hold:*

(1) *If* $\Gamma(M) = \{0\}$ *then* $Tot[M,N] = I[M,N]$ *if and only if,* $[M,N]$ *is I- semi-potent.*
(2) *If* $\Gamma(N) = \{0\}$ *then* $Tot[M,N] = I[M,N]$ *if and only if,* $[M,N]$ *is I- semi-potent.*

In particular, if $\Gamma(M) = \{0\}$ *then* $Tot(E_M) = I(E_M)$ *if and only if,* E_M *is an I-semi-potent ring.*

Proof. (1) Suppose that $\Gamma(M) = \{0\}$. ($\Rightarrow$). let $\alpha \in [M,N]\backslash I[M,N]$ then $\alpha \notin$ Tot$[M,N]$, so $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ for some $\beta \in [N,M]$ and $\beta\alpha \notin I(E_M)$ because $\Gamma(M) = \{0\}$. This shows that $[M,N]$ is I- semi-potent.

($\Leftarrow$). Since $\Gamma(M) = \{0\}$ it is easy to see that $I[M,N] \subseteq$ Tot$[M,N]$. Let $\alpha \in$ Tot$[M,N]$ and suppose $\alpha \notin I[M,N]$ so, for any $\beta \in [N,M]$, either $\alpha\beta = 0$ or $\alpha\beta \neq (\alpha\beta)^2$. Hence $[M,N]$ is not I- semi-potent. Similarly (2) holds. □

Let

$$I\Phi(R) = \{M \in \text{mod} - R : \Gamma(M) = \{0\} \text{ and } \text{Tot}[M,N] = I[M,N];\ \text{for all } N \in \text{mod} - R\}$$
$$I\Gamma(R) = \{N \in \text{mod} - R : \Gamma(N) = \{0\} \text{ and } \text{Tot}[M,N] = I[M,N];\ \text{for all } M \in \text{mod} - R\}$$

We define the following two sets:

(a) $IS\Phi(R)$ the set of all modules $M \in$ mod $- R$ which have the following properties:
   (1) $\Gamma(M) = \{0\}$.
   (2) E_M is an I- semi-potent ring.
   (3) For any $N \in$ mod $- R$;

$$I[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in I(E_M) \text{ for all } \beta \in [N,M]\}.$$

(b) $IS\Gamma(R)$ the set of all modules $N \in$ mod $- R$ which satisfies the following properties:
   (1) $\Gamma(N) = \{0\}$.
   (2) E_N is an I- semi-potent ring.
   (3) For any $M \in$ mod $- R$;

$$I[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in I(E_N) \text{ for all } \beta \in [N,M]\}.$$

Theorem 5.12. *The following are holds:*

(1) $I\Phi(R) = I\ S\Phi(R)$.
(2) $I\Gamma(R) = I\ S\Gamma(R)$.
(3) $I\Phi(R) = I\Gamma(R)$.

Proof. (1) ($\Rightarrow$). Let $M \in I\Phi(R)$. Then $\Gamma(M) = \{0\}$ and Tot$[M,N] = I[M,N]$ for all $N \in$ mod $- R$. So, Tot$(E_M) = I(E_M)$ by Theorem 5.11, E_M is an I- semi-potent ring. On the other hand, by Proposition 5.9(b) for any $N \in$ mod $- R$; $I[M,N] = \{\alpha{:}\alpha \in [M,N];\ \beta\alpha \in I(E_M)$ for all $\beta \in [N,M]\}$. So, $M \in IS\Phi(R)$.

($\Leftarrow$). If $M \in IS\Phi(R)$, then $\Gamma(M) = \{0\}$. Let $N \in$ mod $- R$ and $\alpha \in I[M,N]$, so Im$(\alpha) \subseteq J(N)$. Suppose that $\alpha \notin$ Tot$[M,N]$, there exists $\beta \in [N,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. So, $0 \neq$ Im$(\beta\alpha) \subseteq^{\oplus} M$ and Im$(\beta\alpha) \in \Gamma(M) = \{0\}$, a contradiction.

Thus, $I[M,N] \subseteq \text{Tot}[M,N]$. Let $\alpha \in \text{Tot}[M,N]$, suppose that $\alpha \notin I[M,N]$, since $M \in IS\Phi(R)$ there exists $\beta \in [N,M]$ such that $\beta\alpha \notin I(E_M)$. Since E_M is an I- semi-potent ring there exists $\gamma \in E_M$ such that $\gamma(\beta\alpha)\gamma = \gamma \notin I(E_M)$ thus, $0 \neq (\gamma\beta)\alpha = [(\gamma\beta)\alpha]^2 \in E_M$ and $\gamma\beta \in [N,M]$, a contradiction. Hence $\alpha \in \text{Tot}[M,N]$, therefore $\alpha \in I[M,N]$. Thus, $\text{Tot}[M,N] = I[M,N]$ for any $N \in \text{mod} - R$, so $M \in I\Phi(R)$.

(2) ($\Rightarrow$). Let $N \in I\Gamma(R)$. Then $\Gamma(N) = \{0\}$ and $\text{Tot}[M,N] = I[M,N]$ for all $M \in \text{mod} - R$. So, $\text{Tot}(E_N) = I(E_N)$ by Theorem 5.11, E_N is I- semi-potent. On the other hand, by Proposition 5.9(c) for any $M \in \text{mod} - R$; $I[M,N] = \{\alpha{:}\alpha \in [M,N];\ \alpha\beta \in I(E_N) \text{ for all } \beta \in [N,M]\}$. So, $N \in IS\Gamma(R)$.

($\Leftarrow$). If $N \in IS\Gamma(R)$, then $\Gamma(N) = \{0\}$. Let $M \in \text{mod} - R$, $\alpha \in I[M,N]$, so $\text{Im}(\alpha) \subseteq J(N)$. Suppose that $\alpha \notin \text{Tot}[M,N]$, so there exists $\beta \in [N,M]$ such that $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$. So, $0 \neq \text{Im}(\alpha\beta) \subseteq^{\oplus} N$ and $\text{Im}(\alpha\beta) \in \Gamma(N) = \{0\}$, a contradiction. Thus, $I[M,N] \subseteq \text{Tot}[M,N]$. Let $\alpha \in \text{Tot}[M,N]$, suppose that $\alpha \notin I[M,N]$, since $N \in IS\Gamma(R)$ there exists $\beta \in [N,M]$ such that $\alpha\beta \notin I(E_N)$. Since E_N is I- semi-potent there exists $\gamma \in E_N$ such that $\gamma(\alpha\beta)\gamma = \gamma \notin I(E_N)$ thus, $0 \neq (\alpha\beta)\gamma = [(\alpha\beta)\gamma]^2 \in E_N$ and $\beta\gamma \in [N,M]$, a contradiction. Therefore $\alpha \in I[M,N]$. Thus, $\text{Tot}[M,N] = I[M,N]$ for any $M \in \text{mod} - R$, so $N \in I\Gamma(R)$. (3). By (1) and (2). □

6. Locally injective and locally projective modules

Recall a module Q_R is locally injective [9] if, for every submodule $A \subseteq Q$, which is not large in Q, there exists an injective submodule $0 \neq B \subseteq Q$ with $A \cap B = 0$.

Lemma 6.1. *Let Q_R be a locally injective module. Then for any module $N \in \text{mod} - R$ the following hold:*

(1) $\mathit{Tot}[Q,N] = \Delta[Q,N]$.
(2) $J[Q,N] \subseteq \Delta[Q,N]$.
(3) $\$[Q,N] \subseteq \Delta[Q,N]$.

In particular, $J(E_Q) \subseteq \Delta E_Q = \text{Tot}(E_Q)$ *and* $\$E_Q \subseteq \Delta E_Q$.

Proof. (1) By Kasch [9]. (2). Since $J[Q,N] \subseteq \text{Tot}[Q,N]$, so by (1) $J[Q,N] \subseteq \Delta[Q,N]$. (3). Let $\alpha \in\ \$[Q,N]$ and suppose that $\alpha \notin \Delta[Q,N]$ then $\text{Ker}(\alpha)$ is not large in Q, so there exists an injective module $0 \neq A \subseteq Q$ such that $A \cap \text{Ker}(\alpha) = 0$. Since A is injective there exists $\beta{:}N \to A$ such that $\beta\alpha|_A = i_A$ so $\beta = \beta\alpha\beta$. Thus $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_N$, $\text{Im}(\alpha\,\beta) \subseteq^{\oplus} N$ and $\text{Im}(\alpha\beta) \subseteq \text{Im}(\alpha)\ \ N$, so $\text{Im}(\alpha\beta) = 0$ and $\alpha\beta = 0$, a contradiction. Thus $\alpha \in \Delta[Q,N]$. □

Zhou gave an example of a locally injective module which does not have a semi-potent endomorphism ring [13, Example 4.2]. The following Theorem gives us a necessary and sufficient conditions for the endomorphism ring of a locally injective module to be semi-potent ring.

Theorem 6.2. *Let Q_R be a locally injective module. For any module $N \in$ mod $-$ R the following are equivalent:*

(1) $[Q,N]$ *is a semi-potent.*
(2) $Tot[Q,N] = J[Q,N] = \Delta[Q,N]$.
(3) *For any* $\alpha \in [Q,N] \backslash J[Q,N]$ *there exists* $\beta \in [N,Q]$ *with* $0 \neq \mathrm{Ker}(\beta\alpha) \subseteq^{\oplus} Q$.

In particular, E_Q *is a semi-potent ring if and only if, for any* $\alpha \in E_Q \backslash J(E_Q)$ *there exists* $0 \neq \beta \in E_Q$ *such that* $Ker(\beta\alpha) \subseteq^{\oplus} Q$.

Proof. $(1) \Rightarrow (2)$. Suppose that $[Q,N]$ is semi-potent, by [13, Theorem 2.2] $\mathrm{Tot}[Q,N] = J[Q,N]$ and by Lemma 6.1 $J[Q,N] = \Delta\ [Q,N]$. $(2) \Rightarrow (1)$. Since $J[Q,N] = \Delta[Q,N] = \mathrm{Tot}[Q,N]$, so by [13, Theorem 2.2] $[Q,N]$ is semi-potent. $(1) \Rightarrow (3)$. Let $\alpha \in [Q,N] \backslash J[Q,N]$ then there exists $\beta \in [N,Q]$ such that $0 \neq (\beta\alpha)^2 = \beta\alpha \in E_Q$, so $0 \neq \mathrm{Ker}(\beta\alpha) \subseteq^{\oplus} Q$. $(3) \Rightarrow (2)$. Since Q is a locally injective then by Lemma 6.1 $J[Q,N] \subseteq \Delta\ [Q,N]$. Let $\alpha \in \Delta[Q,N]$ and suppose that $\alpha \notin J[Q,N]$ then there exists $\beta \in [N,Q]$ such that $0 \neq \mathrm{Ker}(\beta\alpha) \subseteq^{\oplus} Q$ and $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta\alpha)$. Since Ker(-$\alpha) \leqslant_e Q$ then $\mathrm{Ker}(\beta\alpha) \leqslant_e Q$ and $\mathrm{Ker}(\beta\alpha) \cap \mathrm{Im}(\beta\alpha) = 0$ so $\mathrm{Im}(\beta\alpha) = 0$ and $\beta\alpha = 0$, a contradiction. Thus, $\alpha \in J[Q,N]$. □

Theorem 6.3. *Let Q_R be a module. The following conditions are equivalent:*

(1) Q *is a locally injective module.*
(2) $Tot[Q,N] = \Delta[Q,N]$ *for all* $N \in$ mod $-$ R.
(3) $Tot[N,Q] = \Delta[N,Q]$ *for all* $N \in$ mod $-$ R.
(4) $[Q,N]$ *is a* Δ- *semi-potent for all* $N \in$ mod $-$ R.
(5) $[N,Q]$ *is a* Δ- *semi-potent for all* $N \in$ mod $-$ R.

Proof. (1) $\Longleftrightarrow$ (2). By Kasch [9]. (2) $\Longleftrightarrow$ (3). By Theorem 5.4. (3) $\Longleftrightarrow$ (4) and (2) $\Longleftrightarrow$ (5) By Theorem 5.3. □

Recall a module P_R is locally projective [9] if, for every submodule $B \subseteq P$, which is not small in P there exists a projective direct summand $0 \neq W \subseteq^{\oplus} P$ with $W \subseteq B$.

Lemma 6.4. *Let P_R be a locally projective module. Then for any module $M \in$ mod $-$ R the following hold:*

(1) $Tot[M,P] = \$[M,P]$.
(2) $J[M,P] \subseteq \$[M,P]$.
(3) $\Delta[M,P] \subseteq \$[M,P]$.

In particular, $J(E_P) \subseteq \$E_P = \mathrm{Tot}(E_P)$ *and* $\Delta E_P \subseteq \$E_P$.

Proof. (1) By Kasch [9]. (2) Since $J[M,P] \subseteq \mathrm{Tot}[M,P]$, so by (1) $J[M,P] \subseteq \$[M,P]$. (3) We have by (1), $[M,P]$ is a \$- semi-potent. Let $\alpha \in \Delta[M,P]$ suppose that $\alpha \notin \$[M,P]$ then there exists $\beta \in [P,M]$ such that $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. Since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta\alpha)$ and

$\alpha \in \Delta\,[M,P]$ then $\mathrm{Ker}(\beta\alpha) \leqslant_e M$, so $\mathrm{Im}(\beta\alpha) = 0$, hence $\mathrm{Ker}(\beta\alpha) \cap \mathrm{Im}(\beta\alpha) = 0$. Thus, $\beta\alpha = 0$ a contradiction, so $\alpha \in \$[M,P]$. □

Theorem 6.5. *Let P_R be a locally projective module. For any module $M \in \mathrm{mod}-R$ the following are equivalent:*

(1) $[M,P]$ *is a semi-potent.*
(2) $Tot[M,P] = J[M,P] = \$[M,P]$.
(3) *For any $\alpha \in [M,P] \backslash J[M,P]$ there exists $\beta \in [P,M]$ with $0 \neq \mathrm{Im}(\alpha\beta) \subseteq^{\oplus} P$.*

In particular, E_P is a semi-potent ring if and only if, for any $\alpha \in E_P \backslash J(E_P)$ there exists $0 \neq \beta \in E_P$ such that $Im(\alpha\beta) \subseteq^{\oplus} P$.

Proof. $(1) \Rightarrow (2)$. Suppose that $[M,P]$ is semi-potent then [13, Theorem 2.2] $\mathrm{Tot}[M,P] = J[M,P]$ and by Lemma 6.4 $J[M,P] = \$[M,P]$. $(2) \Rightarrow (1)$. Since $\mathrm{Tot}[M,P] = J[M,P]$ then by [13, Theorem 2.2] $[M,P]$ is semi-potent. $(1) \Rightarrow (3)$. Let $\alpha \in [M,P] \backslash J[M,P]$ then there exists $\beta \in [P,M]$ such that $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_P$, so $0 \neq \mathrm{Im}(\alpha\beta) \subseteq^{\oplus} P$. $(3) \Rightarrow (2)$. Since P is locally projective then by Lemma 6.4 $J[M,P] \subseteq \$[M,P]$. Let $\alpha \in \$[M,P]$, suppose that $\alpha \notin J[M,P]$ then there exists $\beta \in [P,M]$ such that $0 \neq \mathrm{Im}(\alpha\beta) \subseteq^{\oplus} P$. Since $\alpha \in \$[M,P]$ and $\mathrm{Im}(\alpha\beta) \subseteq \mathrm{Im}(\alpha)$ then $\mathrm{Im}(\alpha\beta)\ P$. Therefore $\mathrm{Im}(\alpha\beta) = 0$ and $\alpha\beta = 0$, a contradiction. Thus, $\alpha \in J[M,P]$. □

Theorem 6.6. *Let P_R be a module. The following conditions are equivalent:*

(1) *P is a locally projective module.*
(2) *$Tot[M,P] = \$[M,P]$ for all $M \in \mathrm{mod}-R$.*
(3) *$Tot[P,M] = \$[P,M]$ for all $M \in \mathrm{mod}-R$.*
(4) *$[P,M]$ is a \$- semi-potent for all $M \in \mathrm{mod}-R$.*
(5) *$[M,P]$ is a \$- semi-potent for all $M \in \mathrm{mod}-R$.*

Proof. $(1) \iff (2)$. By Kasch [9]. $(2) \iff (3)$. By Theorem 5.8. $(3) \iff (4)$ and $(2) \iff (5)$ By Theorem 5.7. □

Corollary 6.7. *The following conditions are equivalent for a ring R:*

(1) *Every module $M \in \mathrm{mod}-R$ with E_M a Δ- semi-potent ring, is injective.*
(2) $\Phi(R) = \Delta\Phi(R)$.
(3) *Every module $M \in \mathrm{mod}-R$ with E_M is a semi-potent ring, is injective.*
(4) *R is a semi-simple Artinian ring.*
(5) *Every module $M \in \mathrm{mod}-R$ with E_M a semi-potent ring, is projective.*
(6) $\Gamma(R) = \$\Gamma(R)$.
(7) *Every module $M \in \mathrm{mod}-R$ with E_M a \$- semi-potent ring, is projective.*

Proof. See [13, Corollary 4.7] and Theorems 6.2, 6.5. □

Corollary 6.8. *The following conditions are equivalent for a ring R:*

(1) *R is a semi-potent ring and $J(R)$ is left T- nilpotent.*
(2) *E_P is a semi-potent ring for every projective module $P \in \mathrm{mod} - R$.*
(3) *E_P is a \$- semi-potent ring for every projective module $P \in \mathrm{mod} - R$.*
(4) *E_F is a semi-potent ring for every free module $F \in \mathrm{mod} - R$.*
(5) *E_F is a \$- semi-potent ring for every free module $F \in \mathrm{mod} - R$.*

Proof. By [13, Theorem 4.10] since for any projective module $P \in \mathrm{mod} - R$; $J(E_P) = \$E_P$, by [11, Proposition 1.1] (See also, [3, Theorem 3.8]). □

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