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Resolution of a high-order parabolic equation in conical time-dependent domains of $\mathbb{R}^3$

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Abstract. New results on the existence, uniqueness and maximal regularity of a solution are given for a two-space dimensional high-order parabolic equation set in conical time-dependent domains. The study is performed in the framework of anisotropic weighted Sobolev spaces. Our method is based on the technique of decomposition of domains.

Keywords: High-order parabolic equations; Conical domains; Anisotropic weighted Sobolev spaces

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1. Introduction

Let Q be an open set of $\mathbb{R}^3$ defined by

$$Q = \left\{(t, x_1, x_2) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_t, 0 < t < T\right\}$$

where T is a finite positive number and for a fixed t in the interval $]0, T[$, Ω_t is a bounded domain of $\mathbb{R}^2$ defined by

$$\Omega_t = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq \sqrt{x_1^2 + x_2^2} < \varphi(t)\right\}.$$

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Here φ is a continuous real-valued function defined on $[0,T]$, Lipschitz continuous on $[0,T]$ and such that

$$\varphi(t) > 0$$

for every $t \in]0,T]$. We assume that

$$\varphi(0) = 0, \tag{1.1}$$

$$\varphi'(t)\,\varphi^m(t) \to 0 \quad \text{as } t \to 0,\, m \in \mathbb{N}^*. \tag{1.2}$$

In Q, consider the boundary value problem

$$\begin{cases} \partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u = f \in L^2_\omega(Q), \\ \partial_{x_j}^k u\Big|_{\partial Q \smallsetminus \Gamma_T} = 0, \quad k = 0,\dots,m-1;\ j = 1,2, \end{cases} \tag{1.3}$$

where $m \in \mathbb{N}^*$, ∂Q is the boundary of Q and Γ_T is the part of the boundary of Q where $t = T$. Here, $L^2_\omega(Q)$ is the space of square-integrable functions on Q with the measure $\omega dt dx_1 dx_2$, where the weight ω is a real-valued function defined on $[0,T]$, differentiable on $]0,T]$, such that

$$\forall t \in [0,T] : \omega(t) > 0, \tag{1.4}$$

$$\omega \text{ is a decreasing function on }]0,T]. \tag{1.5}$$

The difficulty related to this kind of problems comes from the fact that the domain Q considered here is nonstandard since it shrinks at $t = 0$ ($\varphi(0) = 0$), which prevents the domain Q to be transformed into a regular domain without the appearance of some degenerate terms in the parabolic equation, see for example Sadallah [15].

In this work, we will prove that Problem (1.3) has a solution with optimal regularity, that is a solution u belonging to the anisotropic weighted Sobolev space

$$H^{1,2m}_{0,\omega}(Q) := \left\{ u \in H^{1,2m}_\omega(Q) : \partial_{x_j}^k u\Big|_{\partial Q \smallsetminus \Gamma_T} = 0, k = 0,\dots,m-1;\ j = 1,2 \right\},$$

with

$$H^{1,2m}_\omega(Q) = \{u : \partial_t u, \partial^\alpha u \in L^2_\omega(Q), |\alpha| \le 2m\}$$

where

$$\alpha = (i_1, i_2) \in \mathbb{N}^2, \qquad |\alpha| = i_1 + i_2, \qquad \partial^\alpha u = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} u.$$

The space $H^{1,2m}_\omega(Q)$ is equipped with the natural norm, that is

$$\|u\|_{H^{1,2m}_\omega(Q)} = \left(\|\partial_t u\|^2_{L^2_\omega(Q)} + \sum_{|\alpha| \le 2m} \|\partial^\alpha u\|^2_{L^2_\omega(Q)} \right)^{1/2}.$$

Remark 1.1. The boundary conditions of Problem (1.3) are equivalent to

$$\partial_\nu^k u\big|_{\partial Q \smallsetminus \Gamma_T} = 0, \quad k = 0,\dots,m-1,$$

where ∂_ν stands for the normal derivative. This equivalence can be proved, for instance, by induction. So Problem (1.3) is also equivalent to

$$\begin{cases} \partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u = f \in L^2_\omega(Q), \\ \partial_\nu^k u\big|_{\partial Q \setminus \Gamma_T} = 0, \quad k = 0, \dots, m-1. \end{cases} \tag{1.6}$$

Observe that the number of the boundary conditions in (1.3) is $2m$, but they are not independent, while in (1.6), there are m independent boundary conditions.

Our main result is

Theorem 1.1. *Let us assume that φ satisfies condition (1.1) and the weight function ω verifies assumptions (1.4) and (1.5). Then, the $2m$-th order parabolic operator*

$$L = \partial_t + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m}$$

is an isomorphism from $H^{1,2m}_{0,\omega}(Q)$ into $L^2_\omega(Q)$ if one of the following conditions is satisfied
(1) *φ is an increasing function in a neighborhood of 0,*
(2) *φ verifies the condition (1.2).*

The case $m = 1$ corresponding to a second-order parabolic equation is studied in [16] and [9] both in bi-dimensional and multidimensional cases. We can find in Sadallah [15] a study of such kind of problems in the case of one space variable. Further references on the analysis of higher-order parabolic problems in non-cylindrical domains are: Baderko [1,2], Cherepova [4,5], Labbas and Sadallah [10], Galaktionov [6], Mikhailov [13,14] and Kheloufi [8].

The organization of this paper is as follows. In Section 2, first we prove a uniqueness result for Problem (1.3), then we derive some technical lemmas which will allow us to prove an energy type estimate (in a sense to be defined later). In Section 3, we divide the proof of Theorem 1.1 into four steps:

(a) Case of a truncated domain,
(b) An energy type estimate in small in time case,
(c) Passage to the limit,
(d) Case of a large in time conical type domain.

2. Preliminaries

Proposition 2.1. *Under the assumptions (1.4) and (1.5) on the weight function ω, Problem (1.3) is uniquely solvable.*

Proof. Let us consider $u \in H^{1,2m}_{0,\omega}(Q)$ a solution of Problem (1.3) with a null right-hand side term. So,

$$\partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u = 0 \quad \text{in } Q.$$

In addition u fulfils the boundary conditions

$$\partial_{x_j}^k u\Big|_{\partial Q \setminus \Gamma_T} = 0, \quad k = 0, 1, \ldots, m-1;\ j = 1, 2.$$

Using Green's formula, we have

$$\begin{aligned}
&\int_Q \left(\partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u \right) u\, \omega(t)\, dt\, dx_1 dx_2 \\
&\quad = \int_{\partial Q} \left[\frac{1}{2} |u|^2 \nu_t + \sum_{j=1}^{2} \sum_{k=0}^{m-1} (\partial_{x_j}^{2m-k-1} u . \partial_{x_j}^k u)(-1)^{k+m} \nu_{x_j} \right] \omega(t)\, d\sigma \\
&\qquad + \int_Q \left(\left|\partial_{x_1}^m u\right|^2 + \left|\partial_{x_2}^m u\right|^2 \right) dt\, dx_1 dx_2 - \int_Q \frac{1}{2} |u|^2 \omega'(t)\, dt\, dx_1 dx_2,
\end{aligned}$$

where ν_t, ν_{x_1}, ν_{x_2} are the components of the unit outward normal vector at ∂Q. Taking into account the boundary conditions, all the boundary integrals vanish except $\int_{\partial Q} |u|^2 \omega(t)\, \nu_t\, d\sigma$. We have

$$\int_{\partial Q} |u|^2 \omega(t)\, \nu_t d\sigma = \int_{\Gamma_T} |u|^2 \omega(T)\, dx_1 dx_2.$$

Then

$$\begin{aligned}
&\int_Q \left(\partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u \right) u\, \omega(t)\, dt\, dx_1 dx_2 \\
&\quad = \int_{\Gamma_T} \frac{1}{2} |u|^2 \omega(T)\, dx_1 dx_2 - \int_Q \frac{1}{2} |u|^2 \omega'(t)\, dt\, dx_1 dx_2 \\
&\qquad + \int_Q \left(\left|\partial_{x_1}^m u\right|^2 + \left|\partial_{x_2}^m u\right|^2 \right) dt\, dx_1 dx_2\ .
\end{aligned}$$

Consequently

$$\int_Q \left(\partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u \right) u\, \omega(t)\, dt\, dx_1 dx_2 = 0$$

yields

$$\int_Q \left(\left|\partial_{x_1}^m u\right|^2 + \left|\partial_{x_2}^m u\right|^2 \right) dt\, dx_1 dx_2 = 0,$$

because

$$\int_{\Gamma_T} \frac{1}{2} |u|^2 \omega(t)\, dx_1 dx_2 - \int_Q \frac{1}{2} |u|^2 \omega'(t)\, dt\, dx_1 dx_2 \geq 0$$

thanks to the conditions (1.4) and (1.5). This implies that $\left|\partial_{x_1}^m u\right|^2 + \left|\partial_{x_2}^m u\right|^2 = 0$ and consequently $\partial_{x_1}^{2m} u = \partial_{x_2}^{2m} u = 0$. Then, the hypothesis $\partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u = 0$ gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions imply that $u = 0$ in Q. This proves the uniqueness of the solution of Problem (1.3). $\square$

Remark 2.1. In the sequel, we will be interested only in the question of the existence of the solution of Problem (1.3).

The following result is well known (see, for example, [12])

Lemma 2.1. *Let $B(0,1)$ be the unit disk of $\mathbb{R}^2$. Then, the operator*

$$\mathcal{A} : H^{2m}(B(0,1)) \cap H_0^m(B(0,1)) \longrightarrow L^2(B(0,1)),$$
$$v \mapsto \mathcal{A}v = (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} v$$

is an isomorphism. Moreover, there exists a constant $C > 0$ such that

$$\|v\|_{H^{2m}(B(0,1))} \leq C \|\mathcal{A}v\|_{L^2(B(0,1))}, \quad \forall v \in H^{2m}(B(0,1)) \cap H_0^m(B(0,1)).$$

In the above lemma, H^{2m} and H_0^m are the usual Sobolev spaces defined, for instance, in Lions–Magenes [12]. In Section 3, we will need the following result.

Lemma 2.2. *For a fixed $t \in]0,T[$, there exists a constant $C > 0$ such that for each $u \in H^{2m}(\Omega_t)$, we have*

$$\left\|\partial_{x_j}^l u\right\|_{L^2(\Omega_t)}^2 \leq C\varphi^{2(2m-l)}(t) \|\mathcal{A}u\|_{L^2(\Omega_t)}^2, \quad l = 0,1,\dots,2m-1;\ j = 1,2.$$

Proof. It is a direct consequence of Lemma 2.1. Indeed, let $t \in]0,T[$ and define the following change of variables

$$B(0,1) \to \Omega_t, \qquad (x_1,x_2) \longmapsto (\varphi(t)x_1, \varphi(t)x_2) = (x_1', x_2').$$

Set $v(x_1,x_2) = u(x_1',x_2')$, then if $v \in H^{2m}(B(0,1))$, u belongs to $H^{2m}(\Omega_t)$. For $j = 1,2$, we have

$$\begin{aligned}
\left\|\partial_{x_j}^l v\right\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left(\partial_{x_j}^l v\right)^2 (x_1,x_2)\, dx_1 dx_2 \\
&= \int_{\Omega_t} \left(\partial_{x_j'}^l u\right)^2 (x_1',x_2')\, \varphi^{2l}(t) \frac{1}{\varphi^2(t)} dx_1' dx_2' \\
&= \varphi^{2l-2}(t) \int_{\Omega_t} \left(\partial_{x_j'}^l u\right)^2 (x_1',x_2')\, dx_1' dx_2' \\
&= \varphi^{2l-2}(t) \left\|\partial_{x_j'}^l u\right\|_{L^2(\Omega_t)}^2
\end{aligned}$$

where $l \in \{0,1,\dots,2m-1\}$. On the other hand, we have

$$\begin{aligned}
\|\mathcal{A}v\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left[(-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} v(x_1,x_2)\right]^2 dx_1 dx_2 \\
&= \int_{\Omega_t} \left(\sum_{j=1}^{2} \varphi^{2m}(t)\, \partial_{x_j'}^{2m} u\right)^2 (x_1',x_2') \frac{1}{\varphi^2(t)} dx_1' dx_2'
\end{aligned}$$

$$= \varphi^{4m-2}(t) \int_{\Omega_t} \left(\sum_{j=1}^{2} \partial_{x'_j}^{2m} u \right)^2 (x'_1, x'_2)\, dx'_1 dx'_2$$
$$= \varphi^{4m-2}(t) \, \|\mathcal{A}u\|^2_{L^2(\Omega_t)} .$$

Using the inequality

$$\left\| \partial_{x_j}^l v \right\|^2_{L^2(B(0,1))} \leq C \, \|\mathcal{A}v\|^2_{L^2(B(0,1))}$$

of Lemma 2.1, we obtain the desired inequality

$$\left\| \partial_{x'_j}^l u \right\|^2_{L^2(\Omega_t)} \leq C\varphi^{2(2m-l)}(t) \, \|\mathcal{A}u\|^2_{L^2(\Omega_t)} . \quad \square$$

Remark 2.2. In Lemma 2.2 we can replace $\|.\|_{L^2}$ by $\|.\|_{L^2_\omega}$.

3. Proof of Theorem 1.1

3.1. Case of a truncated domain Q_n

In this subsection, we replace Q by $Q_n, n \in \mathbb{N}^*$ and $\frac{1}{n} < T$:

$$Q_n = \left\{ (t, x_1, x_2) \in Q : \frac{1}{n} < t < T \right\} .$$

Theorem 3.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the problem*

$$\begin{cases} \partial_t u_n + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u_n = f_n \in L^2_\omega(Q_n), \\ \partial_{x_j}^k u_n \Big|_{\partial Q_n \setminus \Gamma_T} = 0, \quad k = 0, 1, \ldots, m-1;\ j = 1, 2, \end{cases} \tag{3.1}$$

where $f_n = f|_{Q_n}$ admits a unique solution $u_n \in H^{1,2m}_\omega(Q_n)$.

Proof of Theorem 3.1. The change of variables

$$(t, x_1, x_2) \mapsto (t, y_1, y_2) = \left(t, \frac{x_1}{\varphi(t)}, \frac{x_2}{\varphi(t)} \right)$$

transforms Q_n into the cylinder $P_n = \left]\frac{1}{n}, T\right[\times B(0,1)$, where $B(0,1)$ is the unit disk of $\mathbb{R}^2$. Putting $u_n(t, x_1, x_2) = v_n(t, y_1, y_2)$ and $f_n(t, x_1, x_2) = g_n(t, y_1, y_2)$, then Problem (3.1) is transformed, in P_n into the following variable-coefficient parabolic problem

$$\begin{cases} \partial_t v_n + \dfrac{(-1)^m}{\varphi^{2m}(t)} \sum_{j=1}^{2} \partial_{y_j}^{2m} v_n + \dfrac{\varphi'(t)}{\varphi(t)} \sum_{j=1}^{2} y_j \partial_{y_j} v_n = g_n \\ \partial_{y_j}^k v_n \Big|_{\partial P_n \setminus \Sigma_T} = 0, \quad k = 0, 1, \ldots, m-1;\ j = 1, 2 \end{cases}$$

where Σ_T is the part of the boundary of P_n where $t = T$. The above change of variables conserves the spaces L^2_ω and $H^{1,2m}_\omega$ because $\frac{(-1)^m}{\varphi^{2m}(t)}$ and $\frac{\varphi'(t)}{\varphi(t)}$ are bounded functions when $t \in]\frac{1}{n}, T[$. In other words

$$f_n \in L^2_\omega(Q_n) \Longleftrightarrow g_n \in L^2_\omega(P_n), \qquad u_n \in H^{1,2m}_\omega(Q_n) \Longleftrightarrow v_n \in H^{1,2m}_\omega(P_n).$$

Proposition 3.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the following operator is compact*

$$\frac{\varphi'(t)}{\varphi(t)} \sum_{j=1}^{2} y_j \partial_{y_j} : H^{1,2m}_{0,\omega}(P_n) \longrightarrow L^2_\omega(P_n).$$

Proof. P_n has the "horn property" of Besov (see [3]). So, for $j = 1, 2$

$$\partial_{y_j} : H^{1,2m}_{0,\omega}(P_n) \longrightarrow H^{1-\frac{1}{2m},2m-1}_\omega(P_n), \qquad v \longmapsto \partial_{y_j} v,$$

is continuous. Since P_n is bounded, the canonical injection is compact from $H^{1-\frac{1}{2m},2m-1}_\omega(P_n)$ into $L^2_\omega(P_n)$ (see for instance [3]), where

$$H^{1-\frac{1}{2m},2m-1}_\omega(P_n) = L^2\left(\frac{1}{n}, T; H^{2m-1}(B(0,1))\right) \cap H^{1-\frac{1}{2m}}\left(\frac{1}{n}, T; L^2(B(0,1))\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces, see for instance [12]. Consider the composition

$$\partial_{y_j} : H^{1,2m}_{0,\omega}(P_n) \to H^{1-\frac{1}{2m},2m-1}_\omega(P_n) \to L^2_\omega(P_n), \qquad v \mapsto \partial_{y_j} v \mapsto \partial_{y_j} v,$$

then ∂_{y_j} is a compact operator from $H^{1,2m}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$. Since $\frac{\varphi'(t)}{\varphi(t)}$ is a bounded function for $\frac{1}{n} < t < T$, the operators $\frac{\varphi'(t) y_j}{\varphi(t)} \partial_{y_j}$, $j = 1, 2$ are also compact from $H^{1,2m}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$. Consequently,

$$\frac{\varphi'(t)}{\varphi(t)} \sum_{j=1}^{N} y_j \partial_{y_j}$$

is compact from $H^{1,2m}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$. $\square$

So, thanks to Proposition 3.1, to complete the proof of Theorem 3.1, it is sufficient to show that the operator

$$\partial_t + \frac{(-1)^m}{\varphi^{2m}(t)} \sum_{j=1}^{N} \partial^{2m}_{y_j}$$

is an isomorphism from $H^{1,2m}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$.

Lemma 3.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the operator*

$$\partial_t + \frac{(-1)^m}{\varphi^{2m}(t)} \sum_{j=1}^{2} \partial_{y_j}^{2m}$$

is an isomorphism from $H_{0,\omega}^{1,2m}(P_n)$ into $L_\omega^2(P_n)$.

Proof. Thanks to Remark 1.1, the problem

$$\begin{cases} \partial_t v_n + \dfrac{(-1)^m}{\varphi^{2m}(t)} \displaystyle\sum_{j=1}^{2} \partial_{y_j}^{2m} v_n = g_n \\ \partial_{y_j}^k v_n \Big|_{\partial P_n \setminus \Sigma_T} = 0, \quad k = 0, 1, \ldots, m-1;\ j = 1, 2, \end{cases}$$

is equivalent to the following problem

$$\begin{cases} \partial_t v_n + \dfrac{(-1)^m}{\varphi^{2m}(t)} \displaystyle\sum_{j=1}^{2} \partial_{y_j}^{2m} v_n = g_n \\ \partial_\nu^k v_n|_{\partial P_n \setminus \Sigma_T} = 0, \quad k = 0, \ldots, m-1. \end{cases}$$

Since the coefficient $\frac{1}{\varphi^{2m}(t)}$ is bounded in $\overline{P_n}$, the optimal regularity is given by Ladyzhenskaya, Solonnikov and Ural'tseva [11]. □

We shall need the following result in order to justify the calculus of this section.

Lemma 3.2. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space*

$$\left\{ v \in H^{2m}(P_n) : \partial_{x_j}^k v \Big|_{\partial_p P_n} = 0,\ k = 0, 1, \ldots, m-1;\ j = 1, 2 \right\}$$

is dense in the space

$$\left\{ v \in H^{1,2m}(P_n) : \partial_{x_j}^k v \Big|_{\partial_p P_n} = 0,\ k = 0, 1, \ldots, m-1;\ j = 1, 2 \right\}.$$

Here, $\partial_p P_n$ is the parabolic boundary of P_n and H^{2m} stands for the usual Sobolev space defined, for instance, in Lions–Magenes [12].

The proof of the above lemma may be found in [12].

Remark 3.1. In Lemma 3.2, we can replace P_n by Q_n with the help of the change of variables defined above.

3.2. Case of a "small" conical domain

Now, we return to the conical domain Q and we suppose that the function φ satisfies conditions (1.1) and (1.2).

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we denote $f_n = f|_{Q_n}$ and $u_n \in H_\omega^{1,2m}(Q_n)$ the solution of Problem (1.3) in Q_n. Such a solution exists by Theorem 3.1.

Proposition 3.2. *For T small enough, there exists a constant K_1 independent of n such that*

$$\|u_n\|_{H_\omega^{1,2m}(Q_n)} \leq K_1 \|f_n\|_{L_\omega^2(Q_n)} \leq K_1 \|f\|_{L_\omega^2(Q)}.$$

Remark 3.2. Let $\epsilon > 0$ be a real number which we will choose small enough. The hypothesis (1.2) implies the existence of a real number $T > 0$ small enough such that

$$\forall t \in (0,T), \quad |\varphi'(t)\,\varphi^m(t)| \leq \epsilon. \tag{3.2}$$

In order to prove Proposition 3.2, we need the following result which is a consequence of Lemma 2.2 and Grisvard–Looss [7, Theorem 2.2].

Lemma 3.3. *There exists a constant $C > 0$ independent of n such that*

$$\sum_{|\alpha|=2m} \|\partial^\alpha u_n\|^2_{L_\omega^2(Q_n)} \leq C \|\mathcal{A}u_n\|^2_{L_\omega^2(Q_n)}.$$

Proof of Proposition 3.2. Let us denote the inner product in $L_\omega^2(Q_n)$ by $\langle .,.\rangle$, then we have

$$\begin{aligned}\|f_n\|^2_{L_\omega^2(Q_n)} &= \langle \partial_t u_n + \mathcal{A}u_n, \partial_t u_n + \mathcal{A}u_n\rangle \\ &= \|\partial_t u_n\|^2_{L_\omega^2(Q_n)} + \|\mathcal{A}u_n\|^2_{L_\omega^2(Q_n)} + 2\langle \partial_t u_n, \mathcal{A}u_n\rangle.\end{aligned}$$

Estimation of $2\langle \partial_t u_n, \mathcal{A}u_n\rangle$: We have

$$\partial_t u_n.\mathcal{A}u_n = \sum_{j=1}^{2}\left[\sum_{k=0}^{m-1}\partial_{x_j}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m} + \frac{1}{2}\partial_t\left(\partial_{x_j}^m u_n\right)^2\right].$$

Then

$$\begin{aligned}2\langle \partial_t u_n, \mathcal{A}u_n\rangle &= 2\int_{Q_n}\partial_t u_n.\mathcal{A}u_n.\omega(t)\,dtdx_1dx_2 \\ &= 2\int_{Q_n}\sum_{j=1}^{2}\sum_{k=0}^{m-1}\partial_{x_j}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}.\omega(t)\,dtdx_1dx_2 \\ &\quad + \int_{Q_n}\partial_t\sum_{j=1}^{2}\left(\partial_{x_j}^m u_n\right)^2.\omega(t)\,dtdx_1dx_2 \\ &= 2\int_{\partial Q_n}\sum_{j=1}^{2}\sum_{k=0}^{m-1}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\nu_{x_j}.\omega(t)\,d\sigma \\ &\quad + \int_{\partial Q_n}\sum_{j=1}^{2}\left(\partial_{x_j}^m u_n\right)^2\nu_t.\omega(t)\,d\sigma - \int_{Q_n}\sum_{j=1}^{2}\left(\partial_{x_j}^m u_n\right)^2.\omega'(t)\,dtdx_1dx_2\end{aligned}$$

with $\nu_t, \nu_{x_1}, \nu_{x_2}$ are the components of the unit outward normal vector at ∂Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Q_n where $t = \frac{1}{n}$, we have $\partial_{x_j}^k u_n = 0, k = 0, \ldots, m-1; j = 1, 2$ and consequently the corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_j} = 0$, $j = 1, 2$ and $\nu_t = 1$. Accordingly, the corresponding boundary integral

$$\int_{\Gamma_T} \sum_{j=1}^{2} \left(\partial_{x_j}^m u_n \right)^2 (T, x_1, x_2) \, .\omega \, (T) \, dx_1 dx_2$$

is nonnegative. On the part Γ_1 of ∂Q_n defined by

$$\Gamma_1 = \left\{ (t, x_1, x_2) : \sqrt{x_1^2 + x_2^2} = \varphi \, (t) \right\},$$

we have

$$\begin{aligned} \nu_t &= \frac{-\varphi' \, (t)}{\sqrt{1 + (\varphi')^2 \, (t)}}, \\ \nu_{x_1} &= \frac{\cos \theta}{\sqrt{1 + (\varphi')^2 \, (t)}}, \\ \nu_{x_2} &= \frac{\sin \theta}{\sqrt{1 + (\varphi')^2 \, (t)}}, \end{aligned}$$

and

$$\partial_{x_j}^k u_n(t, \varphi \, (t) \cos \theta, \varphi \, (t) \sin \theta) = 0, \quad k = 0, \ldots, m-1; j = 1, 2.$$

Let us denote

$$I = 2 \int_{\Gamma_1} \sum_{j=1}^{2} \sum_{k=0}^{m-1} \left(\partial_{x_j}^k \partial_t u_n . \partial_{x_j}^{2m-k-1} u_n \right) (-1)^{k+m} \nu_{x_j} . \omega \, (t) \, d\sigma.$$

We have

$$\begin{aligned} I &= 2 \int_{\Gamma_1} \sum_{j=1}^{2} \left(\partial_t u_n . \partial_{x_j}^{2m-1} u_n \right) (-1)^m \nu_{x_j} . \omega \, (t) \, d\sigma \\ &\quad + 2 \int_{\Gamma_1} \sum_{j=1}^{2} \sum_{k=1}^{m-2} \left(\partial_{x_j}^k \partial_t u_n . \partial_{x_j}^{2m-k-1} u_n \right) (-1)^{k+m} \nu_{x_j} . \omega \, (t) \, d\sigma \\ &\quad - 2 \int_{\Gamma_1} \sum_{j=1}^{2} \left(\partial_{x_j}^{m-1} \partial_t u_n . \partial_{x_j}^m u_n \right) \nu_{x_j} . \omega \, (t) \, d\sigma \\ &= I_0 + I_1 + I_{m-1}. \end{aligned}$$

(a) Estimation of $I_0 = 2 \int_{\Gamma_1} \sum_{j=1}^{2} \left(\partial_t u_n . \partial_{x_j}^{2m-1} u_n \right) (-1)^m \nu_{x_j} . \omega \, (t) \, d\sigma$:

We have

$$u_n \, (t, \varphi \, (t) \cos \theta, \varphi \, (t) \sin \theta) = 0.$$

Differentiating with respect to t, we obtain

$$\partial_t u_n = -\varphi'(t)\left(\cos\theta.\partial_{x_1} u_n + \sin\theta.\partial_{x_2} u_n\right) = 0.$$

So, the boundary integral I_0 vanishes.

(b) Estimation of $I_1 = 2\int_{\Gamma_1}\sum_{j=1}^{2}\sum_{k=1}^{m-2}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\nu_{x_j}.\omega(t)d\sigma$: We have

$$\partial_{x_j}^k u_n\left(t,\varphi(t)\cos\theta,\varphi(t)\sin\theta\right) = 0,\quad k=1,\ldots,m-2; j=1,2.$$

Differentiating with respect to t, we obtain

$$\partial_t\partial_{x_1}^k u_n = -\varphi'(t)\left[\cos\theta.\partial_{x_1}^{k+1}u_n + \sin\theta.\partial_{x_2}\partial_{x_1}^k u_n\right],\quad k=1,\ldots,m-2$$

and

$$\partial_t\partial_{x_2}^k u_n = -\varphi'(t)\left[\cos\theta.\partial_{x_1}\partial_{x_2}^k u_n + \sin\theta.\partial_{x_2}^{k+1}u_n\right],\quad k=1,\ldots,m-2.$$

The Dirichlet boundary conditions on Γ_1 lead to

$$\partial_t\partial_{x_1}^k u_n = -\varphi'(t)\sin\theta.\partial_{x_2}\partial_{x_1}^k u_n,\quad k=1,\ldots,m-2$$

and

$$\partial_t\partial_{x_2}^k u_n = -\varphi'(t)\cos\theta.\partial_{x_1}\partial_{x_2}^k u_n,\quad k=1,\ldots,m-2.$$

Now, differentiating the formula

$$\partial_{x_j}^k u_n\left(t,\varphi(t)\cos\theta,\varphi(t)\sin\theta\right) = 0,\quad k=1,\ldots,m-2; j=1,2$$

with respect to θ, we obtain

$$\sin\theta.\partial_{x_1}^{k+1}u_n = \cos\theta.\partial_{x_2}\partial_{x_1}^k u_n,\quad k=1,\ldots,m-2$$

and

$$\cos\theta.\partial_{x_2}^{k+1}u_n = \sin\theta.\partial_{x_1}\partial_{x_2}^k u_n,\quad k=1,\ldots,m-2.$$

The Dirichlet boundary conditions on Γ_1 lead to

$$\partial_{x_1}\partial_{x_2}^k u_n = \partial_{x_2}\partial_{x_1}^k u_n = 0,\quad k=1,\ldots,m-2$$

and consequently

$$\partial_t\partial_{x_1}^k u_n = \partial_t\partial_{x_2}^k u_n = 0,\quad k=1,\ldots,m-2.$$

So, the boundary integral I_1 vanishes.

(c) Estimation of $I_{m-1} = -2\int_{\Gamma_1}\sum_{j=1}^{2}\left(\partial_{x_j}^{m-1}\partial_t u_n.\partial_{x_j}^m u_n\right)\nu_{x_j}.\omega(t)\,d\sigma$
We have

$$\partial_{x_j}^{m-1}u_n\left(t,\varphi(t)\cos\theta,\varphi(t)\sin\theta\right) = 0,\quad j=1,2.$$

Differentiating with respect to t, we obtain

$$\partial_t\partial_{x_1}^{m-1}u_n = -\varphi'(t)\left[\left(\cos\theta.\partial_{x_1}^m u_n + \sin\theta.\partial_{x_2}\partial_{x_1}^{m-1}u_n\right)\right]$$

and

$$\partial_t \partial_{x_2}^{m-1} u_n = -\varphi'(t)\left[\cos\theta.\partial_{x_1}\partial_{x_2}^{m-1}u_n + \sin\theta.\partial_{x_2}^m u_n\right].$$

Differentiating with respect to θ, we obtain

$$\begin{aligned}&\cos\theta.\partial_{x_2}\partial_{x_1}^{m-1}u_n = \sin\theta.\partial_{x_1}^m u_n\\ &\sin\theta.\partial_{x_1}\partial_{x_2}^{m-1}u_n = \cos\theta.\partial_{x_2}^m u_n.\end{aligned}$$

Taking into account these relationships we deduce

$$\begin{aligned}&I_{m-1}\\ &\quad= 2\int_0^{2\pi}\int_{\frac{1}{n}}^T \left[\cos^2\theta.\partial_{x_1}^m u_n + \sin\theta\cos\theta.\partial_{x_1}\partial_{x_2}^{m-1}u_n\right]\partial_{x_1}^m u_n\varphi'(t)\varphi(t).\omega(t)\,dtd\theta\\ &\qquad+2\int_0^{2\pi}\int_{\frac{1}{n}}^T \left[\cos\theta\sin\theta.\partial_{x_2}\partial_{x_1}^{m-1}u_n + \sin^2\theta.\partial_{x_2}^m u_n\right]\partial_{x_2}^m u_n\varphi'(t)\varphi(t).\omega(t)\,dtd\theta\\ &\quad= 2\int_0^{2\pi}\int_{\frac{1}{n}}^T \left[\left(\partial_{x_1}^m u_n\right)^2 + \left(\partial_{x_2}^m u_n\right)^2\right]\varphi'(t)\varphi(t).\omega(t)\,dtd\theta.\end{aligned}$$

Finally

$$\begin{aligned}2\langle\partial_t u_n, \mathcal{A}u_n\rangle &= \int_0^{2\pi}\int_{\frac{1}{n}}^T \left(\sum_{j=1}^2\left(\partial_{x_j}^m u_n\right)^2\right)\varphi'(t)\varphi(t).\omega(t)\,dtd\theta\\ &\quad+\int_{\Gamma_T}\left(\sum_{j=1}^2\left(\partial_{x_j}^m u_n\right)^2\right)(T,x_1,x_2).\omega(T)\,dx_1dx_2\\ &\quad-\int_{Q_n}\left(\sum_{j=1}^2\left(\partial_{x_j}^m u_n\right)^2\right).\omega'(t)\,dtdx_1dx_2. \end{aligned}\tag{3.3}$$

Remark 3.3. Observe that the integrals

$$\int_{\Gamma_T}\left(\sum_{j=1}^2\left(\partial_{x_j}^m u_n\right)^2\right)(T,x_1,x_2).\omega(T)\,dx_1dx_2$$

and

$$-\int_{Q_n}\left(\sum_{j=1}^2\left(\partial_{x_j}^m u_n\right)^2\right).\omega'(t)\,dtdx_1dx_2,$$

which appear in the last formula are nonnegative thanks to the assumptions (1.4) and (1.5) on the weight function ω. This is a good sign for our estimate because we can deduce immediately

$$\begin{aligned}\|f_n\|_{L^2_\omega(Q_n)}^2 \geq \|\partial_t u_n\|_{L^2_\omega(Q_n)}^2 + \|\mathcal{A}u_n\|_{L^2_\omega(Q_n)}^2&\\ +\int_0^{2\pi}\int_{\frac{1}{n}}^T \left(\sum_{j=1}^2\left(\partial_{x_j}^m u_n\right)^2\right)\varphi'(t)\varphi(t).\omega(t)\,dtd\theta.&\end{aligned}$$

So, if φ is an increasing function in the interval $(\frac{1}{n}, T)$, then

$$\int_0^{2\pi} \int_{\frac{1}{n}}^{T} \left(\sum_{j=1}^{2} \left(\partial_{x_j}^m u_n \right)^2 \right) \varphi'(t) \varphi(t) .\omega(t) \, dt d\theta \geq 0.$$

Consequently,

$$\|f_n\|^2_{L^2_\omega(Q_n)} \geq \|\partial_t u_n\|^2_{L^2_\omega(Q_n)} + \|\mathcal{A} u_n\|^2_{L^2_\omega(Q_n)} . \tag{3.4}$$

But, thanks to Lemma 2.2 and since φ is bounded in $(0, T)$, there exists a constant $C' > 0$ such that

$$\left\| \partial_{x_j}^l u_n \right\|^2_{L^2_\omega(Q_n)} \leq C' \|\mathcal{A} u_n\|^2_{L^2_\omega(Q_n)} , \quad l = 0, 1, \ldots, 2m-1; j = 1, 2.$$

Taking into account Lemma 3.3 and estimate (3.4), this proves the desired estimate of Proposition 3.2.

So, it remains to establish the estimate of Proposition 3.2 under the hypothesis (1.2). For this purpose, we need the following lemma

Lemma 3.4. *One has*

$$\begin{aligned} 2\langle \partial_t u_n, \mathcal{A} u_n \rangle = {} & 2 \int_{Q_n} \frac{\varphi'}{\varphi} \left(\sum_{j=1}^{2} x_j \partial_{x_j}^m u_n \right) \mathcal{A} u_n .\omega(t) \, dt dx_1 dx_2 \\ & + \int_{\Gamma_T} \sum_{j=1}^{2} \left(\partial_{x_j}^m u_n \right)^2 (T, x_1, x_2) .\omega(T) \, dx_1 dx_2. \end{aligned}$$

Proof. This result can be obtained by following step by step the proof of [9, Lemma 3.4]. □

Now, we continue the proof of Proposition 3.2. We have

$$\begin{aligned} & \left| \int_{Q_n} \frac{\varphi'}{\varphi} \left(\sum_{j=1}^{2} x_j \partial_{x_j}^m u_n \right) \mathcal{A} u_n .\omega(t) \, dt dx_1 dx_2 \right| \\ & \quad \leq \|\mathcal{A} u_n\|_{L^2_\omega(Q_n)} \sum_{j=1}^{2} \left\| \frac{\varphi'}{\varphi} x_j \partial_{x_j}^m u_n \right\|_{L^2_\omega(Q_n)} , \end{aligned}$$

but Lemma 2.2 yields for $j = 1, 2$

$$\begin{aligned} \left\| \frac{\varphi'}{\varphi} x_j \partial_{x_j}^m u_n \right\|^2_{L^2_\omega(Q_n)} & = \int_{\frac{1}{n}}^{T} \varphi'^2(t) \int_{\Omega_t} \left(\frac{x_j}{\varphi(t)} \right)^2 \left(\partial_{x_j}^m u_n \right)^2 .\omega(t) \, dt dx_1 dx_2 \\ & \leq \int_{\frac{1}{n}}^{T} \varphi'^2(t) \int_{\Omega_t} \left(\partial_{x_j}^m u_n \right)^2 .\omega(t) \, dt dx_1 dx_2 \\ & \leq C^2 \int_{\frac{1}{n}}^{T} \left(\varphi^m(t) \varphi'(t) \right)^2 \int_{\Omega_t} \left(\mathcal{A} u_n \right)^2 .\omega(t) \, dt dx_1 dx_2 \\ & \leq C^2 \epsilon^2 \|\mathcal{A} u_n\|^2_{L^2_\omega(Q_n)} , \end{aligned}$$

since $(\varphi^m(t)\varphi'(t)) \le \epsilon$ thanks to the condition (3.2). Then

$$\left|\int_{Q_n} \frac{\varphi'}{\varphi}\left(\sum_{j=1}^{2} x_j \partial_{x_j}^m u_n\right) \mathcal{A}u_n.\omega(t)\, dt dx_1 dx_2\right| \le 2C\epsilon \left\|\mathcal{A}u_n\right\|^2_{L^2_\omega(Q_n)}.$$

Therefore, Lemma 3.4 shows that

$$\begin{aligned}|2\langle \partial_t u_n, \mathcal{A}u_n\rangle| &\ge -2\left|\int_{Q_n} \frac{\varphi'}{\varphi}\left(\sum_{j=1}^{2} x_j \partial_{x_j}^m u_n\right) \mathcal{A}u_n.\omega(t)\, dt dx_1 dx_2\right| \\ &\quad + \int_{\Gamma_T}\sum_{j=1}^{2}\left(\partial_{x_j}^m u_n\right)^2 (T, x_1, x_2).\omega(T)\, dx_1 dx_2. \\ &\ge -4C\epsilon \left\|\mathcal{A}u_n\right\|^2_{L^2_\omega(Q_n)}.\end{aligned}$$

Hence

$$\begin{aligned}\|f_n\|^2_{L^2_\omega(Q_n)} &= \|\partial_t u_n\|^2_{L^2_\omega(Q_n)} + \|\mathcal{A}u_n\|^2_{L^2_\omega(Q_n)} + 2\langle \partial_t u_n, \mathcal{A}u_n\rangle \\ &\ge \|\partial_t u_n\|^2_{L^2_\omega(Q_n)} + (1-4C\epsilon)\|\mathcal{A}u_n\|^2_{L^2_\omega(Q_n)}.\end{aligned}$$

Then, it is sufficient to choose ϵ such that $1-4C\epsilon > 0$ to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L^2_\omega(Q_n)} \ge K_0 \|u_n\|_{H^{1,2m}_\omega(Q_n)},$$

and since

$$\|f_n\|_{L^2_\omega(Q_n)} \le \|f\|_{L^2_\omega(Q_n)},$$

there exists a constant $K_1 > 0$, independent of n satisfying

$$\|u_n\|_{H^{1,2m}_\omega(Q_n)} \le K_1 \|f_n\|_{L^2_\omega(Q_n)} \le K_1 \|f\|_{L^2_\omega(Q)}.$$

This completes the proof of Proposition 3.2.

3.3. *Passage to the limit*

Choose a sequence $(Q_n)_{n\in\mathbb{N}^*}$ of the domains defined above (see Section 3.1), such that $Q_n \subseteq Q$. Then, we have $Q_n \to Q$, as $n \to \infty$. Consider the solution $u_n \in H^{1,2m}_\omega(Q_n)$ of the Cauchy–Dirichlet problem

$$\begin{cases}\partial_t u_n + (-1)^m \displaystyle\sum_{j=1}^{2} \partial_{x_j}^{2m} u_n = f_n \in L^2_\omega(Q_n), \\ \partial_{x_j}^k u_n\Big|_{\partial Q_n \setminus \Gamma_T} = 0, \quad k = 0, 1, \ldots, m-1;\ j = 1, 2,\end{cases}$$

where $f_n = f|_{Q_n}$. Such a solution u_n exists by Theorem 3.1. Let $\widetilde{u_n}$ be the 0-extension of u_n to Q. In virtue of Proposition 3.2 for T small enough, we know that there exists a constant C such that

$$\|\widetilde{u_n}\|_{L^2_\omega(Q)} + \left\|\widetilde{\partial_t u_n}\right\|_{L^2_\omega(Q)} + \sum_{1\le|\alpha|\le 2m}\left\|\widetilde{\partial^\alpha u_n}\right\|_{L^2_\omega(Q)} \le C\|f\|_{L^2_\omega(Q)}.$$

This means that $\widetilde{u_n}, \widetilde{\partial_t u_n}, \widetilde{\partial^\alpha u_n}$ for $1 \leq |\alpha| \leq 2m$ are bounded functions in $L^2_\omega(Q)$. So, for a suitable increasing sequence of integers n_k, $k = 1, 2, \ldots$, there exist functions

$$u, v \text{ and } v_\alpha \; 1 \leq |\alpha| \leq 2m$$

in $L^2_\omega(Q)$ such that

$$\begin{array}{lll} \widetilde{u_{n_k}} & \rightharpoonup u & \text{weakly in } L^2_\omega(Q), \quad k \to \infty \\ \widetilde{\partial_t u_{n_k}} & \rightharpoonup v & \text{weakly in } L^2_\omega(Q), \quad k \to \infty \\ \widetilde{\partial^\alpha u_{n_k}} & \rightharpoonup v_\alpha & \text{weakly in } L^2_\omega(Q), \quad k \to \infty, \end{array}$$

$1 \leq |\alpha| \leq 2m$. Clearly,

$$v = \partial_t u, \qquad v_\alpha = \partial^\alpha u, \quad 1 \leq |\alpha| \leq 2m$$

in the sense of distributions in Q and so in $L^2_\omega(Q)$. So, $u \in H^{1,2m}_\omega(Q)$ and

$$\partial_t u + (-1)^m \sum_{j=1}^{2} \partial^{2m}_{x_j} u = f \quad \text{in } Q.$$

On the other hand, the solution u satisfies the boundary conditions

$$\partial^k_{x_j} u_n \Big|_{\partial Q_n \setminus \Gamma_T} = 0, \quad k = 0, 1, \ldots, m-1;\; j = 1, 2,$$

since

$$\forall n \in \mathbb{N}^*, \quad u|_{Q_n} = u_n.$$

This proves the existence of a solution to Problem (1.3). This ends the proof of Theorem 1.1 in the case of T small enough.

3.4. The general case

Assume that Q satisfies (1.1). In the case where T is not small enough, we set $Q = D_1 \cup D_2 \cup \Gamma_{T_1}$ where

$$\begin{aligned} D_1 &= \{(t, x_1, x_2) \in Q : 0 < t < T_1\} \\ D_2 &= \{(t, x_1, x_2) \in Q : T_1 < t < T\} \\ \Gamma_{T_1} &= \left\{(T_1, x_1, x_2) \in \mathbb{R}^3 : 0 \leq \sqrt{x_1^2 + x_2^2} \leq \varphi(T_1)\right\} \end{aligned}$$

with T_1 small enough. In the sequel, f stands for an arbitrary fixed element of $L^2_\omega(Q)$ and $f_i = f|_{D_i}$, $i = 1, 2$. We know that (see Section 3.3) there exists a unique solution $w_1 \in H^{1,2m}_\omega(D_1)$ of the problem

$$\begin{cases} \partial_t w_1 + (-1)^m \displaystyle\sum_{j=1}^{2} \partial^{2m}_{x_j} w_1 = f_1 \in L^2_\omega(D_1), \\ \partial^k_{x_j} w_1 \Big|_{\partial D_1 \setminus \Gamma_{T_1}} = 0, \quad k = 0, \ldots, m-1;\; j = 1, 2. \end{cases} \tag{3.5}$$

Hereafter, we denote the trace $w_1|_{\Gamma_{T_1}}$ by ψ which is in the Sobolev space $H_\omega^m(\Gamma_{T_1})$ because $w_1 \in H_\omega^{1,2m}(D_1)$ (see [12]). Now, consider the following problem in D_2

$$\begin{cases} \partial_t w_2 + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} w_2 = f_2 \in L_\omega^2(D_2), \\ w_2|_{\Gamma_{T_1}} = \psi, \\ \partial_{x_j}^k w_2\Big|_{\partial D_2 \smallsetminus (\Gamma_{T_1} \cup \Gamma_T)} = 0, \quad k = 0, \ldots, m-1;\ j = 1, 2. \end{cases} \tag{3.6}$$

We use the following result, which is a consequence of [12, Theorem 4.3, Vol. 2], to solve Problem (3.6).

Proposition 3.3. *Let R be the cylinder $]0,T[\times B(0,1)$ where $B(0,1)$ is the unit disk of $\mathbb{R}^2$, $f \in L_\omega^2(R)$ and $u_0 \in H_\omega^m(\gamma_0)$. Then, the problem*

$$\begin{cases} \partial_t u + (-1)^m \sum_{j=1}^{2} \partial_{x_j}^{2m} u = f \quad \text{in } R, \\ u|_{\gamma_0} = u_0, \\ \partial_{x_j}^k u\Big|_{\gamma_1} = 0, \quad k = 0, \ldots, m-1;\ j = 1, 2, \end{cases}$$

where $\gamma_0 = \{0\} \times B(0,1)$, $\gamma_1 =]0,T[\times \partial B(0,1)$, admits a (unique) solution $u \in H_\omega^{1,2m}(R)$ if and only if the following compatibility conditions are fulfilled

$$\partial_{x_j}^k u_0\Big|_{\partial\gamma_0} = 0, \quad k = 0, \ldots, m-1;\ j = 1, 2.$$

Thanks to the transformation

$$(t, x_1, x_2) \longmapsto (t, y_1, y_2) = (t, \varphi(t) x_1, \varphi(t) x_2),$$

we deduce the following result:

Proposition 3.4. *Problem (3.6) admits a (unique) solution $w_2 \in H_\omega^{1,2m}(D_2)$ if and only if the following compatibility conditions are fulfilled*

$$\partial_{x_j}^k \psi\Big|_{\partial\Gamma_{T_1}} = 0, \quad k = 0, \ldots, m-1;\ j = 1, 2.$$

Remark 3.4. We can observe that the boundary conditions of Problems (3.5) and (3.6) yield

$$w_1|_{\Gamma_{T_1}} = w_2|_{\Gamma_{T_1}}$$

and $\partial_{x_j}^k w_i\Big|_{\Gamma_{T_1}} \in H_\omega^{1-\frac{1}{2m}}(\Gamma_{T_1});\ i, j = 1, 2$. Then the compatibility conditions

$$\partial_{x_j}^k \psi\Big|_{\partial\Gamma_{T_1}} = 0, \quad k = 0, \ldots, m-1;\ j = 1, 2$$

are satisfied since $w_1|_{\Gamma_{T_1}} = \psi$.

Now, define the function u in Q by

$$u := \begin{cases} w_1 & \text{in } D_1 \\ w_2 & \text{in } D_2 \end{cases}$$

where w_1 and w_2 are the solutions of Problem (3.5) and Problem (3.6) respectively. Observe that $w_1|_{\Gamma_{T_1}} = w_2|_{\Gamma_{T_1}}$, see Remark 3.4, so

$$\partial_{x_j}^k w_1\Big|_{\Gamma_{T_1}} = \partial_{x_j}^k w_2\Big|_{\Gamma_{T_1}}, \quad k = 0, \dots, m-1;\ j = 1, 2.$$

This implies that $u \in H_\omega^{1,2m}(Q)$ and u is the (unique) solution of Problem (1.3) for an arbitrary T. This ends the proof of Theorem 1.1.

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