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Remarks on some coupled coincidence point results in partially ordered metric spaces

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Abstract. In this paper we have used a method of reducing coupled coincidence and coupled fixed point results in partially ordered metric spaces to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. Our results generalize, extend, unify and complement coupled coincidence point theorems established by Harjani et al. (2011) [18] and Razani and Parvaneh (2012) [26]. Also, by using this method several tripled coincidence and tripled common fixed point results in partially ordered metric spaces can be reduced to the coincidence and common fixed point results with one variable.

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1. Introduction and preliminaries

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for example: variational inequalities, optimization, approximation theory etc.

The notion of a coupled fixed point was introduced and studied by Opoitsev [21,22] and Opoitsev and Khurodze [23] and then by Guo and Lakhsmikantham in [15]. Bhaskar and Lakhsmikantham were the first to study coupled fixed points in connection to contractive type conditions in [14]. Fixed point theory as and coupled and tripled cases in ordered metric and cone metric spaces was studied in [1,3,5–7,9–11,13,14,16,19].

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The fixed point theorems in partially ordered metric spaces play a major role to prove the existence and uniqueness of solutions for some differential and integral equations.

One of the most interesting fixed point theorems in ordered metric spaces was investigated by Ran and Reurings [25] applied their result to linear and nonlinear matrix equations. Then, many authors obtained several interesting results in ordered metric spaces [3–8,10–20].

Bhaskar and Lakshmikantham [14] initiated the study of a coupled fixed point in ordered metric spaces and applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. For more works in coupled and coincidence point theorems, we refer the reader to [12,13,15,19] and references there in.

At first we need the following definitions and results.

Definition 1.1 (*[14,24]*). The three classes of following mappings are defined as

$$\Psi = \{\psi \mid \psi : [0,\infty) \to [0,\infty) \text{ is continuous and nondecreasing and } \psi^{-1}(\{0\}) = \{0\}\},$$

$$\mathscr{A} = \{\alpha \mid \alpha : [0,\infty) \to [0,\infty) \text{ is continuous and } \alpha(0) = 0\} \text{ and.}$$

$$\mathscr{B} = \{\beta \mid \beta : [0,\infty) \to [0,\infty) \text{ is lower semi-continuous and } \beta(0) = 0\}.$$

For a weak (ψ, α, β)-weak contraction in the frame of metric spaces see [11,24,26,27].

Definition 1.2 [5]. Let $(X, d, \preceq)$ be partially ordered metric space and $F : X^2 \to X, g : X \to X$ two mappings. The mapping F has the mixed g-monotone property if for any $x, y \in X$ hold:

$$gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and}$$
$$gy_1 \preceq gy_2 \Rightarrow F(x, y_2) \succeq F(x, y_1), \quad \text{for } y_1, y_2 \in X.$$

Note that, if $g = i_X$-identity mapping then F has a mixed monotone property.

Remark 1.3. It is clear that if $(X, \preceq)$ is a partially ordered set then $(X^2, \sqsubseteq)$ is also partially ordered with:

$$(x, y) \sqsubseteq (a, b) \Longleftrightarrow x \preceq a \wedge y \succeq b.$$

Definition 1.4 [5]. Let $F : X^2 \to X$ and $g : X \to X$. An element $(x, y) \in X^2$ is called a coupled coincidence point of F and g if

$$F(x, y) = gx \quad \text{and} \quad F(y, x) = gy,$$

while $(gx, gy) \in X^2$ is said a coupled point of coincidence of mappings F and g. Moreover, (x, y) is called a coupled common fixed point of F and g if

$$F(x, y) = gx = x \text{ and } F(y, x) = gy = y.$$

Remark 1.5. It is clear that (x, y) is a coupled coincidence point of F and g if and only if (x, y) is coincidence point for the mappings $T_F : X^2 \to X^2$ and $T_g : X^2 \to X^2$ which are defined by

$$T_F(x,y) = (F(x,y), F(y,x)) \quad \text{and} \quad T_g(x,y) = (gx, gy).$$

Definition 1.6. Mappings $f, g : X \to X$ are said to be compatible in a metric space (X,d) if

$$d(fgx_n, gfx_n) \to 0 \quad \text{as} \quad n \to \infty,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ in (X,d).

Definition 1.7 [10]. Let (X,d) be a metric space and $F : X^2 \to X$ and $g : X \to X$ be two mappings. We say that F and g are compatible if

$$d(gF(x_n, y_n), F(gx_n, gy_n)) \to 0 \quad \text{and}$$
$$d(gF(y_n, x_n), F(gy_n, gx_n)) \to 0, \quad \text{as} \quad n \to \infty,$$

whenever x_n and y_n are such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n.$$

The proof of the following Lemma is immediately.

Lemma 1.8. *Let (X,d) be a metric space. Define $D : X^2 \times X^2 \to \mathbb{R}_+$ by*

$$D((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}.$$

Then (X^2, D) is a new metric space.

Remark 1.9. It is easy to prove that mappings F and g are compatible in (X,d) if and only if T_F and T_g are compatible in (X^2, D).

We note that the used technique in this paper was considered also in [2,17].

Harjani et al. [18] obtained the following theorem for mappings with the mixed monotone property.

Theorem 1.10. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $F : X^2 \to X$ be a mapping having the mixed monotone property on X and continuous such that*

$$\psi(d(F(x,y), F(u,v))) \leqslant \psi(\max\{d(x,u), d(y,v)\}) - \varphi(\max\{d(x,u), d(y,v)\}) \tag{1.1}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, where ψ and φ are altering distance functions. If there exist $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Also, they proved that the above theorem is still valid for F not necessarily continuous, assuming that $(X, d, \preceq)$ is regular.

Razani and Parvaneh [26] proved the following result.

Theorem 1.11. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $F : X^2 \to X$ and $g : X \to X$ be such that $F(X^2) \subset g(X)$ and F is continuous. Assume that*

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) &\leqslant \alpha(\max\{d(gx,gu),d(gy,gv)\}) \\ &\quad - \beta(\max\{d(gx,gu),d(gy,gv)\}),\end{aligned} \tag{1.2}$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$, where $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ are such that, ψ is an altering distance function, α is continuous, ψ is lower semi-continuous, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$. Assume that:

(1) *F has the mixed g-monotone property,*
(2) *g is continuous and commutes with F.*

If there exist $x_0, y_0 \in X$ with $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X.

Also, they proved that the above theorem is still valid for F not necessarily continuous, assuming that $(X, d, \preceq)$ is regular.

2. Main results

Now, we are ready to state and prove our first result.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and $F : X^2 \to X$ and $g : X \to X$. Assume that there exist $\psi \in \Psi, \beta \in \mathcal{B}$, such that*

$$\begin{aligned}&\psi(\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}) \\ &\quad \leqslant \psi(\max\{d(gx,gu),d(gy,gv)\}) - \beta(\max\{d(gx,gu),d(gy,gv)\})\end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ for which $gx \preceq gu \wedge gy \succeq gv$ or $gx \succeq gu \wedge gy \preceq gv$. Assume that F and g satisfy the following conditions:

(1) *$F(X^2) \subset g(X)$;*
(2) *F has the mixed g-monotone property;*
(3) *F and g are continuous and compatible and (X, d) is a complete, or*
(4) *$(X, d, \preceq)$ is a regular and one of $F(X^2)$ or $g(X)$ is a complete;*
(5) *there exist $x_0, y_0 \in X$ such that*

$$gx_0 \preceq F(x_0, y_0) \wedge gy_0 \succeq F(y_0, x_0) \quad or \quad gx_0 \succeq F(x_0, y_0) \wedge gy_0 \preceq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

Remark 2.2.

(a) It is clear that the condition (1.2) is equivalent to the condition (2.1). Here, by using new metric space (X^2, D) we have obtained a method of reducing coupled coincidence and coupled fixed point results in (ordered) metric spaces to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. For other details of coupled case in ordered metric spaces see also [5].

(b) It is worth to notice that Theorem 1.11 holds if F and g are compatible instead commuting (see Step II in [26]). Indeed, since

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = x \quad \text{and} \quad \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = y,$$

then

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \quad \text{and}$$
$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0,$$

because F and g are compatible.

Therefore, now we have

$$\begin{aligned} d(gx, F(x,y)) &\leqslant d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(x,y)) \\ &\leqslant d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(gx_n, gy_n)) + d(F(gx_n, gy_n), F(x,y)) \\ &\to d(gx, gx) + 0 + d(F(x,y), F(x,y)) = 0, \quad \text{as} \quad n\to\infty, \end{aligned}$$

that is $F(x,y) = gx$. Similarly, we obtain that $F(y,x) = gy$.

Assertions similar to the following lemma were used in the frame of metric spaces in the course of proofs of several fixed point results in various papers (see, e.g., [10,24]).

Lemma 2.3. *Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence in (X,d), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tends to ε^+ when $k\to\infty$:*

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_k-1}), \quad d(x_{m_k+1}, x_{n_k}), \quad d(x_{n_k-1}, x_{m_k+1}).$$

The following Lemma is crucial for the proof of Theorem 2.1.

Lemma 2.4. *Let $(X,d,\preceq)$ be a partially ordered metric space and f and g be two self mappings on X. Assume that there exist $\psi\in\Psi$, $\beta\in\mathscr{B}$ such that*

$$\psi(d(fx,fy)) \leqslant \psi(d(gx,gy)) - \beta(d(gx,gy)) \tag{2.2}$$

for all $x,y\in X$ for which $gx\preceq gy$ or $gx\succeq gy$. If the following conditions hold:

(i) *f is a g-nondecreasing with respect to $\preceq$ and $fX\subset gX$;*
(ii) *there exists $x_0\in X$ such that $gx_0\preceq fx_0$;*
(iii) *f and g are continuous and compatible and (X,d) is a complete or;*
(iv) *$(X,d,\preceq)$ is a regular and one of fX or gX is a complete.*

Then f and g have a coincidence point in X.

Proof. If $gx_0 = fx_0$ then x_0 is a coincidence point of f and g. Therefore, let $gx_0 \prec fx_0$. Since $fX \subset gX$ we obtain Jungck sequence $y_n = fx_n = gx_{n+1}$ for all $n = 0, 1, 2, \ldots$ where $x_n \in X$ and by induction we get that $y_n \preceq y_{n+1}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$ then x_n is a coincidence point of f and g. Therefore, suppose that $y_n \neq y_{n+1}$ for each n. Now, we shall prove the following:

(1) $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$;
(2) $\{y_n\}$ is a Cauchy sequence.

Indeed, by putting $x = x_n,\ y = x_{n+1}$ in (2.2) we get

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &= \psi(d(fx_n, fx_{n+1}))\\ &\leqslant \psi(d(gx_n, gx_{n+1})) - \beta(d(gx_n, gx_{n+1}))\\ &= \psi(d(y_{n-1}, y_n)) - \beta(d(y_{n-1}, y_n))\\ &\leqslant \psi(d(y_{n-1}, y_n)),\end{aligned}$$

and since function ψ is nondecreasing, it follows that $d(y_n, y_{n+1}) \leqslant d(y_{n-1}, y_n)$, that is, exists $\lim_{n\to\infty} d(y_n, y_{n+1}) = d^* \geqslant 0$. If $d^* > 0$ we get from the previous relation $\psi(d^*) \leqslant \psi(d^*) - \beta(d^*)$, i.e., $d^* = 0$ which is a contradiction. Hence, we obtain that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Further, using Lemma 2.3 we shall prove that $\{y_n\}$ is a Cauchy sequence. Suppose this is not the case. Then, by Lemma 2.3 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to ε^+ when $k \to \infty$:

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_k-1}), \quad d(x_{m_k+1}, x_{n_k}), \quad d(x_{n_k-1}, x_{m_k+1}).$$

Putting $x = x_{m_k+1}, y = x_{n_k}$ in (2.2) we have

$$\psi(d(fx_{m_k+1}, fx_{n_k})) \leqslant \psi(d(gx_{m_k+1}, gx_{n_k})) - \beta(d(gx_{m_k+1}, gx_{n_k})),$$

that is,

$$\psi(d(y_{m_k+1}, y_{n_k})) \leqslant \psi(d(y_{m_k}, y_{n_k-1})) - \beta(d(y_{m_k}, y_{n_k-1})).$$

Letting $k \to \infty$ and utilizing the property of functions ψ and β, we get $\psi(\varepsilon) \leqslant \psi(\varepsilon) - \beta(\varepsilon) \leqslant 0$, which is a contradiction with $\varepsilon > 0$. We have proved that $\{y_n\}$ is a Cauchy sequence in (X, d).

Now by (iii) since (X, d) is a complete then there exists $z \in X$ such that $y_n \to z$. Then we have

$$d(fx_n, z) \to 0 \text{ and } d(gx_n, z) \to 0 \text{ as } n \to \infty.$$

Further, according to triangle inequality and since f and g are continuous and compatible, we get

$$\begin{aligned}d(fz, gz) &\leqslant d(fz, fgx_n) + d(fgx_n, gz)\\ &\leqslant d(fz, fgx_n) + d(fgx_n, gfx_n) + d(gfx_n, gz) \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

It follows that z is a coincidence point for f and g.

By (iv) follows that $y_n = fx_n = gx_{n+1} \to gz,\ z \in X$ (in both cases when fX or gX is a complete) and then $gx_n \preceq gz$ and by contractive condition (2.2) we have

$$\psi(d(fx_n, fz)) \leqslant \psi(d(gx_n, gz)) - \beta(d(gx_n, gz)).$$

By taking limit as $n \to \infty$ in above inequality, we obtain

$$\psi(d(gz, fz)) \leqslant \psi(d(gz, gz)) - \beta(d(gz, gz)) = 0 - 0 = 0,$$

and hence $fz = gz$. □

Proof of Theorem 2.1. Firstly, (2.1) implies

$$\psi(D(T_F(Y), T_F(V))) \leqslant \psi(D(T_g(Y), \quad T_g(V))) - \beta(D(T_g(Y), T_g(V))) \tag{2.3}$$

for all $Y = (x, y)$ and $V = (u, v)$ from X^2 for which $T_g(Y) \sqsubseteq T_g(V)$ or $T_g(V) \sqsubseteq T_g(Y)$.

Further,

(1) implies that $T_F(X^2) \subset T_g(X^2)$;
(2) implies that T_F is a T_g-nondecreasing with respect to $\sqsubseteq$ and $T_F(X^2) \subset T_g(X^2)$;
(3) implies that T_F and T_g are continuous and compatible and (X^2, D) is a complete or;
(4) implies that $(X^2, D, \sqsubseteq)$ is a regular and one of $T_F(X^2)$ or $T_g(X^2)$ is a complete;
(5) implies that there exists $Y_0 = (x_0, y_0) \in X^2$ such that $T_g(Y_0) \sqsubseteq T_F(Y_0)$ or $T_F(Y_0) \sqsubseteq T_g(Y_0)$.

All conditions of Lemma 2.4 for the ordered metric space (X^2, D) are satisfied. Therefore, the mappings T_F and T_g have a coincidence point in X^2. According to Remark 1.5 the mappings F and g have a coupled coincidence point. The proof of Theorem 2.1 is complete. □

The following examples shows that our results are proper generalizations of the results of Razani and Parvaneh [26, Theorems 2.2 and 2.3] and Bhaskar and Lakshmikantham [14, Theorem 2.1].

Example 2.5. Let $X = \mathbb{R}$ with the usual metric and order. Consider the mappings $F(x, y) = \frac{x^3 - 2y^3}{8}$ and $g(x) = x^3$. All the condition of Theorems 2.1 are satisfied. In particular, the mapping F has the mixed g-monotone property and we will check that F and g are compatible.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim gx_n = a \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim gy_n = b.$$

Then $\frac{a-2b}{8} = a$ and $\frac{b-2a}{8} = b$, wherefrom it follows that $a = b = 0$. Then

$$d(gF(x_n, y_n), F(gx_n, gy_n)) = \left| \left(\frac{x_n^3 - 2y_n^3}{8} \right)^3 - \frac{x_n^9 - 2y_n^9}{8} \right| \to 0, \quad \text{as } n \to \infty$$

and similarly

$$d(gF(y_n, x_n), F(gy_n, gx_n)) \to 0, \quad \text{as } n \to \infty.$$

Also, F and g do not commute and therefore coupled coincidence point of F and g cannot be obtained by Theorem 1.11 [26, Theorems 2.2].

Contractive condition (2.1) is satisfied with $\psi(t) = t$ and $\beta(t) = \frac{1}{2}t$ which follows from

$$\begin{aligned}
d(F(x,y), F(u,v)) &= \left| \frac{x^3 - 2y^3}{8} - \frac{u^3 - 2v^3}{8} \right| \\
&= \frac{1}{8} \left| (x3 - u^3) - 2(y^3 - v^3) \right| \\
&\leqslant \frac{1}{8}(d(gx, gu) + 2d(gy, gv)) \\
&\leqslant \frac{4}{8} \cdot \frac{d(gx, gu) + d(gy, gv)}{2} \\
&\leqslant \frac{1}{2} \max\{d(gx, gu), d(gy, gv)\} \\
&= \max d(gx, gu), d(gy, gv) - \frac{1}{2} \max\{d(gx, gu), d(gy, gv)\}
\end{aligned}$$

and

$$\begin{aligned}
d(F(y,x), F(v,u)) &= \left| \frac{y^3 - 2x^3}{8} - \frac{v^3 - 2u^2}{8} \right| \\
&= \frac{1}{8} |(y3 - v^3) - 2(x^3 - u^3)| \\
&\leqslant \frac{1}{8}(2d(gx, gu) + d(gy, gv)) \\
&\leqslant \frac{4}{8} \cdot \frac{d(gx, gu) + d(gy, gv)}{2} \\
&\leqslant \frac{1}{2} \max\{d(gx, gu), d(gy, gv)\} \\
&= \max\{d(gx, gu), d(gy, gv)\} - \frac{1}{2} \max\{d(gx, gu), d(gy, gv)\}
\end{aligned}$$

for all $x, y, u, v \in X$ for which is $gx \preceq gu \wedge gy \succeq gv$ *or* $gx \succeq gu \wedge gy \preceq gv$.

Hence,

$$\begin{aligned}
&\psi(\max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\}) \\
&\quad \leqslant \psi(\max\{d(gx, gu), d(gy, gv)\}) - \beta(\max\{d(gx, gu), d(gy, gv)\}),
\end{aligned}$$

for all $x, y, u, v \in X$ for which is $gx \preceq gu \wedge gy \succeq gv$ *or* $gx \succeq gu \wedge gy \preceq gv$. There exists a coupled coincidence point $(0, 0)$ of the mappings F and g.

Example 2.6. Let $X = \mathbb{R}$ with the usual metric and order. Define $F : X^2 \to X$ as $F(x,y) = \frac{1}{2}x - \frac{1}{3}y$ for all $x, y \in X$ and $g : X \to X$ with $g(x) = x$ for all $x \in X$.

Let $\psi, \beta : [0, +\infty) \to [0, +\infty)$ be defined by $\psi(t) = t$ and $\beta(t) = \frac{1}{6}t$. Clearly, ψ is an altering distance function, β is lower semi-continuous, $\psi(0) = \beta(0) = 0$ and $\psi(t) > 0$ and $\beta(t) > 0$ for all $t > 0$.

Now, let $x \preceq u$ and $y \succeq v$. So, we obtain

$$\begin{aligned}
&\psi(\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}) \\
&\quad = \max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\} \\
&\quad = \max\left\{\left|\frac{1}{2}(x-u)-\frac{1}{3}(y-v)\right|,\left|\frac{1}{2}(y-v)-\frac{1}{3}(x-u)\right|\right\} \\
&\quad \leqslant \frac{5}{6}\max\{d(x,u),d(y,v)\} \\
&\quad = \psi(\max\{d(x,u),d(y,v)\}) - \beta(\max\{d(x,u),d(y,v)\}).
\end{aligned}$$

Hence, all of the conditions of Theorem 2.1 are satisfied. Moreover, $(0,0)$ is the coupled coincidence point of F and g.

However, inequality

$$d(F(x,y),F(u,v)) \leqslant \frac{k}{2}(d(x,u)+d(y,v)) \tag{2.4}$$

from [14, Theorem 2.1] is not true. Indeed, let $(x,y) = (1,0)$ and $(u,v) = (0,0)$. Then,

$$d(F(1,0),F(0,0)) = \frac{1}{2} > \frac{k}{2} = \frac{k}{2}(d(1,0)+d(0,0))$$

for all $k \in [0,1)$.

Hence, the existence of a coupled coincidence point of F and g cannot be obtained by results from [14].

Remark 2.7. It is clear that (2.3) implies (2.4) for $\psi(t) = t$ and $\beta(t) = (1-k)t$.

The following result is immediately consequence of previous Theorem 2.1 and [3, Theorem 13].

Theorem 2.8. *Let $(X, d, \preceq)$ be a partially ordered metric space and $F : X^2 \to X$ and $g : X \to X$. Assume that there exist $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ where ψ is an altering distance function, α is continuous, ψ is lower semi-continuous, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$, such that*

$$\begin{aligned}
&\psi(\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}) \\
&\quad \leqslant \alpha(\max\{d(gx,gu),d(gy,gv)\}) - \beta(\max\{d(gx,gu),d(gy,gv)\})
\end{aligned} \tag{2.5}$$

for all $x, y, u, v \in X$ for which $gx \preceq gu \wedge gy \succeq gv$ or $gx \succeq gu \wedge gy \preceq gv$. Assume that F and g satisfy the following conditions:

(1) *$F(X^2) \subset g(X)$;*
(2) *F has the mixed g-monotone property;*
(3) *F and g are continuous and compatible and (X, d) is a complete, or*

(4) $(X, d, \preceq)$ *is a regular and one of* $F(X^2)$ *or* $g(X)$ *is a complete;*
(5) *there exist* $x_0, y_0 \in X$ *such that*

$$gx_0 \preceq F(x_0, y_0) \wedge gy_0 \succeq F(y_0, x_0) \text{or} gx_0 \succeq F(x_0, y_0) \wedge gy_0 \preceq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

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