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Remark on the system of nonlinear variational inclusions

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Abstract. We prove the existence of a solution to the system of nonlinear variational inclusions problem. We provide examples of applications related to a coupled best approximations theorem for multivalued mappings and a multivalued coupled coincidence point.

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1. Introduction and preliminaries

In the paper [12] Verma introduced the system of nonlinear variational inclusions (SNVI) problem: finding $(x_0,y_0) \in H_1 \times H_2$ such that

$$0 \in S(x_0, y_0) + M(x_0), \quad 0 \in T(x_0, y_0) + N(y_0), \tag{1}$$

where H_1 and H_2 are real Hilbert spaces, S: $H_1 \times H_2 \to H_1$, T: $H_1 \times H_2 \to H_2$ are mappings and M: $H_1 \to 2^{H_1}$, N: $H_2 \to 2^{H_2}$ are multivalued mappings.

1. If $M(\cdot) = \partial f(\cdot)$ and $N(\cdot) = \partial g(\cdot)$ where $\partial f(\cdot)$ is the subdifferential of a proper, convex and lower semicontinuous functions f: $H_1 \to \mathbb{R} \cup \{+\infty\}$ and g: $H_1 \to \mathbb{R} \cup \{+\infty\}$ then problem SNVI reduces to finding $(x_0,y_0) \in K_1 \times K_2$ such that

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$$\langle S(x_0, y_0), x - x_0 \rangle + f(x) - f(x_0) \geqslant 0 \qquad \text{for all } x \in K_1, \tag{2}$$

$$\langle T(x_0, y_0), y - y_0 \rangle + g(x) - g(y_0) \geqslant 0 \qquad \text{for all } y \in K_2, \tag{3}$$

where K_1 and K_2, respectively, are nonempty closed convex subsets of H_1 and H_2.

2. When $M(x) = \partial_{K_1}(x)$ and ∂_{K_2} denote indicator functions of K_1 and K_2, respectively, the SNVI problem (1) reduces to system of nonlinear variational inequalities problem: finding $(x_0,y_0) \in K_1 \times K_2$ such that

$$\langle S(x_0, y_0), x - x_0 \rangle \geqslant 0 \qquad \text{for all } x \in K_1, \tag{4}$$

$$\langle T(x_0, y_0), y - y_0 \rangle \geqslant 0 \qquad \text{for all } y \in K_2. \tag{5}$$

3. If $H_1 = H_2 = X$, $S(x,y) = - F(x,y)$, $T(x,y) = - F(y,x)$, $M(x) = G(x)$, $N(x) = G(x)$ for all $x,y \in X$ then (1) reduces to finding $(x_0,y_0) \in X \times X$, such that

$$F(x_0, y_0) \in G(x_0), \qquad F(y_0, x_0) \in G(y_0), \tag{6}$$

which is a multivalued coupled coincidence point problem.

4. If G is a single-valued mapping and $G(x) = \{g(x)\}$ for all $x \in X$ then (6) reduces to finding $(x_0,y_0) \in X \times X$, such that

$$F(x_0, y_0) = g(x_0), \quad F(y_0, x_0) = g(y_0), \tag{7}$$

which is known as a coupled coincidence point problem, considered by Lakshmikantham and Ćirić [7].

5. If g is an identity mapping, then (7) is equivalent to finding $(x_0,y_0) \in X \times X$, such that

$$F(x_0, y_0) = x_0, \quad F(y_0, x_0) = y_0, \tag{8}$$

which is known as a coupled fixed point problem, considered by Bhaskar and Lakshmikantham [3].

The aim of this paper is to obtain the results existence of a solution of SNVI problem (1) using the KKM technique.

We need the following definitions and results.

Let $F:X \multimap Y$ be a multivalued mapping from a set X into the power set of a set Y. For $A \subseteq X$, let $F(A) = \cup \{F(x) : x \in A\}$. For any $B \subseteq Y$, the lower inverse and upper inverse of B under F are defined by

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\} \quad \text{and} \quad F^+(B) = \{x \in X : F(x) \subseteq B\},$$

respectively.

A mapping $F:X \multimap Y$ is upper (lower) semicontinuous on X if and only if for every open $V \subseteq Y$, the set $F^+(V)$ ($F^-(V)$) is open. A mapping $F:X \multimap Y$ is continuous if and only if it is upper and lower semicontinuous. A mapping $F:X \multimap Y$ with compact values is continuous if and only if F is a continuous mapping in the Hausdorff distance, see for example [4].

Let X be a normed space. If A and B are nonempty subsets of X, we define

$$A + B = \{a + b : a \in A, b \in B\} \text{ and } \|A\| = \inf\{\|a\| : a \in A\}.$$

We will use the notion a C-convex map for multivalued maps.

Definition 1.1 (Borwein [5]). Let X and Y be real vector spaces, K a nonempty convex subset of X and C is a cone in Y. A multivalued mapping $F{:}K \multimap Y$ is said to be C-convex if,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C \tag{9}$$

for all $x_1,x_2 \in K$ and all $\lambda \in [0,1]$.

A mapping F is convex if it satisfies condition (9) with $C = \{0\}$ (see for example, Nikodem [8], Nikodem and Popa [9]). If F is a single-valued mapping, $Y = \mathbb{R}$ and $C = [0, +\infty)$, we obtain the standard definition of convex functions. The convex multivalued mappings play an important role in convex analysis, economic theory and convex optimization problems see for example [1,2,5,11].

Lemma 1.1 (Nikodem [8]). *If a multivalued mapping $F{:}K \multimap Y$ is C-convex, then*

$$\lambda_1 F(x_1) + \cdots + \lambda_n F(x_n) \subset F(\lambda_1 x_1 + \cdots + \lambda_n x_n) + C, \tag{10}$$

for all $n \in \mathbb{N}, x_1, \ldots, x_n \in K$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\lambda_1 + \ldots + \lambda_n = 1$.

From Lemma 1.1 we easily obtain the following lemma.

Lemma 1.2. *Let K be a convex subset of normed space X if the multivalued mapping $F{:}K \multimap X$ is convex, then*

$$\left\| F\left(\sum_{i=1}^{n} \lambda_i x_i\right) + u \right\| \leqslant \sum_{i=1}^{n} \lambda_i \|F(x_i) + u\| \tag{11}$$

for all $n \in \mathbb{N}, x_1, \ldots, x_n \in K, u \in X$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\lambda_1 + \ldots + \lambda_n = 1$.

Definition 1.2 (Prolla [10]). Let X be a normed space and C a nonempty convex subset of X. A map $g{:}C \to X$ is almost affine if for all $x,y \in C$ and $u \in C$

$$\|g(\lambda x + (1-\lambda)y) - u\| \leqslant \lambda \|g(x) - u\| + (1-\lambda)\|g(y) - u\|$$

for each λ with $0 < \lambda < 1$.

Remark 1.1. If $F: K \to K$ is single valued and almost-affine mapping then the condition (11) holds.

Definition 1.3. Let K be a nonempty subset of a topological vector space X. A multivalued mapping $H: K \to 2^X$ is called a KKM mapping if, for every finite subset $\{x_1,x_2, \ldots ,x_n\}$ of K,

$$co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} H(x_i),$$

where co denotes the convex hull.

Lemma 1.3 (Ky Fan [6], Lemma 1). *Let X be a topological vector space, K be a nonempty subset of X and $H{:}K \to 2^X$ a mapping with closed values and KKM mapping. If $H(x)$ is compact for at least one $x \in K$ then $\cap_{x\in K} H(x) \neq \emptyset$.*

2. Main result

Lemma 2.1. *Let X be a normed space, K a nonempty convex compact subset of X, $S,T : K\times K \to X$ continuous mappings and $M,N : K \to 2^X$ continuous convex mappings with compact values. Then there exists $(x_0,y_0) \in K\times K$ such that*

$$\|M(x_0) + S(x_0, y_0)\| + \|N(y_0) + T(x_0, y_0)\| = \inf_{(x,y)\in K\times K}\{\|M(x) + S(x_0, y_0)\| + \|N(y) + T(x_0, y_0)\|\}.$$

Proof. Define a multivalued mapping $H : K\times K \to 2^{K\times K}$ by

$$H(z,t) = \{(x,y) \in K\times K : ||M(x) + S(x,y)|| + ||N(y) + T(x,y))|| \leqslant ||M(z) + S(x,y)|| + ||N(t) + T(x,y)||\} \quad \text{for each } (z,t) \in K\times K.$$

We have that $(z,t) \in H(z,t)$, hence $H(z,t)$ is nonempty for all $(z,t) \in K\times K$.

The mappings S,T,M and N are continuous and we have that $H(z,t)$ is closed for each $(z,t) \in K\times K$.

Since $K\times K$ is a compact set we have that $H(z,t)$ is compact for each $(z,t) \in K\times K$.

Mapping H is a KKM map. Namely, suppose for any $(z_i,t_i) \in K\times K, i \in \{1, \ldots ,n\}$, there exists

$$(z_0, t_0) \in co\{(z_1, t_1), \ldots, (z_n, t_n)\}, \tag{12}$$

such that

$$(z_0, t_0) \notin \bigcup_{i=1}^{n} H(z_i, t_i). \tag{13}$$

From (12) we obtain that there exist $\lambda_i \geqslant 0, i \in \{1, \ldots ,n\}$, such that

$$(z_0, t_0) = \sum_{i=1}^{n}\lambda_i(z_i, t_i) \text{ and } \sum_{i=1}^{n}\lambda_i = 1.$$

Since M is convex mapping, from Lemma 1.2, we have

$$\|M(z_0) + S(z_0, t_0)\| \leqslant \sum_{i=1}^{n}\lambda_i\|M(z_i) + S(z_0, t_0)\|.$$

In a similar way, since N is convex mapping, we have

$$\|N(t_0) + T(z_0, t_0)\| \leqslant \sum_{i=1}^{n} \lambda_i \|N(t_i) + T(z_0, t_0)\|.$$

In other hand, from (13) we obtain

$$\|M(z_0) + S(z_0, t_0)\| + \|N(t_0) + T(z_0, t_0)\| > \|M(z_i) + S(z_0, t_0)\| + \|N(t_i) + T(z_0, t_0)\|$$

for all $i \in \{1, \ldots, n\}$. This is a contradiction and H is KKM mapping. From Lemma 1.3 it follows that there exists $(x_0, y_0) \in K \times K$ such that

$$(x_0, y_0) \in H(x, y) \text{ for all } (x, y) \in K \times K.$$

So,

$$\|M(x_0) + S(x_0, y_0)\| + \|N(y_0) + T(x_0, y_0)\| \leqslant \|M(x) + S(x_0, y_0)\| + \|N(y) + T(x_0, y_0)\|,$$

for all $(x, y) \in K \times K$. □

Applying Lemma 2.1, we have the following theorem on existence solutions the SNVI problem (1).

Theorem 2.1. *In addition to the hypotheses of Lemma* 2.1 *suppose that for every* $(x, y) \in K \times K$

$$0 \in M(K) + S(x, y) \text{ and } 0 \in N(K) + T(x, y). \tag{14}$$

Then there exists $(x_0, y_0) \in K \times K$ *such that*

$$0 \in S(x_0, y_0) + M(x_0) \text{ and } 0 \in T(x_0, y_0) + N(y_0).$$

Proof. From Lemma 2.1, we have that there exists $(x_0, y_0) \in K \times K$ such that

$$\|M(x_0) + S(x_0, y_0)\| + \|N(y_0) + T(x_0, y_0)\| = \inf_{(x,y) \in K \times K} \{\|M(x) + S(x_0, y_0)\| + \|N(y) + T(x_0, y_0)\|\}.$$

From condition (14) we obtain that

$$\inf_{(x,y) \in K \times K} \{\|M(x) + S(x_0, y_0)\| + \|N(y) + T(x_0, y_0)\|\} = 0,$$

so, we have

$$\|M(x_0) + S(x_0, y_0)\| + \|N(y_0) + T(x_0, y_0)\| = 0,$$

hence,

$$0 \in M(x_0) + S(x_0, y_0) \text{ and } 0 \in N(y_0) + T(x_0, y_0).$$

□

3. A coupled coincidence point

Applying Theorem 2.1, we have the following multivalued coupled coincidence point theorem.

Theorem 3.1. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping and $G : K \to 2^X$ continuous convex mapping with compact values such that $F(K \times K) \subseteq G(K)$. Then F and G have a multivalued coupled coincidence point.*

Proof. Put

$$S(x,y) = -F(x,y),\ T(x,y) = -F(y,x) \text{ for } x,y \in K,$$
$$M(x) = G(x),\ N(y) = G(y) \text{ for } x,y \in K.$$

Then S,T,M and N satisfy all of the requirements of Theorem 2.1. Therefore, there exists $(x_0,y_0) \in K$ such that

$$0 \in -F(x_0, y_0) + G(x_0) \text{ and } 0 \in -F(y_0, x_0) + G(y_0)$$

i.e.

$$F(x_0, y_0) \in G(x_0) \text{ and } F(y_0, x_0) \in G(y_0).$$

□

Corollary 3.1. *Let X be a normed space, K a nonempty convex compact subset of $X, F : K \times K \to X$ continuous mapping and $g : K \to X$ continuous almost-affine mapping such that $F(K \times K) \subseteq g(K)$. Then F and g have a coupled coincidence point.*

Proof. Let $G(x) = \{g(x)\}$ and apply Theorem 3.1. □

Corollary 3.2. *Let X be a normed space, K a nonempty convex compact subset of $X, F : K \times K \to K$ continuous mapping. Then F has a coupled fixed point.*

Proof. Let $G(x) = \{x\}$ and apply Theorem 3.1. □

4. A coupled best approximations

Theorem 4.1. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping and $G : K \to 2^X$ continuous convex mapping with compact values. Then there exists $(x_0,y_0) \in K \times K$ such that*

$$\begin{aligned}&\|G(x_0) - F(x_0, y_0)\| + \|G(y_0) - F(y_0, x_0)\| \\ &\quad = \inf_{(x,y)\in K\times K}\{\|G(x) - F(x_0, y_0)\| + \|G(y) - F(y_0, x_0)\|\}. \qquad (15)\end{aligned}$$

Proof. Put

$$S(x,y) = -F(x,y), \quad T(x,y) = -F(y,x) \text{ for } x,y \in K,$$
$$M(x) = G(x), \quad N(y) = G(y) \text{ for } x,y \in K.$$

Then S,T,M and N satisfy all of the requirements of Lemma 2.1. Therefore, there exists $(x_0,y_0) \in K \times K$ such that (15) holds. □

Corollary 4.1. *Let X be a normed space, K a nonempty convex compact subset of X, $F: K \times K \to X$ continuous map and $g: K \to X$ continuous almost-affine map. Then there exists $(x_0, y_0) \in K \times K$ such that*

$$\|g(x_0) - F(x_0,y_0)\| + \|g(y_0) - F(y_0,x_0)\| = \inf_{(x,y)\in K\times K}\{\|g(x) - F(x_0,y_0)\| + \|g(y) - F(y_0,x_0)\|\}.$$

Corollary 4.2. *Let X be a normed space, K a nonempty convex compact subset of X, $F: K \times K \to X$ continuous mapping. Then there exists $(x_0,y_0) \in K \times K$ such that*

$$\|x_0 - F(x_0,y_0)\| + \|y_0 - F(y_0,x_0)\| = \inf_{(x,y)\in K\times K}\{\|x - F(x_0,y_0)\| + \|y - F(y_0,x_0)\|\}. \tag{16}$$

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