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Redheffer type inequalities for modified Bessel functions

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Abstract. In this short note, we give new proofs of Redheffer's inequality for modified Bessel functions of first kind published by Ling Zhu (2011). In addition, using the Grosswald formula we prove new Redheffer type inequality for the modified Bessel functions of the second kind.

Keywords: Sharpening Redheffer type inequalities; Modified Bessel functions

1. Introduction

This following inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad \text{for all } x \in \mathbb{R} \tag{1}$$

is known in literature as Redheffer's inequality [5]. J. P. Williams [7] proved the inequality (1). Chen et al. [2] obtained the following three Redheffer type inequalities for the functions $\cos x$, $\frac{\sinh x}{x}$ and $\cosh x$

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \quad x \in [0, \frac{\pi}{2}]. \tag{2}$$

$$\cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad x \in [0, \frac{\pi}{2}[. \tag{3}$$

$$\frac{\sinh x}{x} \geq \frac{\pi^2 + x^2}{\pi^2 - x^2}, \quad x \in [0, \pi]. \tag{4}$$

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Recently, some extensions of inequalities (3) and (4) involving modified Bessel function have been shown in Baricz [1]. Define the function $\mathcal{I}_p : \mathbb{R} \longrightarrow [1, +\infty[$ by

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) \frac{I_p(x)}{x^p} = \sum_{n \geq 0} \frac{\left(\frac{1}{4}\right)^n}{(p+1)_n n!} x^{2n}$$

where $(p+1)_n = (p+1)(p+2)\cdots(p+n) = \frac{\Gamma(p+n+1)}{\Gamma(p+1)}$ is the well-known Pochhammer (or Appel) symbol defined in terms of Euler's gamma function, and $I_p(x)$ is the modified Bessel function. Recall that in 2007 Baricz [1] proved that for all $p > -1$, the following inequality

$$\mathcal{I}_p(x) \leq \frac{j_{p,1}^2 + x^2}{j_{p,1}^2 - x^2}, \quad x \in]0, j_{p,1}[$$

where $j_{p,n}$ is the nth positive zero of the Bessel function $J_p(x)$.

In 2008, L. Zhu and J. Sun [9] extended and sharpened inequalities (3) and (4) as follows.

Theorem 1. *Let* $0 < x < r$. *Then*

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\alpha \leq \frac{\sinh x}{x} \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\beta \tag{5}$$

holds if and only if $\alpha \leq 0$ *and* $\beta \geq \frac{r^2}{12}$.

Theorem 2. *Let* $0 \leq x < r$. *Then*

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\alpha \leq \cosh x \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\beta \tag{6}$$

holds if and only if $\alpha \leq 0$ *and* $\beta \geq \frac{r^2}{4}$.

Next, let us recall the following result which will be used in the sequel.

Lemma 1. *Let* $f, g : [a, b] \longrightarrow \mathbb{R}$ *two continuous functions which are differentiable on* (a, b). *Further, let* $g' \neq 0$ *on* (a, b). *If* $\frac{f'}{g'}$ *is increasing (or decreasing) on* (a, b), *then the functions* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *and* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *are also increasing (or decreasing) on* (a, b).

Proof. Denoting by $\phi(x) = \frac{f(x)-f(a)}{g(x)-g(a)}$, a simple calculation reveals that the numerator of ϕ' equals

$$\left\{\frac{f'(x)}{g'(x)} - \frac{f(x) - f(a)}{g(x) - g(a)}\right\} g'(x)(g(x) - g(a))$$

from which the stated result follows upon applying Cauchy's mean value theorem and the monotonicity hypotheses in the lemma. ■

2. The Redheffer type inequalities involving modified Bessel functions

In this theorem, we give a new proof of Redheffer type inequality involving modified Bessel functions published by Ling Zhu in [8].

Theorem 3 *([8]). Let $0 < x < r$ and $p > -1$ then the following inequalities*

$$\left(\frac{r^2+x^2}{r^2-x^2}\right)^{\alpha} \leq \mathcal{I}_p(x) \leq \left(\frac{r^2+x^2}{r^2-x^2}\right)^{\beta} \tag{7}$$

holds, if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{8(p+1)}$.

Proof. We consider the function $f_p :]0, r[\longrightarrow \mathbb{R}$ defined by

$$f_p(x) = \frac{r^2}{8(p+1)} \log\left(\frac{r^2+x^2}{r^2-x^2}\right) - \log\left(\mathcal{I}_p(x)\right).$$

Then $f_p(0) = 0$, and

$$f_p'(x) = \frac{r^4 x}{2(p+1)(r^4-x^4)} - \frac{\mathcal{I}_p'(x)}{\mathcal{I}_p(x)}.$$

Now, using the relation [6, p. 79]

$$\frac{d}{dx}\left(\frac{I_p(x)}{x^p}\right) = \frac{I_{p+1}(x)}{x^p} \tag{8}$$

and the Mittag-Leffler expansion for the modified Bessel functions of first kind, which becomes [3, Eq. 7.9.3]

$$\frac{I_{p+1}(x)}{I_p(x)} = \sum_{n=1}^{\infty} \frac{2x}{j_{p,n}^2 + x^2},$$

we have

$$f_p'(x) = \frac{r^4 x}{2(p+1)(x^4-r^4)} - \sum_{n=1}^{\infty} \frac{2x}{j_{p,n}^2 + x^2}.$$

Using now the Rayleigh formula [6, p. 502]

$$\sum_{n=1}^{\infty} \frac{1}{j_{p,n}^2} = \frac{1}{4(p+1)}$$

we get

$$
\begin{aligned}
f_p'(x) &= \frac{2r^4x}{(r^4-x^4)}\sum_{n=1}^{\infty}\frac{1}{j_{p,n}^2} - \sum_{n=1}^{\infty}\frac{2x}{j_{p,n}^2+x^2} \\
&= 2xr^4\left[\sum_{n=1}^{\infty}\frac{1}{j_{p,n}^2(r^4-x^4)} - \sum_{n=1}^{\infty}\frac{1}{r^4(j_{p,n}^2+x^2)}\right] \\
&= 2x^3\sum_{n=1}^{\infty}\frac{r^4+j_{p,n}^2x^2}{j_{p,n}^2(r^4-x^4)(j_{p,n}^2+x^2)}.
\end{aligned}
\tag{9}
$$

Therefore the function f_p is increasing on $]0,r[$ for all $p>-1$, and hence $f_p(x)\ge f_p(0)=0$, which implies the right-hand side of (7). To prove the left-hand side of (7), from the recurrence formula (8) we conclude that the function $x\longmapsto \mathcal{I}_p(x)$ is increasing on $]0,r[$ for all $p>-1$ and hence $\mathcal{I}_p(x)\ge 1$. We shall establish the result for the boundary cases $\alpha=0$ and $\beta=\frac{r^2}{8(p+1)}$ and show that these are sharp bounds. We consider the function $g_p:]0,r[\longrightarrow\mathbb{R}$, defined by

$$
g_p(x)=\frac{\log\left(\mathcal{I}_p(x)\right)}{\log\left(\frac{x^2+r^2}{x^2-r^2}\right)}.
$$

We note that $\lim_{x\longrightarrow r} g_p(x)=0=\alpha$ and using the l'Hospital rule we have

$$
\begin{aligned}
\lim_{x\longrightarrow 0} g_p(x) &= \lim_{x\longrightarrow 0}\frac{\mathcal{I}_p'(x)}{\mathcal{I}_p(x)}\cdot\frac{r^4-x^4}{4xr^4} \\
&= \sum_{n=1}^{\infty}\frac{2x}{j_{p,n}^2+x^2}\cdot\frac{r^4-x^4}{4xr^4}=\beta.
\end{aligned}
\tag{10}
$$

Therefore $\alpha=0$ and $\beta=\frac{r^2}{8(p+1)}$ are indeed the best possible constants. Alternatively, inequality (7) can be proved by using the monotone form of l'Hospital's rule. Namely, it is enough to observe that

$$
x\longmapsto\frac{\frac{d}{dx}\log\left(\mathcal{I}_p(x)\right)}{\frac{d}{dx}\log\left(\frac{x^2+r^2}{x^2-r^2}\right)}=\frac{1}{2r^2}\sum_{n=1}^{\infty}\frac{r^4-x^4}{j_{p,n}^2+x^2}
$$

is decreasing on $]0,r[$ as each terms in the above series is decreasing. Therefore g_p is decreasing too on $]0,r[$ by Lemma 1 and hence

$$
\alpha=\lim_{x\longrightarrow r} g_p(x)\le g_p(x)\le\lim_{x\longrightarrow 0} g_p(x)=\beta,
$$

which gives the inequality (7). So the proof of Theorem 3 is complete. ■

Now, for $p>0$ let us consider the function $\mathcal{K}_p:(0,\infty)\longrightarrow(0,1)$ defined by

$$
\mathcal{K}_p(x)=\frac{x^pK_p(x)}{2^{p-1}\Gamma(p)},
$$

where $K_p(x)$ is the modified Bessel functions of the second kind.

Theorem 4. *Let $n \in \mathbb{N}_0$ and $0 < x < r$, then the following inequality*

$$e^x \mathcal{K}_{n+\frac{1}{2}}(x) \leq \left(\frac{r+x}{r-x}\right)^{\gamma_n}, \tag{11}$$

holds, if and only if $\gamma_n \geq \frac{-r}{2}\beta_n$, where $\beta_n = \sum_{j=1}^{n} \frac{1}{\alpha_j}$ and $\alpha_1, \ldots, \alpha_n$ are the zeros of $\mathcal{K}_{n+\frac{1}{2}}$.

Proof. Let $n \in \mathbb{N}_0,\ r > 0$ and $\alpha_1, \ldots, \alpha_n$ are the zeros of $\mathcal{K}_{n+\frac{1}{2}}(x)$ (see [6], p. 511–513). We consider the function h_n defined by

$$h_n(x) = e^x \mathcal{K}_{n+\frac{1}{2}}(x),$$

where $0 < \alpha_j < x < r$ for all $j = 1, \ldots, n$. Then $h_n(0) = 0$. From the Grosswald formula in [4] we deduce that the function

$$x \longmapsto \frac{\frac{d}{dx}\log h_n(x)}{\frac{d}{dx}\log\left(\frac{r+x}{r-x}\right)} = \frac{1}{2r}\sum_{j=1}^{n}\frac{r^2-x^2}{x-\alpha_j}$$

is decreasing on $(0, r)$ as each terms in the above series are decreasing. Thus, the function $h_n(x)$ is also decreasing by Lemma 1. Furthermore,

$$\lim_{x \longrightarrow 0} h_n(x) = -\frac{r}{2}\sum_{j=1}^{n}\frac{1}{\alpha_j},$$

which completes the proof. ■

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