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Rate of convergence for generalized Baskakov operators

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Abstract In the present paper, we consider the generalized Baskakov operators having the weight functions of Beta basis functions. We study the rate of convergence for functions having derivatives of bounded variation.

Introduction

To approximate Lebesgue integrable functions on the interval $[0,\infty)$, Gupta [6] introduced the integral modification of the well known Baskakov operators by taking the weight functions of Beta basis functions. It was observed in [6] that by taking weights of Beta basis functions, one can have better approximation than the usual Baskakov–Durrmeyer operators [7]. In [6] the author has estimated an asymptotic formula and error estimation in simultaneous approximation for the Baskakov–Beta operators. In recent years a lot of work has been done on such

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operators, we refer to some of the important papers on the recent developments on similar types of operators (see [1–5,8], etc.). We can define the Baskakov–Beta operators in generalized form as:

$$V_{n,r}(f,x)=\frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)f(t)dt, \qquad (1.1)$$

where $n\in N$, $r\in N^0$, $n>r$ and the Baskakov and Beta basis functions are respectively defined as

$$p_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},\quad b_{n,k}(t)=\frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}},$$

and $B(m,n)=\frac{(m-1)!(n-1)!}{(n+m-1)!}$.

The rate of convergence for certain Durrmeyer type operators and their Bézier variants is one of the important areas of research in recent years. Zeng and collaborators have done commendable work in this direction and they estimated the rate of convergence for bounded/bounded variation functions (see [9–11]). In the present article, we extend the studies and here we estimate the rate of convergence for functions having derivatives of bounded variation.

Auxiliary results

In the sequel, we need the following results:

Lemma 1 [6]. *Let the $m\in N^0$, $x\in[0,\infty)$, and suppose that*

$$T_{n,r,m}(x)=\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)(t-x)^m dt.$$

Then,

$$T_{n,r,0}(x)=1;\quad T_{n,r,1}(x)=\frac{(1+r)+x(1+2r)}{n-r-1},\quad n>r+1$$

and

$$T_{n,r,2}(x)=\frac{2(2r^2+4r+n+1)x^2+2(2r^2+5r+n+2)x+r^2+3r+2}{(n-r-1)(n-r-2)},$$

$n>r+2.$

Also for $n>m+r+1$, there holds the recurrence relation:

$$(n-m-r-1)T_{n,r,m+1}(x)=x(1+x)\left[T'_{n,r,m}(x)+2mT_{n,r,m-1}(x)\right]+[(m+r+1)(1+2x)-x]T_{n,r,m}(x).$$

Consequently for all $x \in [0, \infty)$, *we have*

$$T_{n,r,m}(x) = O(n^{-[(m+1)/2]}),$$

where $[\alpha]$ *denotes the integral part of* α.

Remark 1. From Lemma 1, taking n to be sufficiently large, $x \in (0, \infty)$, we observe that

$$\frac{2x(1+x)}{n-r-2} \leqslant T_{n,r,2}(x) \leqslant \frac{Cx(1+x)}{n-r-2}, \quad \text{for } (C > 2).$$

Remark 2. Applying the Cauchy–Schwarz inequality and keeping the same conditions as in Remark 1 for x, n and C, we derive from Lemma 1, that

$$\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} b_{n-r,k+r}(t)|t-x|dt \leqslant [T_{n,r,2}(x)]^{\frac{1}{2}} \leqslant \sqrt{\frac{Cx(1+x)}{n-r-2}}.$$

Lemma 2. *Suppose that* $x \in (0, \infty)$ *and* $C > 2$, *then for sufficiently large* n, *we have*

$$\lambda_{n,r}(x,y) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{y} b_{n-r,k+r}(t)dt \leqslant \frac{Cx(1+x)}{(n-r-2)(x-y)^2}, \quad 0 \leqslant y < x$$

$$1 - \lambda_{n,r}(x,z) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_z^{\infty} b_{n-r,k+r}(t)dt \leqslant \frac{Cx(1+x)}{(n-r-2)(z-x)^2}, \quad x < z < \infty.$$

Proof. The proof follows directly from Remark 1, as for the first inequality, for sufficiently large n, we have

$$\begin{aligned}
\lambda_{n,r}(x,y) &= \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{y} b_{n-r,k+r}(t)dt \\
&\leqslant \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{y} b_{n-r,k+r}(t) \frac{(t-x)^2}{(y-x)^2} dt = \frac{T_{n,r,2}(x)}{(y-x)^2} \\
&\leqslant \frac{Cx(1+x)}{(n-r-2)(x-y)^2}.
\end{aligned}$$

Similarly, we can prove the second inequality. □

Lemma 3. *Let us consider that* f *be* s *times differentiable on* $[0, \infty)$ *such that* $f^{(s-1)}(t) = O(t^q)$, *as* $t \to \infty$ *where* q *is a positive integer. Then for any* $r, s \in N^0$ *and* $n > \max\{q, r+s+1\}$, *we have*

$$D^s V_{n,r}(f,x) = V_{n,r+s}(D^s f,x), \quad D \equiv \frac{d}{dx}.$$

Proof. First

$$\begin{aligned}
D[p_{n,k}(x)] &= \binom{n+k-1}{k} D\frac{x^k}{(1+x)^{n+k}} \\
&= \binom{n+k-1}{k}\left[k\frac{x^{k-1}}{(1+x)^{n+k}} - (n+k)\frac{x^k}{(1+x)^{n+k+1}}\right] \\
&= \frac{(n+k-1)!}{(k-1)!(n-1)!}\frac{x^{k-1}}{(1+x)^{n+k}} - \frac{(n+k-1)!}{k!(n-1)!}(n+k)\frac{x^k}{(1+x)^{n+k+1}} \\
&= n\binom{(n+1)+(k-1)-1}{k-1}\frac{x^{k-1}}{(1+x)^{n+k}} - \binom{n+k}{k}\frac{x^k}{(1+x)^{n+k+1}}.
\end{aligned}$$

Thus

$$D[p_{n,k}(x)] = n[p_{n+1,k-1}(x) - p_{n+1,k}(x)]. \tag{2.1}$$

Proceeding along similar lines, we have

$$D[b_{n,k}(x)] = n[b_{n+1,k-1}(x) - b_{n+1,k}(x)] \tag{2.2}$$

The identities (2.1) and (2.2), are true even for the case $k = 0$, as we observe that $b_{n+1,negative}(x) = 0$ and $p_{n+1,negative}(x) = 0$. We shall prove the result by using the principle of mathematical induction. Using (2.1) and (2.2), we have

$$\begin{aligned}
D[V_{n,r}(f,x)] &= \frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty} Dp_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)f(t)dt \\
&= \frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}(n+r)[p_{n+r+1,k-1}(x) - p_{n+r+1,k}(x)] \\
&\quad \times \int_0^{\infty} b_{n-r,k+r}(t)f(t)dt \\
&= \frac{(n+r)!(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty} p_{n+r+1,k}(x)\int_0^{\infty}[b_{n-r,k+r+1}(t) \\
&\quad - b_{n-r,k+r}(t)]f(t)dt.
\end{aligned}$$

Using (2.2), and integrating by parts we have

$$\begin{aligned}
DV_{n,r}(f,x) &= \frac{(n+r)!(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}p_{n+r+1,k}(x)\int_0^{\infty}-\frac{D[b_{n-r-1,k+r+1}(t)]}{n-r-1}f(t)dt\\
&= \frac{(n+r)!(n-r-2)!}{((n-1)!)^2}\sum_{k=0}^{\infty}p_{n+r+1,k}(x)\int_0^{\infty}b_{n-r-1,k+r+1}(t)f^{(1)}(t)dt\\
&= V_{n,r+1}(Df,x),
\end{aligned}$$

which means that the identity is satisfied for $s = 1$. Let us suppose that the result holds for $s = l$, i.e.,

$$\begin{aligned}
D^l V_{n,r}(f,x) &= V_{n,r+l}(D^l f,x)\\
&= \frac{(n+r+l-1)!(n-r-l-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}p_{n+r+l,k}(x)\\
&\quad\times\int_0^{\infty}b_{n-r-l,k+r+l}(t)D^{(l)}f(t)dt.
\end{aligned}$$

Now,

$$\begin{aligned}
D^{l+1}V_{n,r}(f,x) &= \frac{(n+r+l-1)!(n-r-l-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}Dp_{n+r+l,k}(x)\\
&\quad\times\int_0^{\infty}b_{n-r-l,k+r+l}(t)D^l f(t)dt\\
&= \frac{(n+r+l-1)!(n-r-l-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}(n+r+l)[p_{n+r+l+1,k-1}(x)\\
&\quad - p_{n+r+l+1,k}(x)]\int_0^{\infty}b_{n-r-l,k+r+l}(t)D^l f(t)dt\\
&= \frac{(n+r+l)!(n-r-l-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}p_{n+r+l+1,k}(x)\\
&\quad\times\int_0^{\infty}[b_{n-r-l,k+r+l+1}(t)-b_{n-r-l,k+r+l}(t)]D^l f(t)dt\\
&= \frac{(n+r+l)!(n-r-l-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}p_{n+r+l+1,k}(x)\\
&\quad\times\int_0^{\infty}-\frac{D[b_{n-r-l-1,k+r+l+1}(t)]}{n-r-l-1}D^l f(t)dt.
\end{aligned}$$

Integrating by parts the last integral, we get

$$D^{l+1}V_{n,r}(f,x) = \frac{(n+r+l)!(n-r-l-2)!}{((n-1)!)^2}\sum_{k=0}^{\infty} p_{n+r+l+1,k}(x) \times \int_0^{\infty} b_{n-r-l-1,k+r+l+1}(t)D^{l+1}f(t)dt.$$

Therefore,

$$D^{l+1}V_{n,r}(f,x) = V_{n,r+l+1}(D^{l+1}f(x)).$$

Thus the result is true for $s = l + 1$, hence by mathematical induction, proof of the lemma is complete. □

Rate of convergence

The class of absolutely continuous functions f defined on $(0,\infty)$ is defined by $B_q(0,\infty)$, $q > 0$ and satisfying:

(i) $|f(t)| \leqslant C_1 t^q$, $C_1 > 0$,
(ii) having a derivative f' on the interval $(0,\infty)$ which coincide a.e. with a function which is of bounded variation on every finite sub-interval of $(0,\infty)$. It can be observed that for all functions $f \in B_q(0,\infty)$ possess for each $C > 0$ the representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \quad x \geqslant c.$$

Theorem 1. *Let $f \in B_q(0,\infty)$, $q > 0$ and $x \in (0,\infty)$. Then for $C > 2$ and n sufficiently large, we have*

$$\left|\frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!}V_{n,r}(f,x) - f(x)\right|$$
$$\leqslant \frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x) + \frac{C(1+x)}{(n-r-2)x}(|f(2x)-f(x)$$
$$-xf'(x^+)| + |f(x)|) + O(n^{-q}) + |f'(x^+)|\frac{C(1+x)}{n-r-2} + \frac{1}{2}\sqrt{\frac{Cx(1+x)}{n-r-2}}|f'(x^+)$$
$$-f'(x^-)| + \frac{1}{2}|f'(x^+)+f'(x^-)|\frac{(1+r)+x(1+2r)}{n-r-1},$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and the auxiliary function f_x is defined by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leqslant t < x; \\ 0, & t = x; \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

Proof. On applying the mean value theorem, we get

$$\left| \frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!} V_{n,r}(f,x) - f(x) \right|$$
$$\leqslant \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} b_{n-r,k+r}(t)|f(t) - f(x)|dt$$
$$= \int_0^{\infty} \left| \int_x^t \sum_{k=0}^{\infty} p_{n+r,k}(x) b_{n-r,k+r}(t) f'(u) du \right| dt.$$

Also, using the identity

$$f'(u) = \frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) + \frac{f'(x^+) - f'(x^-)}{2} sgn(u - x)$$
$$+ \left[f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right] \chi_x(u),$$

where

$$\chi_x(u) = \begin{cases} 1, & u = x; \\ 0, & u \neq x. \end{cases}$$

We can see that

$$\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} \left(\int_x^t \left(f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right) \chi_x(u) du \right) b_{n-r,k+r}(t) dt = 0.$$

Now, by using the above identities, we have

$$\left| \frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!} V_{n,r}(f,x) - f(x) \right|$$
$$\leqslant \left| \int_0^{\infty} \left(\int_x^t \sum_{k=0}^{\infty} p_{n+r,k}(x) b_{n-r,k+r}(t) \left(\frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) \right) du \right) dt \right|$$
$$+ \left| \int_0^{\infty} \left(\int_x^t \sum_{k=0}^{\infty} p_{n+r,k}(x) b_{n-r,k+r}(t) \frac{[f'(x^+) - f'(x^-)]}{2} sgn(u - x) du \right) dt \right|. \tag{3.1}$$

Also, we have

$$\left|\int_0^\infty \left(\int_x^t \frac{[f'(x^+)-f'(x^-)]}{2} sgn(u-x)du\right)\sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)dt\right| \leqslant \frac{|f'(x^+)-f'(x^-)|}{2}[T_{n,r,2}(x)]^{1/2} \quad (3.2)$$

and

$$\int_0^\infty \left(\int_x^t \frac{[f'(x^+)+f'(x^-)]}{2} du\right)\sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)dt = \frac{[f'(x^+)+f'(x^-)]}{2} T_{n,r,1}(x). \quad (3.3)$$

Combining (3.1)–(3.3), we have

$$\begin{aligned}\left|\frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!}V_{n,r}(f,x)-f(x)\right| &\leqslant \left|\int_x^\infty \left(\int_x^t (f')_x(u)du\right)\sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)dt\right.\\ &\quad \left. + \int_0^x \left(\int_x^t (f')_x(u)du\right)\sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)dt\right|\\ &\quad + \frac{|f'(x^+)-f'(x^-)|}{2}[T_{n,r,2}(x)]^{1/2} + \frac{|f'(x^+)+f'(x^-)|}{2}T_{n,r,1}(x)\\ &= |A_{n,r}(f,x)+B_{n,r}(f,x)| + \frac{|f'(x^+)-f'(x^-)|}{2}[T_{n,r,2}(x)]^{1/2}\\ &\quad + \frac{|f'(x^+)+f'(x^-)|}{2}T_{n,r,1}(x). \end{aligned} \quad (3.4)$$

Applying Remark 2 and Lemma 1, in (3.4), we have

$$\begin{aligned}\left|\frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!}V_{n,r}(f,x)-f(x)\right| &\leqslant |A_{n,r}(f,x)|+|B_{n,r}(f,x)| + \frac{|f'(x^+)-f'(x^-)|}{2}\sqrt{\frac{Cx(1+x)}{n-r-2}}\\ &\quad + \frac{|f'(x^+)+f'(x^-)|}{2}\frac{(1+r)+x(1+2r)}{n-r-1}. \end{aligned} \quad (3.5)$$

The estimation of the terms $A_{n,r}(f,x)$ and $B_{n,r}(f,x)$ will lead to proof of the theorem.

First,

$$\begin{aligned}
|A_{n,r}(f,x)| &= \left|\int_x^\infty \left(\int_x^t (f')_x(u)du\right)\sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)dt\right| \\
&= \left|\int_{2x}^\infty \left(\int_x^t (f')_x(u)du\right)\sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)dt\right. \\
&\quad \left. + \int_x^{2x}\left(\int_x^t (f')_x(u)du\right)d_t(1-\lambda_{n,r}(x,t))\right| \\
&\leqslant \left|\sum_{k=0}^\infty p_{n+r,k}(x)\int_{2x}^\infty (f(t)-f(x))b_{n-r,k+r}(t)dt\right| \\
&\quad + |f'(x^+)|\left|\sum_{k=0}^\infty p_{n+r,k}(x)\int_{2x}^\infty b_{n-r,k+r}(t)(t-x)dt\right| \\
&\quad + \left|\int_x^{2x}(f')_x(u)du\right||1-\lambda_{n,r}(x,2x)| \\
&\quad + \int_x^{2x}|(f')_x(t)||1-\lambda_{n,r}(x,t)|dt \\
&\leqslant \sum_{k=0}^\infty p_{n+r,k}(x)\int_{2x}^\infty b_{n-r,k+r}(t)C_1t^{2q}dt \\
&\quad + \frac{|f(x)|}{x^2}\sum_{k=0}^\infty p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)(t-x)^2dt \\
&\quad + |f'(x^+)|\int_{2x}^\infty \sum_{k=0}^\infty p_{n+r,k}(x)b_{n-r,k+r}(t)|t-x|dt \\
&\quad + \frac{C(1+x)}{(n-r-2)x}|f(2x)-f(x)-xf'(x^+)| \\
&\quad + \frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}((f')_x).
\end{aligned} \tag{3.6}$$

For estimating the integral $(n-r-1)\sum_{k=0}^\infty p_{n+r,k}(x)\int_{2x}^\infty b_{n-r,k+r}(t)C_1t^{2q}dt$ in (3.6) above, we proceed as follows:

Obviously $t \geqslant 2x$ implies that $t \leqslant 2(t-x)$ and it follows from Lemma 1, that

$$\begin{aligned}
\sum_{k=0}^\infty p_{n+r,k}(x)\int_{2x}^\infty b_{n-r,k+r}(t)t^{2q}dt &\leqslant C_1 2^{2q}\sum_{k=0}^\infty p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)(t-x)^{2q}dt \\
&= C_1 2^{2q}T_{n,r,2q}(x) = O(n^{-q})(n\to\infty).
\end{aligned}$$

To estimate the second term, we use Remark 1, thus

$$\frac{|f(x)|}{x^2}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)(t-x)^2dt=\frac{|f(x)|}{x^2}\cdot\frac{Cx(1+x)}{n-r-2}.$$

Applying Schwarz inequality and Remark 1, third term in right hand side of (3.6) is estimated as follows:

$$\begin{aligned}&|f'(x^+)|\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_{2x}^{\infty}b_{n-r,k+r}(t)|t-x|dt\\&\leqslant\frac{|f'(x^+)|}{x}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)(t-x)^2dt=|f'(x^+)|\frac{C(1+x)}{n-r-2}.\end{aligned}$$

By collecting the estimates, we have

$$\begin{aligned}|A_{n,r}(f,x)|\leqslant{}&\mathcal{O}(n^{-q})+|f'(x^+)|\cdot\frac{C(1+x)}{n-r-2}+\frac{C(1+x)}{(n-r-2)x}(|f(2x)-f(x)\\&-xf'(x^+)|+|f(x)|)+\frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x).\end{aligned}\tag{3.7}$$

On applying, Lemma 2 with $y=x-\frac{x}{\sqrt{n}}$, and integrating by parts, we have

$$\begin{aligned}|B_{n,r}(f,x)|&=\left|\int_0^x\int_x^t(f')_x(u)du\,d_t(\lambda_{n,r}(x,t))\right|=\int_0^x\lambda_{n,r}(x,t)(f')_x(t)dt\\&\leqslant\left(\int_0^y+\int_y^x\right)|(f')_x(t)||\lambda_{n,r}(x,t)|dt\\&\leqslant\frac{Cx(1+x)}{n-r-2}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt+\int_y^x\bigvee_t^x((f')_x)dt\\&\leqslant\frac{Cx(1+x)}{n-r-2}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x)\\&=\frac{Cx(1+x)}{n-r-2}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^{x}((f')_x)du+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x)\\&\leqslant\frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x),\end{aligned}\tag{3.8}$$

where $u=\frac{x}{x-t}$.

The required result is obtained on combining (3.5), (3.7) and (3.8). □

As a consequence of Lemma 3, we have the following corollary:

Corollary 1. *Let $f^{(s)} \in DB_q(0,\infty)$, $q > 0$ and $x \in (0,\infty)$. Then for $C > 2$ and for n sufficiently large, we have*

$$\begin{aligned}
&\left|\frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!}D^s V_{n,r}(f,x) - f^{(s)}(x)\right| \\
&\leqslant \frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((D^{s+1}f)_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((D^{s+1}f)_x) + \frac{C(1+x)}{x(n-r-2)} \\
&\quad \times (|D^s f(2x) - D^s f(x) - xD^{s+1}f(x^+)| + |D^s f(x)|) + O(n^{-q}) \\
&\quad + \frac{C(1+x)}{n-r-2}|D^{s+1}f(x^+)| + \frac{1}{2}\sqrt{\frac{Cx(1+x)}{n}}|D^{s+1}f(x^+) - D^{s+1}f(x^-)| \\
&\quad + \frac{1}{2}|D^{s+1}f(x^+) + D^{s+1}f(x^-)|\frac{(1+r)+x(1+2r)}{n-r-1},
\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$D^{s+1}f_x(t) = \begin{cases} D^{s+1}f(t) - D^{s+1}f(x^-), & 0 \leqslant t < x; \\ 0, & t = x; \\ D^{s+1}f(t) - D^{s+1}f(x^+), & x < t < \infty. \end{cases}$$

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