Arab Journal of
Mathematical Sciences

CrossMark

Arab J Math Sci 23 (2017) 231–241

Potential eventual positivity of sign patterns with the underlying broom graph[☆]

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Received 10 July 2016; accepted 6 November 2016
Available online 30 November 2016

Abstract. A sign pattern is a matrix whose entries belong to the set $\{+, -, 0\}$. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix or be potentially eventually positive if there exist at least one real matrix A with the same sign pattern as $\mathcal{A}$ and a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. Identifying the necessary and sufficient conditions for an n-by-n sign pattern to be potentially eventually positive, and classifying the n-by-n sign patterns that allow an eventually positive matrix were posed as two open problems by Berman, Catral, Dealba, et al. In this article, we focus on the potential eventual positivity of a collection of the n-by-n tree sign patterns $\mathcal{A}_{n,4}$ whose underlying graph $G(\mathcal{A}_{n,4})$ consists of a path P with 4 vertices, together with $(n-4)$ pendent vertices all adjacent to the same end vertex of P. Some necessary conditions for the n-by-n tree sign patterns $\mathcal{A}_{n,4}$ to be potentially eventually positive are established. All the minimal subpatterns of $\mathcal{A}_{n,4}$ that allow an eventually positive matrix are identified. Consequently, all the potentially eventually positive subpatterns of $\mathcal{A}_{n,4}$ are classified.

Keywords: Sign pattern; Eventually positive matrix; Tree; Checkerboard block sign pattern

2010 Mathematics Subject Classification: 15A48; 05C50; 15A18

☆ The authors' work are supported by the National Natural Science Foundation of China under Grant number 71301060 and 41301048, and a Foundation from Huaiyin Institute of Technology.

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Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.11.002

1. Introduction

A *sign pattern* is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in the set $\{+, -, 0\}$. An n-by-n real matrix A with the same sign pattern as $\mathcal{A}$ is called a *realization* of $\mathcal{A}$. The set of all realizations of sign pattern $\mathcal{A}$ is called the *qualitative class* of $\mathcal{A}$ and is denoted by $Q(\mathcal{A})$. A *subpattern* of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a *proper subpattern* of $\mathcal{A}$. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is said to be a *superpattern* of $\mathcal{B}$. A pattern $\mathcal{A}$ is *reducible* if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is *irreducible* if it is not reducible; see, e.g. [3] and [4] for more details.

A sign pattern matrix $\mathcal{A}$ is said to *require* a certain property P referring to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and *allow* P or be *potentially* P if there is some $A \in Q(\mathcal{A})$ that has property P.

Recall that an n-by-n real matrix A is said to be *eventually positive* if there exists a nonnegative integer k_0 such that $A^k > 0$ for all $k \geq k_0$; see, e.g., [7]. Eventually positive matrices have applications to dynamical systems in situations where it is of interest to determine whether an initial trajectory reaches positivity at a certain time and remains positive thereafter; see e.g., [8]. An n-by-n sign pattern $\mathcal{A}$ is said to *allow an eventually positive matrix* or be *potentially eventually positive (PEP)*, if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [2] and the references therein.

Sign patterns that allow an eventually positive matrix were studied first in [2], where a sufficient condition and some necessary conditions for a sign pattern to be potentially eventually positive were established. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be potentially eventually positive remains open. Also open is the classification of sign patterns that are potentially eventually positive.

Recall that an n-by-n real matrix A is said to be *power-positive* if there exists a nonnegative integer k such that $A^k > 0$. An n-by-n sign pattern $\mathcal{A}$ is said to *allow a power-positive matrix* or be *potentially power-positive (PPP)*, if there exists some $A \in Q(\mathcal{A})$ such that A is power-positive; see, e.g., [5]. A relation between potentially eventually positive sign patterns and potentially power-positive sign patterns was established in [5]. An n-by-n sign pattern $\mathcal{A}$ is said to be a *minimal potentially eventually positive sign pattern (MPEP sign pattern)* if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP; see, e.g. [12] for more details. An n-by-n sign pattern $\mathcal{A}$ is said to a *minimal potentially power-positive (MPPP) sign pattern*, if $\mathcal{A}$ is potentially power-positive and no proper subpattern of $\mathcal{A}$ is potentially power-positive; see, [10] for example. A relation between the minimal potentially eventually positive sign patterns and the minimal potentially power-positive sign patterns was investigated in [10]. At present, there are a few literatures on the potential eventual positivity of sign pattern matrices with certain underlying combinatorial structures. A family of potentially eventually positive sign patterns with reducible positive part were constructed in [1]. The potentially eventually positive double star sign patterns of order n were identified and classified in [12]. More recently, the n-by-n minimal potentially eventually positive tridiagonal sign patterns were identified and all n-by-n potentially eventually positive tridiagonal sign patterns were classified in [11].

In this article, we focus on the eventual positivity of a collection of the n-by-n ($n \geq 4$) tree sign patterns $\mathcal{A}_{n,4}$ whose underlying broom graph $G(\mathcal{A}_{n,4})$ consists of a path P with 4 vertices, together with $(n-4)$ pendent vertices all adjacent to the same end vertex of P. Our work is organized as follows. In Section 2, some preliminary results for the tree sign patterns $\mathcal{A}_{n,4}$ to allow an eventually positive matrix are established. In Section 3, all the minimal potentially eventually positive subpatterns of $\mathcal{A}_{n,4}$ are identified as five specific tree sign patterns, and hence all the potentially eventually positive subpatterns of $\mathcal{A}_{n,4}$ are classified.

2. Preliminary results

We begin this section with introducing some necessary graph theoretical concepts which can be seen from [3,7] and the references therein.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is *combinatorially symmetric* if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices 1, 2, …, n and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the *graph* of the pattern $\mathcal{A}$. A combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a *tree sign pattern* if $G(\mathcal{A})$ is a tree. Similarly, *path* (or *tridiagonal*) and *double* star sign patterns can be defined.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has *signed digraph* $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \dots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). A (directed) *simple cycle of length* k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$ such that the vertices $i_1, \dots, i_k$ are distinct. Recall that a digraph $D = (V, E)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. It is well known that a digraph D is primitive if and only if there exists a natural number k such that for all $V_i \in V$, $V_j \in V$, there is a walk of length k from V_i to V_j. A nonnegative sign pattern $\mathcal{A}$ is *primitive* if its signed digraph $\Gamma(\mathcal{A})$ is primitive; see, e.g. [2] for more details.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *positive part* of $\mathcal{A}$ is defined to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. The *negative part* of $\mathcal{A}$ can be defined similarly. In [2], it has been shown that if sign pattern $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP. Here, we cite some necessary conditions for an n-by-n sign pattern to be potentially eventually positive in [2] as Lemmas 1–4 in order to state our work clearly.

Lemma 1. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then every superpattern of $\mathcal{A}$ is PEP.*

Lemma 2. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then the sign pattern $\hat{\mathcal{A}}$ obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$ is also PEP.*

Lemma 3. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix $A \in Q(\mathcal{A})$ such that*

(1) $\rho(A) = 1$.

(2) $A\mathbf{1} = \mathbf{1}$, *where* $\mathbf{1}$ *is the* $n \times 1$ *all ones vector.*

(3) *If* $n \geq 2$, *the sum of all the off-diagonal entries of* A *is positive.*

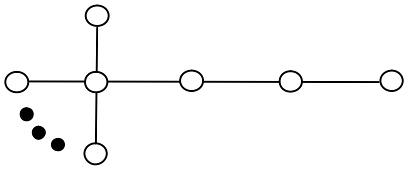

Fig. 1. The underlying graph.

We denote a sign pattern consisting entirely of positive (respectively, negative) entries by $[+]$ (respectively, $[-]$). Let $[+]_i$ be a square block sign pattern of order i consisting entirely of positive entries. For block sign patterns, we have the following Lemma 4.

Lemma 4. *If $\mathcal{A}$ is the checkerboard block sign pattern*

$$\begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

Now we turn to all combinatorially symmetric sign patterns with the underlying graph shown in Fig. 1.

Note that the graph is a broom graph that consists of a path P with 4 vertices, together with $(n-4)$ pendant vertices all adjacent to the same end vertex of P; see [9] for more details.

Let $\mathfrak{A}_{n,4}$ be a collection of sign patterns all with the same underlying broom graph shown in Fig. 1 and the diagonal of which is free. Throughout the paper, let $\mathcal{A}_{n,4} \in \mathfrak{A}_{n,4}$. Since sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}^T$ or $\mathcal{P}^T\mathcal{A}\mathcal{P}$ is potentially eventually positive, for any permutation pattern $\mathcal{P}$. Thus, without loss of generality, let the n-by-n tree sign patterns $\mathcal{A}_{n,4}$ be of the following form

$$\begin{bmatrix} ? & * & \cdots & * & & \\ * & ? & & & & \\ \vdots & & \ddots & & & \\ * & & & ? & * & \\ & & & * & ? & * \\ & & & & * & ? \end{bmatrix},$$

where $?$ denotes an entry from $\{+,-,0\}$, $*$ denotes a nonzero entry and the unspecified entries are all zeros.

The following propositions are necessary for an n-by-n tree sign pattern $\mathcal{A}_{n,4}$ to be potentially eventually positive.

Proposition 1. *If an n-by-n tree sign patterns $\mathcal{A}_{n,4}$ is potentially eventually positive, then $\mathcal{A}_{n,4}$ is symmetric.*

Proof. Since tree sign pattern $\mathcal{A}_{n,4}$ is potentially eventually positive, let real matrix $A = [a_{ij}] \in Q(\mathcal{A}_{n,4})$ be eventually positive. By Lemma 3, let $a_{22} = 1 - a_{21}$, $a_{33} = 1 - a_{31}, \ldots, a_{n-3,n-3} = 1 - a_{n-3,1}$, $a_{n-2,n-2} = 1 - a_{n-2,1} - a_{n-2,n-1}$ $a_{n-1,n-1} = 1 - a_{n-1,n-2} - a_{n-1,n}$ and $a_{n,n} = 1 - a_{n,n-1}$. To complete the proof, it suffices to show that $a_{21}a_{12} > 0$, $a_{31}a_{13} > 0, \ldots, a_{1,n-3}a_{n-3,1} > 0$, $a_{1,n-2}a_{n-2,1} > 0$, $a_{n-2,n-1}a_{n-1,n-2} > 0$ and $a_{n-1,n}a_{n,n-1} > 0$. Suppose the positive left eigenvector of A is $w = (w_1, w_2, \ldots, w_n)^T$. Then by $w^T A = w^T$, we have the following equalities:

$$w_{n-1}a_{n-1,n} + w_n(1 - a_{n,n-1}) = w_n, \tag{1}$$

$$w_{n-2,}a_{n-2,n-1} + w_{n-1}(1 - a_{n-1,n-2} - a_{n-1,n}) + w_n a_{n,n-1} = w_{n-1}, \tag{2}$$

$$w_1 a_{1,n-2} + w_{n-2}(1 - a_{n-2,1} - a_{n-2,n-1}) + w_{n-1}a_{n-1,n-2} = w_{n-2}, \tag{3}$$

and

$$w_1 a_{1k} + w_k(1 - a_{k1}) = w_k,\ k = 2, 3, \ldots, n-3. \tag{4}$$

By Equality (4), we have $w_1 a_{1k} = w_k a_{k1}$. Then $a_{1k}a_{1k} > 0$ for $k = 2, 3, \ldots, n-3$. By Equality (1), we have

$$w_{n-1}a_{n-1,n} = w_n a_{n,n-1}. \tag{5}$$

It follows that $a_{n-1,n}a_{n,n-1} > 0$. By Equalities (2) and (5), we have

$$w_{n-2,}a_{n-2,n-1} = w_{n-1}a_{n-1,n-2}. \tag{6}$$

So $a_{n-2,n-1}a_{n-1,n-2} > 0$. By Equalities (3) and (6), we have

$$w_1 a_{1,n-2} = w_{n-2}a_{n-2,1}. \tag{7}$$

Thus, $a_{1,n-2}a_{n-2,1} > 0$. It follows that tree sign pattern $\mathcal{A}_{n,4}$ is symmetric. □

Theorem 1. *If an n-by-n tree sign pattern $\mathcal{A}_{n,4}$ is potentially eventually positive, then all nonzero off-diagonal entries of $\mathcal{A}_{n,4}$ is $+$.*

Proof. By Proposition 1, the potentially eventually positive tree sign pattern $\mathcal{A}_{n,4}$ is symmetric. To complete the proof, it suffices to show that $\alpha_{n-2,n-1} = +$, $\alpha_{n-1,n} = +$ and $\alpha_{1k} = +$, for $k = 2, 3, \ldots, n-3, n-2$. To state clearly, let s be the number of k such that $\alpha_{1k} = -$, $2 \leq k \leq n-3$. And let t be the number of negative entries in the set $\{\alpha_{1,n-2}, \alpha_{n-2,n-1}, \alpha_{n-1,n}\}$. Next, we show that $s = 0$ and $t = 0$ to complete the proof.

Claim 1. *If $\mathcal{A}_{n,4}$ is potentially eventually positive, then $s = 0$.*

Proof of Claim 1. By a way of contradiction, assume that $s > 0$. Then without loss of generality, let $\alpha_{1k} = -$ for $k = 2, 3, \ldots, s$ and $\alpha_{1k} = +$ for $k = s+1, s+2, \ldots, n-3$. For t, there are four possibilities to be considered.

Case 1. $t = 0$.

Then the sign pattern $\widehat{\mathcal{A}_{n,4}}$ obtained from $\mathcal{A}_{n,4}$ by changing all 0 and $-$ diagonal entries to $+$ is potentially eventually positive. But $\widehat{\mathcal{A}_{n,4}}$ is a proper subpattern of the checkerboard block sign pattern

$$\begin{bmatrix} [+]_1 & [-] & [+] \\ [-] & [+]_s & [-] \\ [+] & [-] & [+]_{n-s-1} \end{bmatrix},$$

and thus $\widehat{\mathcal{A}_{n,4}}$ cannot be potentially eventually positive by Lemmas 4 and 1; a contradiction.

Case 2. $t = 1$.

If $\alpha_{1,n-2} = -$. Then sign pattern

$$\mathcal{A}_{n,4} = \left[\begin{array}{c|ccc|ccc|ccc}
? & - & \cdots & - & + & \cdots & + & - & & \\ \hline
- & ? & & & & & & & & \\
\vdots & & \ddots & & & & & & & \\
- & & & ? & & & & & & \\ \hline
+ & & & & ? & & & & & \\
\vdots & & & & & \ddots & & & & \\
+ & & & & & & ? & & & \\ \hline
- & & & & & & & ? & + & \\
 & & & & & & & + & ? & + \\
 & & & & & & & & + & ?
\end{array}\right].$$

The sign pattern $\widehat{\mathcal{A}_{n,4}}$ obtained from $\mathcal{A}_{n,4}$ by changing all 0 and $-$ diagonal entries to $+$ is potentially eventually positive by Lemma 2. But $\widehat{\mathcal{A}_{n,4}}$ is a proper subpattern of the checkerboard block sign pattern

$$\begin{bmatrix} [+]_1 & [-] & [+] & [-] \\ [-] & [+]_s & [-] & [+] \\ [+] & [-] & [+]_{n-s-4} & [-] \\ [-] & [+] & [-] & [+]_3 \end{bmatrix},$$

and thus $\widehat{\mathcal{A}_{n,4}}$ cannot be potentially eventually positive by Lemmas 4 and 1; a contradiction.

The other two Subcases $\alpha_{n-2,n-1} = -$ and $\alpha_{n-1,n} = -$ can be shown similarly. For completeness, we list the corresponding checkerboard block sign patterns

$$\begin{bmatrix} [+]_1 & [-] & [+] & [-] \\ [-] & [+]_s & [-] & [+] \\ [+] & [-] & [+]_{n-s-3} & [-] \\ [-] & [+] & [-] & [+]_2 \end{bmatrix}, \text{ and } \begin{bmatrix} [+]_1 & [-] & [+] & [-] \\ [-] & [+]_s & [-] & [+] \\ [+] & [-] & [+]_{n-s-2} & [-] \\ [-] & [+] & [-] & [+]_1 \end{bmatrix}.$$

Case 3. $t = 2$.

If $\alpha_{1,n-2} = \alpha_{n-2,n-1} = -$. Then sign pattern

$$
\mathcal{A}_{n,4} = \left[\begin{array}{c|ccc|ccc|c|cc}
? & - & \cdots & - & + & \cdots & + & - & & \\ \hline
- & ? & & & & & & & & \\
\vdots & & \ddots & & & & & & & \\
- & & & ? & & & & & & \\ \hline
+ & & & & ? & & & & & \\
\vdots & & & & & \ddots & & & & \\
+ & & & & & & ? & & & \\ \hline
- & & & & & & & ? & - & \\ \hline
 & & & & & & & - & ? & + \\
 & & & & & & & & + & ?
\end{array}\right].
$$

The sign pattern $\widehat{\mathcal{A}_{n,4}}$ obtained from $\mathcal{A}_{n,4}$ by changing all 0 and $-$ diagonal entries to $+$ is potentially eventually positive by Lemma 2. But $\widehat{\mathcal{A}_{n,4}}$ is a proper subpattern of the checkerboard block sign pattern

$$
\begin{bmatrix}
[+]_1 & [-] & [+] & [-] & [+] \\
[-] & [+]_s & [-] & [+] & [-] \\
[+] & [-] & [+]_{n-s-4} & [-] & [+] \\
[-] & [+] & [-] & [+]_1 & [-] \\
[+] & [-] & [+] & [-] & [+]_2
\end{bmatrix},
$$

and thus $\widehat{\mathcal{A}_{n,4}}$ cannot be potentially eventually positive by Lemmas 4 and 1; a contradiction.

The other two Subcases $\alpha_{1,n-2} = \alpha_{n-1,n} = -$, and $\alpha_{n-2,n-1} = \alpha_{n-1,n} = -$ can be shown similarly. For completeness, we list the corresponding checkerboard block sign patterns

$$
\begin{bmatrix}
[+]_1 & [-] & [+] & [-] & [+] \\
[-] & [+]_s & [-] & [+] & [-] \\
[+] & [-] & [+]_{n-s-4} & [-] & [+] \\
[-] & [+] & [-] & [+]_2 & [-] \\
[+] & [-] & [+] & [-] & [+]_1
\end{bmatrix},
$$

and

$$
\begin{bmatrix}
[+]_1 & [-] & [+] & [-] & [+] \\
[-] & [+]_s & [-] & [+] & [-] \\
[+] & [-] & [+]_{n-s-3} & [-] & [+] \\
[-] & [+] & [-] & [+]_1 & [-] \\
[+] & [-] & [+] & [-] & [+]_1
\end{bmatrix}.
$$

Case 4. $t = 3$.

Then a contradiction that sign pattern $\mathcal{A}_{n,4}$ is not potentially eventually positive can be obtained similarly by considering the checkerboard block sign pattern

$$\begin{bmatrix} [+]_1 & [-] & [+] & [-] & [+] & [-] \\ [-] & [+]_s & [-] & [+] & [-] & [+] \\ [+] & [-] & [+]_{n-s-4} & [-] & [+] & [-] \\ [-] & [+] & [-] & [+]_1 & [-] & [+] \\ [+] & [-] & [+] & [-] & [+]_1 & [-] \\ [-] & [+] & [-] & [+] & [-] & [+]_1 \end{bmatrix}.$$

Claim 2. *If $\mathcal{A}_{n,4}$ is potentially eventually positive, then $t = 0$.*

Proof of Claim 2. By Claim 1, if $\mathcal{A}_{n,4}$ is potentially eventually positive, then $s = 0$. That is, $\alpha_{1k} = +$ for all $k = 2, 3, \ldots, n-3$. To complete the proof of Claim 2, it suffices to show that $t \neq 1$, $t \neq 2$ and $t \neq 3$. By a similar discussion as the proof of Claim 1, a contradiction can be obtained when $t = 1$, $t = 2$ and $t = 3$, respectively. Below, we list the corresponding checkerboard block sign patterns for completeness.

For $t = 1$, the checkerboard block sign patterns to be considered are

$$\begin{bmatrix} [+]_{n-3} & [-] \\ [-] & [+]_3 \end{bmatrix}, \begin{bmatrix} [+]_{n-2} & [-] \\ [-] & [+]_2 \end{bmatrix}, \text{ and } \begin{bmatrix} [+]_{n-1} & [-] \\ [-] & [+]_1 \end{bmatrix}.$$

For $t = 2$, the checkerboard block sign patterns to be considered are

$$\begin{bmatrix} [+]_{n-3} & [-] & [+] \\ [-] & [+]_1 & [-] \\ [+] & [-] & [+]_2 \end{bmatrix}, \begin{bmatrix} [+]_{n-3} & [-] & [+] \\ [-] & [+]_2 & [-] \\ [+] & [-] & [+]_1 \end{bmatrix}, \text{ and } \begin{bmatrix} [+]_{n-2} & [-] & [+] \\ [-] & [+]_1 & [-] \\ [+] & [-] & [+]_1 \end{bmatrix}.$$

For $t = 3$, the checkerboard block sign pattern to be considered is

$$\begin{bmatrix} [+]_{n-3} & [-] & [+] & [-] \\ [-] & [+]_1 & [-] & [+] \\ [+] & [-] & [+]_1 & [-] \\ [-] & [+] & [-] & [+]_1 \end{bmatrix}. \qquad \square$$

3. The potential eventual positivity of tree sign patterns $\mathfrak{A}_{n,4}$

Recall that an n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially eventually positive sign pattern if $\mathcal{A}$ is potentially eventually positive and no proper subpattern of $\mathcal{A}$ is potentially eventually positive. To identify all the minimal potentially eventually positive subpatterns of $\mathcal{A}$, it is necessary to discuss the numbers of diagonal entries of potentially eventually positive sign patterns.

Proposition 2. *If an n-by-n tree sign pattern $\mathcal{A}_{n,4}$ is potentially eventually positive, then $\mathcal{A}_{n,4}$ has at least one positive diagonal entry. That is, there exists some $i \in \{1, 2, \ldots, n\}$ such that $\alpha_{ii} = +$.*

Proof. By a way of contradiction, assume that $\alpha_{ii} = -$ or 0, for all $k = 1, 2, \ldots, n$. Since tree sign pattern $\mathcal{A}_{n,4}$ is potentially eventually positive, all nonzero off-diagonal entries are $+$ by Theorem 1. Thus, by Lemma 1, without loss of generality, let

$$\mathcal{A}_{n,4} = \left[\begin{array}{c|ccc|c|c} - & + & \cdots & + & & \\ \hline + & - & & & & \\ \vdots & & \ddots & & & \\ + & & & - & + & \\ \hline & & & + & - & + \\ \hline & & & & + & - \end{array}\right].$$

It is clear that $\mathcal{A}_{n,4}$ is a proper subpattern of the checkerboard block sign pattern

$$\begin{bmatrix} [-]_1 & [+] & [-] & [+] \\ [+] & [-]_{n-3} & [+] & [-] \\ [-] & [+] & [-]_1 & [+] \\ [+] & [-] & [+] & [-]_1 \end{bmatrix},$$

and hence is not potentially eventually positive; a contradiction. Thus, tree sign pattern $\mathcal{A}_{n,4}$ has at least one positive diagonal entry. $\square$

For the sake of convenience, let $\mathcal{A}^{i}_{n,4}$ be the tree sign pattern $\mathcal{A}_{n,4}$ with all nonzero off-diagonal entries $+$, $\alpha_{ii} = +$ and $\alpha_{jj} = 0$ for all $j \neq i$, $i \in \{1, 2, \ldots, n\}$. For example,

$$\mathcal{A}^{n-1}_{n,4} = \begin{bmatrix} 0 & + & \cdots & + & & \\ + & 0 & & & & \\ \vdots & & \ddots & & & \\ + & & & 0 & + & \\ & & & + & + & + \\ & & & & + & 0 \end{bmatrix}.$$

Theorem 2. *$\mathcal{A}^{1}_{n,4}$, $\mathcal{A}^{2}_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$ are minimal potentially eventually positive sign patterns.*

Proof. Tree sign patterns $\mathcal{A}^{1}_{n,4}$, $\mathcal{A}^{2}_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$ are potentially eventually positive for their positive parts are primitive, respectively. If the diagonal entries of $\mathcal{A}^{1}_{n,4}$, $\mathcal{A}^{2}_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$ are changed to be 0, then the corresponding subpatterns are not potentially eventually positive by Proposition 2. If some nonzero off-diagonal entries of $\mathcal{A}^{1}_{n,4}$, $\mathcal{A}^{2}_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$ are changed to be 0, then the corresponding subpatterns are not irreducible, and thus are not potentially eventually positive. It follows that no proper subpatterns of $\mathcal{A}^{1}_{n,4}$, $\mathcal{A}^{2}_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$ are potentially eventually positive. So $\mathcal{A}^{1}_{n,4}$, $\mathcal{A}^{2}_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$ are minimal potentially eventually positive sign patterns. $\square$

The following proposition follows readily from Theorem 2 and Proposition 2.

Proposition 3. *If an n-by-n tree sign pattern $\mathcal{A}_{n,4}$ is a minimal potentially eventually positive sign pattern, then $\mathcal{A}_{n,4}$ has exactly one positive diagonal entry and all other diagonal entries are* 0.

In the following theorems, we identify all minimal potentially eventually positive subpatterns of $\mathcal{A}_{n,4}$ and classify all potentially eventually positive subpatterns of $\mathcal{A}_{n,4}$.

Theorem 3. *An n-by-n tree sign pattern $\mathcal{A}_{n,4}$ is a minimal potentially eventually positive sign pattern if and only if $\mathcal{A}_{n,4}$ is equivalent to one of sign patterns $\mathcal{A}^1_{n,4}$, $\mathcal{A}^2_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$.*

Proof. The sufficiency follows from Theorem 2. For the necessity, if tree sign pattern $\mathcal{A}_{n,4}$ is a minimal potentially eventually positive sign pattern, then all nonzero off-diagonal entries are positive by Theorem 1 and $\mathcal{A}_{n,4}$ has exactly one positive diagonal entry by Proposition 3. Thus, up to equivalence, $\mathcal{A}_{n,4}$ is one of sign patterns $\mathcal{A}^1_{n,4}$, $\mathcal{A}^2_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$. $\square$

Theorem 4. *An n-by-n tree sign pattern $\mathcal{A}_{n,4}$ is potentially eventually positive if and only if $\mathcal{A}_{n,4}$ is equivalent to a superpattern of one of sign patterns $\mathcal{A}^1_{n,4}$, $\mathcal{A}^2_{n,4}$, $\mathcal{A}^{n-2}_{n,4}$, $\mathcal{A}^{n-1}_{n,4}$ and $\mathcal{A}^{n}_{n,4}$.*

Proof. Theorem 4 follows readily from Theorem 3. $\square$

Recall that an arbitrary n-by-n sign pattern $\mathcal{A}$ is said to require an eventually positive matrix (REP, for short), if every matrix $A \in Q(\mathcal{A})$ is eventually positive; see e.g., [6]. It is obvious that an arbitrary sign pattern $\mathcal{A}$ is REP, then $\mathcal{A}$ is potentially eventually positive. But the converse is not true. We end this paper by drawing an interesting conclusion about minimal potentially eventually positive tree sign patterns and REP tree sign patterns with exactly one positive diagonal entry.

Proposition 4. *If an n-by-n tree sign pattern $\mathcal{A}_{n,4}$ has exactly one positive diagonal entries, then the following statements are equivalent:*

(1) $\mathcal{A}_{n,4}$ *is a minimal potentially eventually positive sign pattern;*
(2) $\mathcal{A}_{n,4}$ *is REP;*
(3) $\mathcal{A}_{n,4}$ *is nonnegative and primitive.*

Proof. Proposition 4 follows readily from Theorem 3 and Theorem 2.3 in [6]. $\square$

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