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Positive solutions of multi-point boundary value problem of fractional differential equation☆

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Abstract. By means of two fixed-point theorems on a cone in Banach spaces, some existence and multiplicity results of positive solutions of a nonlinear fractional differential equation boundary value problem are obtained. The proofs are based upon some properties of Green's function, which are also the key of the paper.

Keywords: Fractional differential equation; Positive solution; Fixed-point theorem; Green's function

2010 Mathematics Subject Classification: 34A08

1. INTRODUCTION

The purpose of this paper is to consider the existence and multiplicity of positive solutions of the nonlinear fractional differential equation boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^{\alpha}u(t) = a(t)f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m} \beta_i u(\xi_i), \end{cases} \tag{1}$$

where D_{0+}^{α} is the Riemann–Liouville differential operator of order $2 < \alpha \leq 3$ and $m \geq 1$ is integer and $\xi_i, \beta_i > 0, f(\cdot,\cdot), a(\cdot)$ satisfying

(H1) $\beta_i > 0$ for $1 \leq i \leq m$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ and $\sum_{i=1}^{m} \beta_i \xi_i^{\alpha-1} < 1$;

(H2) $a(t) \in L[0,1]$ is non-negative and not identically zero on any compact subset of $(0,1)$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

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Fractional calculus arises in many mathematical models in engineering and scientific disciplines. In fact, fractional-order models are more accurate than integer-order models in physics, mechanics, chemistry, aerodynamics, etc., see [3,6,7,5]. During the last few decades, many papers and books on fractional calculus are devoted to the solvability of initial fractional differential equations, see [12,1,9]. In recent years, many researchers focused on the solutions, especially the positive solutions of fractional differential equation boundary value problems, we refer to [4,2] and their references.

Very recently, the following BVP

$$\begin{cases} D_{0+}^{\alpha}u(t) = a(t)f(t,u(t),u'(t)), \quad 0<t<1, \\ u(0)=u'(0)=0, \qquad u(1)-\sum_{i=1}^{m}\beta_i u(\xi_i)=\lambda, \end{cases} \tag{2}$$

where $2<\alpha\leq 3$, has been studied in [11]. By employing the Leggett–Williams fixed-point theorem, the author in [11] obtained the existence of three positive solutions for BVP (2). He proved the following conclusion (a key lemma, which is about some properties of Green's function $G(t,s)$ corresponding to BVP (2) and these properties are critical in employing the Leggett–Williams fixed-point theorem).

Conclusion (*See [11], Lemma 5*)**.** $G(t,s)$ satisfies the following conditions:

(i) $G(t,s)\geq 0$, $G(t,s)\leq G(s,s)$ for all $s,t\in[0,1]$;

(ii) there exists a positive function $g\in C(0,1)$ such that $\min_{\gamma\leq t\leq\delta}G(t,s)\geq g(s)G(s,s)$, $s\in(0,1)$, where $0<\gamma<\delta<1$ and

$$g(s)=\begin{cases} \dfrac{\delta^{\alpha-1}(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{s^{\alpha-1}(1-s)^{\alpha-1}}, & s\in(0,m_1], \\ \left(\dfrac{\gamma}{\delta}\right)^{\alpha-1}, & s\in[m_1,1), \end{cases}$$

where $\gamma<m_1<\delta$;

(iii) $\max_{0\leq t\leq 1}\int_0^1 G(t,s)ds=\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}$.

We show that $G(t,s)$ mentioned above is

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\leq t\leq s\leq 1, \end{cases} \tag{3}$$

where $2<\alpha\leq 3$.

In the proof of Lemma 5 in [11], the author concludes that for $2<\alpha\leq 3$,

$$\Gamma(\alpha)G(t,s)=t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}$$

is decreasing with respect to t for $t\geq s$. But, we declare that the conclusion is wrong because if we choose $\alpha=3$, $s=\frac{1}{2}$, then for $\frac{1}{2}\leq t\leq 1$, it is obvious that

$$\Gamma(\alpha)G(t,s)=t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}=\frac{1}{4}(-3t^2+4t-1)$$

is increasing in $[\frac{1}{2},\frac{2}{3}]$ and decreasing in $[\frac{2}{3},1]$. Thus (i) cannot be obtained, hence (ii) and (iii) are all invalid since their proofs are based upon (i). We refer to [11] for more details. In fact, the conclusions above are definitely wrong.

Coincidentally, the same mistake has appeared in [13], where the following BVP was studied,

$$\begin{cases} D_{0+}^{\alpha}x(t) = q(t)f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = x''(0) = \cdots = x^{n-2}(0) = 0, \qquad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{4}$$

where $n - 1 < \alpha \leq n$ and $n \geq 2$. The expression of Green's function corresponding to BVP (4) is the same as (3), the only difference is that $n - 1 < \alpha \leq n$ and $n \geq 2$. The author also concluded that $G(t, s)$ is decreasing with respect to t for $t \geq s$ and increasing with respect to t for $t \leq s$. Unfortunately, we can also verify that this conclusion is wrong.

In this paper, we will give some proper properties of Green's function $G(t, s)$ in Lemma 2.2 which are also the key of the paper. We believe BVP (2) and BVP (4) can be restudied based upon the proper properties of $G(t, s)$.

The paper is organized as follows. After this section, some definitions and lemmas will be established in Section 2. In Section 3, we give our main results in Theorems 3.1 and 3.2.

2. Preliminaries

For convenience, we present some necessary definitions from fractional calculus theory and lemmas. We also state two fixed-point theorems due to Guo–Krasnosel'skii and Leggett–Williams.

Definition 2.1 ([11]). Let $f \in L^1(R^+)$. The Riemann–Liouville fractional integral of order $\alpha > 0$ for f is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 ([11]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function f is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad n = [\alpha] + 1,$$

where the function $f(\cdot)$ has absolutely continuous derivatives up to order $(n-1)$ on R^+.

Definition 2.3. Let $(E, \|\cdot\|)$ be a Banach space. A non-empty closed convex set $K \subset E$ is said to be a cone if the following conditions are satisfied:

(i) if $y \in K$ and $\lambda \geq 0$, then $\lambda y \in K$;
(ii) if $y \in K$ and $-y \in K$, then $y = 0$.

Definition 2.4. $(E, \|\cdot\|)$ is a Banach space and $K \subset E$ is a cone. The map θ is said to be a non-negative continuous concave function on cone K if $\theta : K \to [0, \infty)$ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$$

for any $x, y \in K$ and $t \in [0, 1]$.

Lemma 2.1 ([11]). *Let* $x \in C^+[0,1] := \{x \in C[0,1],\ x(t) \geq 0, t \in [0,1]\}$. *Then the following BVP*

$$\begin{cases} D_{0+}^{\alpha} u(t) = a(t) f(t, x(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \displaystyle\sum_{i=1}^{m} \beta_i u(\xi_i) \end{cases} \tag{5}$$

has a solution

$$u(t) = \int_0^1 G(t,s) a(s) f(s, x(s)) ds + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i, s) a(s) f(s, x(s)) ds t^{\alpha-1},$$

where $\Delta = \frac{1}{1-\delta}, \delta = \sum_{i=1}^{m} \beta_i \xi_i^{\alpha-1} < 1$ *and* $G(t,s)$ *is defined as in* (3).

Lemma 2.2. $G(t,s)$ *has the following properties:*

(i) *For any* $(t,s) \in [0,1] \times [0,1]$, $G(t,s) \geq 0$.

(ii) *Given* $s \in [0,1]$, *then for any* $t \in [0,1]$,

$$G(t,s) \leq G(t_0, s) = \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha) \left[1 - (1-s)^{\frac{\alpha-1}{\alpha-2}}\right]^{\alpha-2}},$$

where $t_0 = \frac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}} \in [s, 1)$.

(iii) *Given* $s \in [0,1]$, *then for any* $t \in [0,1]$, $G(t,s) \geq \rho(t) G(t_0, s)$, *where*

$$\rho(t) = \begin{cases} t(1-t), & 1 \geq t \geq \dfrac{1}{2}, \\ t^2, & 0 \leq t \leq \dfrac{1}{2}. \end{cases}$$

(iv) *Given* $s \in [0,1]$, *then for any* $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $G(t,s) \geq \frac{1}{16} G(t_0, s)$.

Proof. (i) For any $(t,s) \in [0,1] \times [0,1]$, when $s \leq t$,

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)} \left[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \left[(1-s)^{\alpha-1} - \left(1 - \frac{s}{t}\right)^{\alpha-1}\right] \geq 0; \end{aligned}$$

when $t \leq s$, $G(t,s) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-1} \geq 0$.

Now, we prove (ii). For a given $s \in [0,1]$, when $t \in [s, 1]$,

$$\Gamma(\alpha) G(t,s) = t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1},$$

and thus,

$$\Gamma(\alpha) G_t'(t,s) = (\alpha - 1) \left[t^{\alpha-2}(1-s)^{\alpha-1} - (t-s)^{\alpha-2}\right] \begin{cases} \leq 0, & t \geq t_0, \\ \geq 0, & t \leq t_0, \end{cases}$$

where $t_0 = \frac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}} \in [s, 1)$ (if $s = 0$, let $t_0 = \lim_{s\to 0} \frac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}} = \frac{\alpha-2}{\alpha-1}$). So

$$\max_{t\in[s,1]} G(t, s) = G(t_0, s).$$

When $t \in [0, s]$, since $\left[1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right]^{\alpha-2} \leq 1$, we get by (3) that

$$\begin{aligned}\Gamma(\alpha)G(t, s) &= t^{\alpha-1}(1 - s)^{\alpha-1} \leq s^{\alpha-1}(1 - s)^{\alpha-1} \\ &\leq \frac{s^{\alpha-1}(1 - s)^{\alpha-1}}{\left[1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right]^{\alpha-2}} = \Gamma(\alpha)G(t_0, s).\end{aligned}$$

Above all, for a given $s \in [0, 1]$, we have $G(t, s) \leq G(t_0, s)$ for any $t \in [0, 1]$.

Next, we prove (iii). For a given $s \in [0, 1]$, when $t \in (s, 1)$,

$$\begin{aligned}\Gamma(\alpha)G''_{tt}(t, s) &= (\alpha - 1)(\alpha - 2)\left[t^{\alpha-3}(1 - s)^{\alpha-1} - (t - s)^{\alpha-3}\right] \\ &= (\alpha - 1)(\alpha - 2)\left[\frac{(1 - s)^2}{(t - ts)^{3-\alpha}} - \frac{1}{(t - s)^{3-\alpha}}\right] \\ &\leq 0,\end{aligned}$$

which means that $G(t, s)$ is concave about t on $[s, 1]$.

For any $t \in [s, t_0]$, by the concavity of $G(t, s)$, we have

$$\begin{aligned}G(t, s) &\geq \frac{G(t_0, s) - G(s, s)}{t_0 - s}(t - s) + G(s, s) \\ &= G(s, s) - \frac{G(t_0, s) - G(s, s)}{t_0 - s}s + \frac{G(t_0, s) - G(s, s)}{t_0 - s}t.\end{aligned} \tag{6}$$

Since $2 < \alpha \leq 3$, we know $1 \geq \left(1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right)^{3-\alpha}$ and thus

$$(1 - s)^{\frac{\alpha-1}{\alpha-2}} \geq \left[\left(1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right)^{2-\alpha} - 1\right]\left(1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right);$$

then we get

$$s^{\alpha-1}(1 - s)^{\alpha-1} - \frac{s^{\alpha-1}(1 - s)^{\alpha-1}\left[\left(1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right)^{2-\alpha} - 1\right]\left(1 - (1 - s)^{\frac{\alpha-1}{\alpha-2}}\right)}{(1 - s)^{\frac{\alpha-1}{\alpha-2}}} \geq 0,$$

which means that

$$G(s, s) - \frac{G(t_0, s) - G(s, s)}{t_0 - s}s \geq 0.$$

Then, for any $t \in [s, t_0]$, by (6) we get

$$G(t, s) \geq \left[G(s, s) - \frac{G(t_0, s) - G(s, s)}{t_0 - s}s + \frac{G(t_0, s) - G(s, s)}{t_0 - s}\right]t.$$

$$
\begin{aligned}
&= \left[(G(t_0,s) - G(s,s)) \frac{1-t_0}{t_0-s} + G(t_0,s) \right] t \\
&\geq G(t_0,s)t \\
&\geq G(t_0,s)t(1-t). \qquad (7)
\end{aligned}
$$

For any $t \in [t_0, 1]$, by the concavity of $G(t,s)$, we have

$$
\begin{aligned}
G(t,s) &\geq \frac{G(t_0,s) - G(1,s)}{t_0 - 1}(t-1) + G(1,s) \\
&\geq \frac{G(t_0,s)}{1-t_0}(1-t) \\
&\geq G(t_0,s)t(1-t). \qquad (8)
\end{aligned}
$$

From (7) and (8) we have

$$
G(t,s) \geq G(t_0,s)t(1-t) \quad \text{for any } t \in [s,1]. \qquad (9)
$$

On the other hand, let $h(s) = 1 - (1-s)^{\frac{\alpha-1}{\alpha-2}} - s^{\frac{\alpha-1}{\alpha-2}}$, $s \in [0,1]$. It is not difficult to find that $h''(s) \leq 0$ and $h(0) = h(1) = 0$. Thus $h(s) \geq 0$ for any $s \in [0,1]$, which means that

$$
\frac{s^{\alpha-1}}{\left(1 - (1-s)^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}} \leq 1.
$$

Now, for $t \in [0,s]$, by (3),

$$
\begin{aligned}
\Gamma(\alpha)G(t,s) &= t^{\alpha-1}(1-s)^{\alpha-1} \\
&\geq t^2(1-s)^{\alpha-1} \\
&\geq t^2(1-s)^{\alpha-1} \frac{s^{\alpha-1}}{\left(1 - (1-s)^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}} \\
&= \Gamma(\alpha)G(t_0,s)t^2. \qquad (10)
\end{aligned}
$$

From (9) and (10), for a given $s \in [0,1]$, we have

$$
G(t,s) \geq \rho(t)G(t_0,s), \quad t \in [0,1], \qquad (11)
$$

where

$$
\rho(t) = \begin{cases} t^2, & 0 \leq t \leq \frac{1}{2}, \\ t(1-t), & \frac{1}{2} \leq t \leq 1, \end{cases}
$$

which proves (iii).

The result in (iv) is obvious since $\min_{t \in [\frac{1}{4}, \frac{3}{4}]} \rho(t) = \frac{1}{16}$.

Lemma 2.3. *If $x \in C^+[0,1]$, then the solution $u(t)$ of BVP (5) is non-negative and satisfies*

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \frac{1}{16}\|u\|.$$

Proof. Let $x \in C^+[0,1]$, by Lemma 2.1, we have

$$u(t) = \int_0^1 G(t,s)a(s)f(s,x(s))ds + \Delta\sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,x(s))dst^{\alpha-1}. \tag{12}$$

From (i) of Lemma 2.2, we know $G(t,s) \geq 0$. Combining $\sum_{i=1}^m \beta_i\xi_i^{\alpha-1} < 1$ with the fact that $a(t)$ and $f(t,x(t))$ are non-negative, we can easily get that $u(t)$ is non-negative by (12).

By (ii) of Lemma 2.2, we obtain that

$$\|u\| \leq \int_0^1 G(t_0,s)a(s)f(s,x(s))ds + \Delta\sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,x(s))ds. \tag{13}$$

On the other hand, for any $t \in [\frac{1}{4},\frac{3}{4}]$, by (iv) of Lemma 2.2, we get

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)a(s)f(s,x(s))ds + \Delta\sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,x(s))dst^{\alpha-1} \\
&\geq \int_0^1 \frac{1}{16}G(t_0,s)a(s)f(s,x(s))ds \\
&\quad + \Delta\sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,x(s))ds\left(\frac{1}{4}\right)^{\alpha-1} \\
&\geq \frac{1}{16}\left[\int_0^1 G(t_0,s)a(s)f(s,x(s))ds + \Delta\sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,x(s))ds\right] \\
&\geq \frac{1}{16}\|u\|,
\end{aligned} \tag{14}$$

which means that

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \frac{1}{16}\|u\|.$$

In the paper, let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$. If $u \in E$ satisfies BVP (1) and $u(t) \geq 0$ for any $t \in [0,1]$, then u is called a non-negative solution of BVP (1). If u is a non-negative solution of BVP (1) with $\|u\| > 0$, then u is called a positive solution of BVP (1). Let $P \subseteq E$ be defined as

$$P = \left\{u \in E | u(t) \geq 0, \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \frac{1}{16}\|u\|\right\},$$

then P is a cone in E.

Define operator $T : P \to C[0,1]$ as

$$(Tu)(t) = \int_0^1 G(t,s)a(s)f(s,u(s))ds + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))dst^{\alpha-1}, \qquad (15)$$

with $G(t,s)$ defined in (3).

It is clear, from Lemma 2.1, that the fixed points of operator T in P coincide with the non-negative solutions of BVP (1).

Lemma 2.4. *$T : P \to P$ is completely continuous.*

Proof. For each $u \in P$, since $G(t,s) \geq 0$, by (15), one gets $(Tu)(t) \geq 0$ for any $t \in [0,1]$. Using Lemma 2.3, we get $T(P) \subseteq P$. The continuity of T is obvious since $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Next, we show that T is uniformly bounded and equi-continuous.

Let $D \subset P$ be bounded, which is to say there exists a positive constant $q > 0$ such that $\|u\| \leq q$ for all $u \in D$. Let $k = \max_{t\in[0,1],u\in[0,q]} f(t,u)$.

Firstly, for any $u \in D$,

$$\begin{aligned} 0 \leq (Tu)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\quad + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))dst^{\alpha-1} \\ &\leq k\left(\int_0^1 G(t_0,s)a(s)ds + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i,s)a(s)ds\right). \end{aligned}$$

Hence, $T(D)$ is uniformly bounded.

Secondly, for any $\varepsilon > 0$, since $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$ and $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, then there exists $\eta > 0$ such that for any $t_1, t_2 \in [0,1]$, when $|t_1 - t_2| < \eta$, we have $|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{2k\int_0^1 a(s)ds}$ and $|t_1^{\alpha-1} - t_2^{\alpha-1}| < \frac{\varepsilon}{2k\Delta\sum_{i=1}^{m}\beta_i\int_0^1 G(\xi_i,s)a(s)ds}$. Then, for each $u \in D$, one has

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &= \left|\int_0^1 (G(t_1,s) - G(t_2,s))a(s)f(s,u(s))ds \right. \\ &\quad \left. + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds(t_1^{\alpha-1} - t_2^{\alpha-1})\right| \\ &\leq \int_0^1 |(G(t_1,s) - G(t_2,s))|a(s)kds \\ &\quad + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i,s)a(s)kds|t_1^{\alpha-1} - t_2^{\alpha-1}| \\ &< \varepsilon, \end{aligned}$$

which means that $T(D)$ is equi-continuous.

By means of the Arzela–Ascoli theorem, we know that $T : P \to P$ is compact, and thus is completely continuous.

Theorem 2.1 (*[8]*). *Let E be a Banach space and $K \subseteq E$ be a cone in E. Assume Ω_1 and Ω_2 are two bounded open balls of E at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$ *hold. Then T has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.*

Theorem 2.2 (*[10]*). *Let K be a cone in a Banach space X. Let $K_c = \{x \in K \mid \|x\| < c\}$. θ is a non-negative continuous concave function on K with $\theta(x) \leq \|x\|$ for any $x \in \overline{K_c}$. Let $K(\theta, b, d) = \{x \in K | b \leq \theta(x), \|x\| \leq d\}$. Suppose $A : \overline{K_c} \longrightarrow \overline{K_c}$ is a completely continuous operator and there exist constants $0 < a < b < d \leq c$ such that*

(c1) $\{x \in K(\theta, b, d) \mid \theta(x) \geq b\} \neq \emptyset$ *and* $\theta(Ax) > b$ *for any* $x \in K(\theta, b, d)$;

(c2) $\|Ax\| < a$ *for any* $x \in \overline{K_a}$;

(c3) $\theta(Ax) > b$ *for any* $x \in K(\theta, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2 and x_3 in K with $\|x_1\| < a, b < \theta(x_2), a < \|x_3\|$ and $\theta(x_3) < b$.

3. Main result

In this section, in order to establish some results of existence and multiplicity of positive solutions for BVP (1), we will impose growth conditions on f which allow us to apply Theorems 2.1 and 2.2.

Throughout this section, we shall use the following notations:

$$M = \left(\int_0^1 G(t_0, s)a(s)ds + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i, s)a(s)ds\right)^{-1};$$

$$N = \left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0, s)a(s)ds + \Delta \sum_{i=1}^{m} \beta_i \int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i, s)a(s)ds\right)^{-1}.$$

Theorem 3.1. *Assume that there exist two positive constants $r_2 > r_1 > 0$ such that*

(H1) $f(t, u) \leq Mr_2$ *for any* $(t, u) \in [0, 1] \times [0, r_2]$;

(H2) $f(t, u) \geq 16Mr_1$ *for any* $(t, u) \in [0, 1] \times [0, r_1]$.

Then BVP (1) has at least one positive solution $u \in P$ with $r_1 \leq \|u\| \leq r_2$.

Proof. We divide the proof into two steps.

Step 1. Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$. For any $u \in P \bigcap \partial\Omega_1$, we have $0 \leq u(s) \leq r_1$ for any $s \in [0, 1]$. It follows from (H2) and (iv) of Lemma 2.2 that for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t, s)a(s)f(s, u(s))ds + \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i, s)a(s)f(s, u(s))ds t^{\alpha-1} \\ &\geq 16Mr_1 \left(\int_0^1 \frac{1}{16} G(t_0, s)a(s)ds + \left(\frac{1}{4}\right)^{\alpha-1} \Delta \sum_{i=1}^{m} \beta_i \int_0^1 G(\xi_i, s)a(s)ds\right) \\ &= r_1 = \|u\|,\end{aligned}$$

which means that

$$\|Tu\| \geq \|u\| \quad \text{for any } u \in P \cap \partial\Omega_1.$$

Step 2. Let $\Omega_2 = \{u \in E : \|u\| < r_2\}$. For any $u \in P \bigcap \partial\Omega_2$, we have $0 \leq u(s) \leq r_2$ for any $s \in [0,1]$. It follows from (H1) and (ii) of Lemma 2.2 that for $t \in [0,1]$,

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))ds + \Delta \sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))dst^{\alpha-1} \\ &\leq Mr_2 \left(\int_0^1 G(t_0,s)a(s)ds + \Delta \sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)ds \right) \\ &= r_2 = \|u\|,\end{aligned}$$

which means that

$$\|Tu\| \leq \|u\| \quad \text{for any } u \in P \cap \partial\Omega_2.$$

By (ii) of Theorem 2.1, we get that T has a fixed point u in P with $r_1 \leq \|u\| \leq r_2$, which is also a positive solution of BVP (1).

Theorem 3.2. *Assume that there exist three positive constants a, b, c with $0 < a < b < c$ such that*

(A1) $f(t,u) < Ma$ *for any* $(t,u) \in [0,1] \times [0,a]$;

(A2) $f(t,u) > 16Nb$ *for any* $(t,u) \in [\frac{1}{4}, \frac{3}{4}] \times [b,c]$;

(A3) $f(t,u) \leq Mc$ *for any* $(t,u) \in [0,1] \times [0,c]$.

Then BVP (1) has at least one non-negative solution u_1 and two positive solutions u_2, u_3 in P with

$$\max_{t\in[0,1]} |u_1(t)| < a, \qquad b < \min_{t\in[\frac{1}{4},\frac{3}{4}]} |u_2(t)|,$$
$$a < \max_{t\in[0,1]} |u_3(t)|, \qquad \min_{t\in[\frac{1}{4},\frac{3}{4}]} |u_3(t)| < b.$$

Proof. We will show that all conditions of Theorem 2.2 are satisfied.

Define a function θ on cone P by

$$\theta(u) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t),$$

then θ is a non-negative continuous concave function on cone P. For any $u \in \overline{P_c}$, it is obvious that $\theta(u) \leq \|u\|$ and $0 \leq u(t) \leq \|u\| \leq c$. Thus by $(A3)$ we have

$$\begin{aligned}\|Tu\| = \max_{t\in[0,1]} \Bigg(&\int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &+ \Delta \sum_{i=1}^m \beta_i \int_0^1 G(\xi_i,s)a(s)f(s,u(s))dst^{\alpha-1} \Bigg)\end{aligned}$$

$$\leq Mc\left(\int_0^1 G(t_0,s)a(s)ds + \Delta\sum_{i=1}^{m}\beta_i\int_0^1 G(\xi_i,s)a(s)ds\right)$$
$$= c.$$

Hence $T:\overline{P_c}\to\overline{P_c}$ is completely continuous.

Firstly, we check condition (c1) of Theorem 2.2. We choose $u(t)=\frac{b+c}{2}, t\in[0,1]$. It is easy to see that $u(t)\in P(\theta,b,c)$ and $\theta(u)=\frac{b+c}{2}>b$, which means that $\{P(\theta,b,c)|\theta(u)>b\}\neq\emptyset$. For any $u\in P(\theta,b,c)$, we have $b\leq u(t)\leq c$ for any $t\in[\frac{1}{4},\frac{3}{4}]$, so from assumption (A2), we get that $f(s,u(s))>16Nb, s\in[\frac{1}{4},\frac{3}{4}]$. Thus by (iv) of Lemma 2.2 we have

$$\begin{aligned}
\theta(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]}\left(\int_0^1 G(t,s)a(s)f(s,u(s))ds\right.\\
&\quad\left.+\Delta\sum_{i=1}^{m}\beta_i\int_0^1 G(\xi_i,s)a(s)f(s,u(s))dst^{\alpha-1}\right)\\
&\geq \frac{1}{16}\int_0^1 G(t_0,s)a(s)f(s,u(s))ds\\
&\quad+\left(\frac{1}{4}\right)^{\alpha-1}\Delta\sum_{i=1}^{m}\beta_i\int_0^1 G(\xi_i,s)a(s)f(s,u(s))ds\\
&\geq \frac{1}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)a(s)f(s,u(s))ds\\
&\quad+\left(\frac{1}{4}\right)^{\alpha-1}\Delta\sum_{i=1}^{m}\beta_i\int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i,s)a(s)f(s,u(s))ds\\
&> \frac{1}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)a(s)16Nbds+\frac{1}{16}\Delta\sum_{i=1}^{m}\beta_i\int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i,s)a(s)16Nbds\\
&= b.
\end{aligned}$$

(c2) of Theorem 2.2 is not difficult to proved by (A1).

(c3) is also obvious since $d=c$ and thus (c1) implies (c3) here.

By Theorem 2.2, T has at least three fixed points u_1,u_2,u_3 in P with

$$\max_{t\in[0,1]}|u_1(t)|<a,\qquad b<\min_{t\in[\frac{1}{4},\frac{3}{4}]}|u_2(t)|,$$
$$a<\max_{t\in[0,1]}|u_3(t)|,\qquad \min_{t\in[\frac{1}{4},\frac{3}{4}]}|u_3(t)|<b,$$

which means BVP (1) has at least one non-negative solution u_1 and two positive solutions u_2,u_3.

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