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Perturbations of Fredholm linear relations in Banach spaces with application to 3 × 3-block matrices of linear relations

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Abstract. The main focus in this paper consists of extending the main results of Diagana (2015) to linear relations. Moreover, we apply our obtained results to study some properties of 3×3-block matrices of linear relations in the form,

$$\mathcal{A} := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}.$$

Namely, we show that under some suitable conditions on relations B, C, F, D, G and H, the relation $\mathcal{A}$ is closed if and only if A, E and K are closed. In addition, we show that if A, E, K are Fredholm linear relations, then so is the linear relation $\mathcal{A}$.

Keywords: Closed linear relation; Fredholm linear relation; 3×3 matrix linear relation

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1. Introduction

In this paper, the symbols X, Y, Z stand for infinite dimensional Banach spaces over the same field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

A multivalued linear operator (or a linear relation) is a mapping $T \subset X \times Y$ which goes from a subspace $\mathcal{D}(T) \subset X$ called the domain of T, into the collection of nonempty subsets

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of Y such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all nonzero scalars α_1, α_2 and $x_1,\ x_2 \in \mathcal{D}(T)$.

For $x \in X\backslash\mathcal{D}(T)$, we define $Tx = \emptyset$. With this notation, we have

$$\mathcal{D}(T) = \Big\{x \in X : Tx \neq \emptyset\Big\}.$$

The collection of linear relations as defined above will be denoted by $L\mathcal{R}(X,Y)$. A linear relation $T \in L\mathcal{R}(X,Y)$ is uniquely determined by and identified with its graph, $G(T)$, which is defined by

$$G(T) = \Big\{(x,y) \in X \times Y : x \in \mathcal{D}(T),\ y \in Tx\Big\}.$$

The inverse of $T \in L\mathcal{R}(X,Y)$ is the linear relation T^{-1} defined by

$$G(T^{-1}) = \Big\{(y,x) \in Y \times X :\ (x,y) \in G(T)\Big\}.$$

Let $T \in L\mathcal{R}(X,Y)$. The symbols $\mathcal{R}(T)$, $\mathcal{N}(T)$ and $T(0)$ stand for the range, the null space, and the multivalued part of T, which are defined respectively by

$$R(T) := \Big\{y :\ (x,y) \in T\Big\},$$
$$N(T) := \Big\{x \in \mathcal{D}(T) :\ (x,0) \in T\Big\}, \quad \text{and}$$
$$T(0) := \Big\{y :\ (0,y) \in T\Big\}.$$

Observe that $Tx = y + T(0)$, for any $y \in Tx$. A linear relation T is said to be surjective if $R(T) = Y$. Similarly, T is said to be injective, if $N(T) = \{0\}$. Now if T is both injective and surjective, then we say that T is bijective.

If $T,\ S \in L\mathcal{R}(X,Y)$, then their algebraic sum $T + S$ is also a linear relation defined by

$$T + S := \Big\{(x, u+v) :\ (x,u) \in T,\ (x,v) \in S\Big\}.$$

Similarly, if $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$, then their composition TS is also a linear relation, which is defined by

$$ST := \Big\{(x,z) \in X \times Z :\ (x,y) \in T \text{ and } (y,z) \in S \text{ for some } y \in Y\Big\}.$$

If M is a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$, then $T_{|M\cap\mathcal{D}(T)} := T_{|M}$ is defined by

$$T_{|M} := \Big\{(x,y) \in T :\ x \in M\Big\}.$$

The quotient map from Y onto $Y/\overline{T(0)}$ is denoted by Q_T. It easy to see that $Q_T T$ is single valued so that we can define

$$\|Tx\| := \|Q_T Tx\| \quad \text{for all } x \in\ \mathcal{D}(T) \quad \text{and} \quad \|T\| := \|Q_T T\|.$$

We say that $T \in L\mathcal{R}(X,Y)$ is continuous if $\|T\| < \infty$; bounded if it is continuous and $\mathcal{D}(T) = X$; open if T^{-1} is continuous equivalently if its minimum modulus $\gamma(T)$ is a positive number, where

$$\gamma(T) = \sup\Big\{\lambda \geq 0 : \lambda d(x, \mathcal{N}(T)) \leq \|Tx\|,\ \ x \in \mathcal{D}(T)\Big\}.$$

A linear relation T is said to be closed if its graph is closed. Similarly, T is called closable if $\overline{T}$ is an extension of T where the closure of T, $\overline{T}$, is defined by $G(\overline{T}) := \overline{G(T)}$.

We denote the class of all bounded linear relations from X to Y by $S\mathcal{R}(X,Y)$. The collection of all closed linear relations from X to Y is denoted by $C\mathcal{R}(X,Y)$. The set $K\mathcal{R}(X,Y)$ will denote the class of compact linear relations from X to Y where $T \in L\mathcal{R}(X,Y)$ is called compact if $\overline{Q_T T B_X}$ is compact where B_X is the unit ball of X.

If M and N are subspaces of X and if X' is the (topological) dual of X, then we define,

$$M^{\perp} := \Big\{ x' \in X' : \ x'(x) = 0 \ \text{ for all } x \in M \Big\}$$

and

$$N^{\top} := \Big\{ x \in X : \ x'(x) = 0 \ \text{ for all } x' \in N \Big\}.$$

The conjugate of $T \in L\mathcal{R}(X,Y)$ is the linear relation T' defined by

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X',$$

so that $(y',x') \in G(T')$ if and only if $y'(y) = x'(x)$ for all $(x,y) \in G(T)$.

If $T \in L\mathcal{R}(X,Y)$, then we write $\alpha(T) := \dim N(T)$, $\beta(T) := \dim Y/R(T)$, $\overline{\beta}(T) := \dim Y/\overline{R(T)}$, and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite.

Definition 1.1. A linear relation $T \in C\mathcal{R}(X,Y)$ is said to be upper semi-Fredholm and denoted $A \in \mathcal{F}_+(X,Y)$, if there exists a closed finite codimensional subspace M of X such that the restriction $T|_M$ has a single-valued continuous inverse.

A linear relation T is said to be lower semi-Fredholm and denoted $T \in \mathcal{F}_-(X,Y)$, if its conjugate T' is upper semi-Fredholm.

In the case when both X and Y are Banach spaces, we extend the classes of closed single-valued Fredholm type operators given earlier to include closed multivalued operators. Note that the definitions of the classes $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ are consistent with

$$\Phi_+(X,Y) = \{T \in C\mathcal{R}(X,Y) : \ \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\},$$
$$\Phi_-(X,Y) = \{T \in C\mathcal{R}(X,Y) : \ \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

A linear relation T on X is said to be a semi-Fredholm (respectively, a Fredholm) relation if $T \in \Phi_+(X) \cup \Phi_-(X)$, (respectively, $\Phi_+(X) \cap \Phi_-(X)$).

The concept of linear relation appeared in the literature some decades ago through not only the need of considering adjoints (conjugates) of non-densely defined linear differential operators (see, e.g., J. von Neumann [30]) but also the need of considering the inverses of certain operators, used, for example, in the study of some Cauchy problems associated with parabolic type equations in Banach spaces (see, e.g., [18]).

The spectral theory for linear relations, especially the theory for the essential spectra, has important applications to several problems arising in operator theory, see, e.g., [1,4,6,3,12, 9–11,20–22,26,27]. First, the spectral theory of ordered pair of operators. Many properties of the spectrum and the essential spectra of the pair (G,F) of closed operators are obtained as applications to the corresponding properties of the linear relations $F^{-1}G$ and GF^{-1}.

Second, the study of linear bundles. Let $T, S : \ X \longrightarrow Y$ be bounded operators. The map $P(\lambda) : T + \lambda S, \ \ \lambda \in \mathbb{C}$ is called a linear bundle. It is well-known that many problems arising in mathematical physics including quantum theory are reduced to the study of certain reversibility conditions for operators $P(\lambda)$ and this last study is reduced to the study of the spectral properties of the linear relations $S^{-1}T$ and $T^{-1}S$.

It is worth mentioning that the study of the essential spectra of a linear relation may provide some useful tools for the study of operators as the class of all bounded Fredholm operators in a Banach space is the class of inverses of closed Fredholm linear relations that are bijective. Let us mention that many problems arising in mathematical physics are described through systems of partial or ordinary differential equations or the linearization of those.

This paper is devoted to the study of some properties of unbounded linear relations. Among other things, we show that most of the results obtained by Diagana [17] and others remain valid for operators in the context of multivalued linear operators. In this work, we give some sufficient conditions so that if $A,\ B,\ C$ are three unbounded linear relations with A being a Fredholm linear relation, then their algebraic sum $A+B+C$ is also a Fredholm linear relation.

The paper is organized as follows. Section 2 contains preliminary and auxiliary results that will be needed in the sequel. In Section 3 we study sufficient conditions for the Fredholmness of the algebraic sum $A+B+C$ in the case when $A,\ B,\ C$ are three unbounded linear relations with A being a Fredholm linear relation (Theorems 3.2 and 3.3). In Section 4, we apply the results obtained in Section 3 to study some properties for some 3×3-block matrices of linear relations. Finally, in Section 5, we outline some open questions.

2. Auxiliary results

The goal of this section consists of establishing some preliminary results which will be needed in the sequel. For that, we begin by giving some auxiliary results from the theory of linear relations in Banach spaces.

Lemma 2.1 *([15, Proposition II.5.3], [2, Lemma 5.3]). Let $T \in L\mathcal{R}(X,Y)$. Then*

(i) *T is closed if and only if $Q_T T$ is closed and $T(0)$ is closed. In particular, $N(T)$ is closed if T is closed.*

(ii) *Assume that $T \in L\mathcal{R}(X,Y)$ is closed. Then,*
  - (ii_1) *$R(T)$ is closed if and only if so is $R(Q_T T)$.*
  - (ii_2) *$T \in \Phi_+(X,Y)$ if and only if $Q_T T \in \Phi_+(X, Y/T(0))$ if and only if $T' \in \Phi_-(Y', X')$. In such case, $i(T) = -i(T')$.*
  - (ii_3) *$T \in \Phi_-(X,Y)$ if and only if $Q_T T \in \Phi_-(X, Y/T(0))$ if and only if $T' \in \Phi_+(Y', X')$. In such case, $i(T) = -i(T')$.* $\diamond$

Lemma 2.2 *([16, Corollary 3.2]). Let $X,\ Y Z$ be three vector spaces, $T \in L\mathcal{R}(X,Y),\ S \in L\mathcal{R}(X,Y),\ \mathcal{D}(S) = Y$ and suppose that T and S have finite indices. Then, ST has a finite index and:*

$$i(ST) = i(S) + i(T) - \dim\Big(T(0) \cap S^{-1}(0)\Big).$$

Definition 2.1. Let X be a Banach space and let $T \in L\mathcal{R}(X)$. The graph operator, $G_T \in L\mathcal{R}(X_T, X)$ is defined by

$$\mathcal{D}(G_T) = X_T \quad \text{and} \quad G_T\varphi = \varphi \quad \text{for } \varphi \in X_T. \quad \diamond$$

Definition 2.2 (*[15, Definition VII.2.1]*). A relation $S \in L\mathcal{R}(X,Y)$ is said to be T- bounded if $\mathcal{D}(S) \supset \mathcal{D}(T)$ and there exist real numbers a, b for which the inequality

$$\|S\varphi\| \leq a\|\varphi\| + b\|T\varphi\| \tag{2.1}$$

for all $\varphi \in \mathcal{D}(T)$. In that case, the infimum of all the numbers b such that (2.1) holds is called the T-bound of S. $\diamond$

Remark 2.1. The inequality (2.1) is equivalent to,

$$\|S\varphi\|^2 \leq a_1^2\|\varphi\|^2 + b_2^2\|T\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(T)$$

where $a_1 = \sqrt{a^2 + ab}$ and $b_1 = \sqrt{b^2 + ab}$.

Lemma 2.3. *Let S, $T \in L\mathcal{R}(X,Y)$ such that $S(0) \subset \overline{T(0)}$ and $\overline{\mathcal{D}(T)} \subset \mathcal{D}(S)$. Then,*

(i) $T - S + S = T$.
(ii) *$Q_T(S)$ is a single valued operator and $\|Q_T(S)\| \leq \|Q_S(S)\|$.* $\diamond$

Proof. (i) Let $(\varphi,\psi) \in G(T+S-S)$ then $\varphi \in \mathcal{D}(T-S+S) = \mathcal{D}(T)$ and $\psi \in (T-S+S)\varphi$, so that, $\psi \in T\varphi + S(0)$. On the other hand, using the fact $S(0) \subset T(0)$ it follows that $T\varphi + S(0) \subset T\varphi + T(0)$ which yields $\psi \in T\varphi$ and $\varphi \in \mathcal{D}(T)$, that is $(\varphi,\psi) \in G(T)$. Therefore, $G(T+S-S) \subseteq G(T)$. Conversely, let $(\varphi,\psi) \in G(T)$, so that $\varphi \in \mathcal{D}(T)$ and $T\varphi = \psi + T(0)$. On the other hand, from $(T+S-S)(0) = T(0)$ and $\mathcal{D}(T-S+S) = \mathcal{D}(T)$ we deduce that $\varphi \in \mathcal{D}(T-S+S)$ and $(T+S-S)\varphi = \psi + (T+S-S)(0)$. That is, $G(T) \subseteq G(T+S-S)$.

(ii) Using [15, Proposition II.1.3] it follows that $Q_T(T)$ is single valued. Since $S(0) \subset \overline{T(0)}$, then $Q_T(S(0)) \subset Q_T(\overline{T(0)}) = 0$. Consequently, $Q_T(S)$ is single valued, and

$$\|Q_T(S)\| = d(S\varphi, T(0)) = d(S\varphi, \overline{T(0)}) \leq d(S\varphi, S(0)) = \|S\varphi\| = \|Q_S(S)\|. \quad \square$$

Let T be a closed linear relation on a Banach space X. For $x \in \mathcal{D}(T)$ the graph norm of x is defined by $\|x\|_T := \|x\| + \|Tx\|$. It follows from the closedness of T that $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space.

3. Main results

Lemma 3.1. *If S is T-bounded with T-bound $\delta < 1$ and $S(0) \subset T(0)$, then S is $(T+S)$-bounded with T-bound $\leq \frac{\delta}{1-\delta}$.* $\diamond$

Proof. First of all, it should be mentioned that the linear relation $T+S$ is well-defined as $\mathcal{D}(T+S) = \mathcal{D}(S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ with $\mathcal{D}(T+S) \subset \mathcal{D}(S)$. Using the fact that S is T- bounded, it follows that there exist $a > 0$, $\delta \leq b < 1$, such that for all $\varphi \in \mathcal{D}(T)$,

$$\begin{array}{rcl}\|S\varphi\| & \leq & a\|\varphi\| + b\|T\varphi\| \\ & = & a\|\varphi\| + b\|T\varphi + S\varphi - S\varphi\| \quad \text{(Lemma 2.3(ii))} \\ & \leq & a\|\varphi\| + b\|T\varphi + S\varphi\| + b\|S\varphi\|.\end{array}$$

Since $b < 1$, it follows that

$$\|S\varphi\| \leq \frac{a}{1-b}\|\varphi\| + \frac{b}{1-b}\|(T+S)\varphi\|, \quad \varphi \in \mathcal{D}(T). \quad \square$$

Theorem 3.1. *Let $S, T \in L\mathcal{R}(X,Y)$ such that $\mathcal{D}(S) \supset \mathcal{D}(T)$, $S(0) \subset \overline{T(0)}$ and*

$$\|S\varphi\| \leq a\|\varphi\| + b\|T\varphi\|, \quad \varphi \in \mathcal{D}(T), \tag{3.1}$$

for some constants a, b with $b < 1$. Then,

$$T \in C\mathcal{R}(X,Y) \quad \textit{if and only if } T+S \in C\mathcal{R}(X,Y). \quad \diamond$$

Proof. Suppose $T \in C\mathcal{R}(X,Y)$. Let us consider two cases for S and T.

Case 1: S and T are single valued. Then, using (3.1), we obtain that

$$\|T\varphi\| = \|(T+S-S)\varphi\| \leq \|(T+S)\varphi\| + \|S\varphi\|, \quad \varphi \in \mathcal{D}(T),$$

which in turn yields

$$(1-b)\|T\varphi\| \leq \|(T+S)\varphi\| + a\|\varphi\|, \quad \varphi \in \mathcal{D}(T).$$

Thus if $\{\varphi_n\}_{n\in\mathbb{N}}$ is sequence in $\mathcal{D}(T+S) = \mathcal{D}(T)$ such that $\varphi_n \to \varphi$ and $(T+S)\varphi_n \to \psi$. Then, there exists $N_0 \in \mathbb{N}$ such that for all $n,\ m \geq N_0$ we have

$$(1-b)\|T\varphi_n - T\varphi_m\| \leq a\|\varphi_n - \varphi_m\| + \|(T+S)\|\|\varphi_n - \varphi_n\|,$$

and therefore $\{T\varphi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space Y. Thus, $\{T\varphi_n\} \to \varphi_0$. Since T is closed, $\varphi \in \mathcal{D}(T)$ and $T\varphi = \varphi_0$. Moreover $S\varphi_n = (T+S)\varphi_n - T\varphi_n \to \psi - \varphi_0$ as $n \to \infty$. But

$$\|S(\varphi_n - \varphi)\| \leq \Big(a\|\varphi_n - \varphi\| + b\|T\varphi_n - \varphi_0\|\Big) \to 0 \quad \text{as } n \to \infty,$$

which shows that $S\varphi = \psi - \varphi_0$ and hence $(T+S)\varphi = \psi$ and $\varphi \in \mathcal{D}(T+S)$.

Case 2: S and T are linear relations. Since $S(0) \subset \overline{T(0)} = T(0)$, it is clear that $Q_T = Q_{T+S}$. Then $Q_{T+S}(T+S) = Q_T(T) + Q_T(S)$ and using Lemma 2.3 we deduce that $Q_T(S)$ is single valued and that

$$\|Q_T(S)\varphi\| \leq \|Q_S(S)\varphi\| = \|S\varphi\| \leq a\|\varphi\| + b\|T\varphi\|, \quad \varphi \in \mathcal{D}(T).$$

Now

$$\|Q_T(S)\varphi\| \leq a\|\varphi\| + b\|Q_T(T)\varphi\|, \quad \varphi \in \mathcal{D}(T).$$

On the other hand $Q_T(T)$ is closed, then $Q_T(T) + Q_T(S)$ is closed single valued. This means, since $(T+S)(0) = T(0)$ is closed, that $T+S$ is closed. Conversely, assume that

$T+S$ is closed. It follows from Lemma 3.1 that,

$$\|-S\varphi\| = \|S\varphi\| \leq \frac{a}{1-b}\|\varphi\| + \frac{b}{1-b}\|(T+S)\varphi\|, \quad \varphi \in \mathcal{D}(T).$$

In view of the above, $T+S-S$ is closed. Now by Lemma 2.3, T is closed. $\square$

Theorem 3.2. *Let $S, T, K \in L\mathcal{R}(X,Y)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S) \subset \mathcal{D}(K)$, $K(0) \subset S(0) \subset \overline{T(0)}$ and such that:*

(i) *there exist two constants $a_1,\ b_1 > 0$*

$$\|S\varphi\| \leq a_1\|\varphi\| + b_1\|T\varphi\|, \quad \varphi \in \mathcal{D}(T),$$

(ii) *there exist two constants $a_2,\ b_2 > 0$ such that $b_1(1+b_2) < 1$ and*

$$\|K\varphi\| \leq a_2\|\varphi\| + b_2\|S\varphi\|, \quad \varphi \in \mathcal{D}(S).$$

Then, $T \in C\mathcal{R}(X,Y)$ if and only if $T+S+K \in C\mathcal{R}(X,Y)$. $\diamond$

Proof. Let us consider two cases for S, T and K.

Case 1: S, T and K are operators, then for all $\varphi \in \mathcal{D}(T)$ we have

$$\begin{aligned}\|(T+S+K)\varphi\| &\leq \|T\varphi\| + \|S\varphi\| + \|K\varphi\| \\ &\leq \|T\varphi\| + a_1\|\varphi\| + b_1\|T\varphi\| + a_2\|\varphi\| + b_2\|S\varphi\| \\ &= (1+b_1)\|T\varphi\| + (a_1+a_2)\|\varphi\| + b_2(a_1\|\varphi\| + b_1\|T\varphi\|),\end{aligned}$$

and hence,

$$\|(T+S+K)\varphi\| \leq (a_1+a_2+a_1b_2)\|\varphi\| + (1+b_1+b_1b_2)\|T\varphi\|. \tag{3.2}$$

Similarly, for all $\varphi \in \mathcal{D}(T)$, we have

$$\begin{aligned}\|(S+K)\varphi\| &\leq \|S\varphi\| + \|K\varphi\| \\ &\leq a_1\|\varphi\| + b_1\|T\varphi\| + a_2\|\varphi\| + b_2\|S\varphi\| \\ &= b_1\|T\varphi\| + (a_1+a_2)\|\varphi\| + b_2(a_1\|\varphi\| + b_1\|T\varphi\|),\end{aligned}$$

and hence

$$\|(S+K)\varphi\| \leq (a_1+a_2+a_1b_2)\|\varphi\| + b_1(1+b_2)\|T\varphi\|. \tag{3.3}$$

Combining Eqs. (3.2) and (3.3), it follows that for all $\varphi \in \mathcal{D}(T)$,

$$\begin{aligned}\|(T+S+K)\varphi\| &\geq \|T\varphi\| - \|S\varphi\| + \|K\varphi\| \\ &\geq \|T\varphi\| - (a_1+a_2+a_1b_2)\|\varphi\| - b_1(1+b_2)\|T\varphi\| \\ &= -(a_1+b_2+a_1b_2)\|\varphi\| + [1-b_1(1+a_2)]\|T\varphi\|\end{aligned}$$

which yields,

$$[1-b_1(1+a_2)]\|T\varphi\| \leq \|(T+S+K)\varphi\| + (a_1+b_2+a_1b_2)\|\varphi\|.$$

Setting $\Theta = 1 - b_1(1 + a_2)$ $(0 < \Theta < 1)$ and $\Psi = a_1 + b_2 + a_1 b_2$ $(\Psi > 0)$, gives

$$\|T\varphi\| \leq \Theta^{-1}\Big(\|(T + S + K)\varphi\| + \Psi\|\varphi\|\Big). \tag{3.4}$$

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ is sequence in $\mathcal{D}(T+S+K) = \mathcal{D}(T)$ such that $\varphi_n \to \varphi$ and $(T+S+K)\varphi_n \to \psi$. Then, By Eq. (3.4) there exists $N_1 \in \mathbb{N}$ such that for all $n,\ m \geq N_1$ we have

$$\|T\varphi_n - T\varphi_m\| \leq \Theta^{-1}\Big(\|(T + S + K)(\varphi_n - \varphi_m)\| + \Psi\|\varphi_n - \varphi_m\|\Big). \tag{3.5}$$

So, $\{T\varphi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space Y and therefore there exist $\psi_1 \in Y$ such that $T\varphi_n \to \psi_1$. Since T is closed it follows that $\varphi \in \mathcal{D}(T)$ and $T\varphi = \psi_1$. From Eq. (3.2) it follows that

$$\begin{aligned} &\|(T + S + K)(\varphi_n - \varphi)\| \\ &\quad \leq \Big((a_1 + a_2 + a_1 b_2)\|\varphi_n - \varphi\| + (1 + b_1 + b_1 b_2)\|T\varphi_n - \varphi\|\Big) \to 0 \end{aligned} \tag{3.6}$$

then, by letting $n \to +\infty$, then $(T+S+K)\varphi_n \to \psi = (T+S+K)\varphi$. Therefore, $A+B+C$ is a closed linear operator.

Case 2: T, S and K are linear relations. Since $K(0) \subset S(0) \subset T(0)$, it is clear that $Q_{T+S+K} = Q_T$. Then

$$Q_{T+S+K}(T + S + K) = Q_T(T + S + K) = Q_T(T) + Q_T(S) + Q_T(K).$$

Since $Q_T(T)$ is a closed operator (by Lemma 2.1), using Lemma 2.3 we deduce that $Q_T(S)$, $Q_T(K)$ are single valued and that $\|Q_T(S)\| \leq \|Q_S(S)\|$ and $\|Q_T(K)\| \leq \|Q_K(K)\|$.

Hence,

$$\|Q_T(S)\| \leq \|Q_S(S)\| = \|S\varphi\| \leq a_1\|\varphi\| + b_1\|T\varphi\|, \quad \varphi \in \mathcal{D}(T),$$

and hence,

$$\|Q_T(S)\| \leq \|Q_S(S)\varphi\| \leq a_1\|\varphi\| + b_1\|Q_T(T)\varphi\|, \quad \varphi \in \mathcal{D}(T). \tag{3.7}$$

Similarly, we have

$$\|Q_T(K)\| \leq a_2\|\varphi\| + b_2\|Q_S(S)\varphi\|, \quad \varphi \in \mathcal{D}(S),$$

where $b_1(1 + a_2) < 1$. Consequently, $Q_T(T) + Q_T(S) + Q_T(K)$ is a closed operator. Since $T(0)+S(0)+K(0) = T(0)$ is closed then, $T+S+K$ is a closed linear relation. Conversely, assume that $T + K + S$ is closed.

$$\|-S\varphi\| \leq \big(a_1 + b_1\Psi\big)\|\varphi\| + b_1\Theta^{-1}\|(T + K + S)\varphi\|, \quad \varphi \in \mathcal{D}(T),$$

and

$$\|-K\varphi\| \leq a_2\|\varphi\| + b_2\| - S\varphi\|, \quad \varphi \in \mathcal{D}(S) \quad \text{and} \quad b_1\Theta^{-1}(1 + a_2) < 1.$$

Now, applying what precedes to the relations $T+S+K$, $-K$ and $-S$ we deduce that $T+S+K-K-S$ is closed operator. Since $(K+S)(0) \subset T(0)$ then

$$T = T + (K+S) - (K+S) = (T+K+S) - K - S.$$

Therefore, T is closed. $\square$

Theorem 3.3. *Let $S, T, K \in L\mathcal{R}(X,Y)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S) \subset \mathcal{D}(K)$, $K(0) \subset S(0) \subset \overline{T(0)}$ and let $\widehat{T}$ be the bijection associated with T. Suppose*

(a) *there exists a constant α_1 such that*

$$\|S\varphi\| \leq \alpha_1 \big(\|\varphi\| + \|T\varphi\|\big), \quad \varphi \in \mathcal{D}(T),$$

(b) *there exists a constant β_1 such that $\alpha_1(1+\beta_1) < 1$, $(1+\beta_1) < \gamma(\widehat{T})$ and*

$$\|K\varphi\| \leq \beta_1 \big(\|\varphi\| + \|S\varphi\|\big), \quad \varphi \in \mathcal{D}(T).$$

If $T \in \Phi(X,Y)$ then the sum $T+S+K \in \Phi(X,Y)$ and satisfies the following properties:

(i) $\alpha(T+S+K) \leq \alpha(T)$*; and*

(ii) $\overline{\beta}(T+S+K) \leq \overline{\beta}(T)$.

Proof. From Theorem 3.2 it follows that $T+S+K$ is a closed linear relation. Let $T_1,\ S_1\ K_1$ be the restrictions of the relation $T,\ S,\ K$ to X_T. Obviously, T is a Fredholm linear relation and $S_1,\ K_1$ is a bounded linear relation. Moreover, it is easy to prove that (see, [15, Theorem III.5.3]),

$$\|S_1 + K_1\| \leq \Big(\beta_1 + \alpha_1(1+\beta_1)\Big) \leq \gamma(\widehat{T}) = \gamma(\widehat{T'}),$$

which combined with [15, Theorem V.5.12] and [15, Theorem V.3.2] yields $S_1 + K_1 + T_1$ is a Fredholm linear relation.

Properties (i) and (ii) are straightforward consequences of [15, Theorem III.7.4]. $\square$

Corollary 3.1. *Let $S,\ T,\ K$ be three operators such that $\mathcal{D}(T) \subset \mathcal{D}(S) \subset \mathcal{D}(K)$ and let $\widehat{T}$ be the bijection associated with T. Suppose,*

(a) *there exists a constant α_1 such that*

$$\|S\varphi\| \leq \alpha_1 \big(\|\varphi\| + \|T\varphi\|\big), \quad \varphi \in \mathcal{D}(T),$$

(b) *there exists a constant β_1 such that $\alpha_1(1+\beta_1) < 1$, $(1+\beta_1) < \gamma(\widehat{T})$ and*

$$\|K\varphi\| \leq \beta_1 \big(\|\varphi\| + \|S\varphi\|\big), \quad \varphi \in \mathcal{D}(T).$$

If $T \in \Phi(X,Y)$ then the sum $T+S+K \in \Phi(X,Y)$ and also satisfies the following properties

(i) $\alpha(T+S+K) \leq \alpha(T)$*;*

(ii) $\beta(T+S+K) \leq \beta(T)$*; and*

(iii) $i(T+S+K) = i(T)$.

4. Application

Noteworthy progress has been made during the past years (see, e.g., [8,14,23,24,28]) in studying the spectra of 2×2 block operator matrices $\mathcal{A}_0$ which act on the product $X \times Y$ of Banach spaces and of the form

$$\mathcal{A}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

A comprehensive account of the investigations undertaken in that regard as well as a wide panorama of the tools used to study the spectrum of block operator matrices are presented in Tretter [29]. In general, the operators involved in $\mathcal{A}_0$ are unbounded and $\mathcal{A}_0$ need not to be a closed nor a closable operator, even if its entries are closed. However, under some conditions $\mathcal{A}_0$ becomes closable and its closure $\mathcal{A}$ can be determined.

In [25], Moalla, Damak and Jeribi extended and obtained results for a large class of operators and described various essential spectra of $\mathcal{A}$ and applied their findings to describe the essential spectra of two-group transport operators with general boundary conditions in L_p-spaces.

In [13], Bátkai, Binding, Dijksma, Hryniv and Langer studied a 2×2 block operator matrix and described its essential spectrum under the assumption that $\mathcal{D}(A) \subset \mathcal{D}(C)$, that $D(B) \cap D(D)$ is sufficiently large, and that the domain of the operator matrix is defined by an additional relation of the form $\Gamma_X x = \Gamma_Y y$ between the two components of its elements.

In [7], Ammar, Jeribi, and Moalla, considered the case of 3×3 block operator matrix and proposed an abstract approach to study the essential spectra of the operator. Recently, Álvarez, Ammar and Jeribi [5] investigated a detailed treatment of some of the subsets of the essential spectra of a 2×2 block of matrix of linear relations.

In this section, we consider in the product of Banach spaces $X \times Y \times Z$ the linear relation defined by a 3×3 block matrix

$$\mathcal{A} := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}, \tag{4.1}$$

where the entries of the matrix are in general unbounded linear relations.

The relation A acts on X with domain $\mathcal{D}(A)$, the relation E acts on Y with domain $\mathcal{D}(E)$, and the relation K acts on Z with domain $\mathcal{D}(K)$. The intertwining relation B is defined on the domain $\mathcal{D}(B) \subset Y$ to X, the relation H is defined on the domain $\mathcal{D}(H) \subset Y$ to Z, the relation C is defined on the domain $\mathcal{D}(C) \subset Z$ to X, the relation F is defined on the domain $\mathcal{D}(F) \subset Z$ to Y, the relation D is defined on the domain $\mathcal{D}(D) \subset X$ to Y, and the relation G is defined on the domain $\mathcal{D}(G) \subset X$ to Z. The relation $\mathcal{A}$ is defined on

$$\begin{cases} G(\mathcal{A}) = \left\{ \left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \in (X \times Y \times Z)^2 : \begin{array}{l} v_1 \in Au_1 + Bu_2 + Cu_3 \\ v_2 \in Du_1 + Eu_2 + Fu_3 \\ v_3 \in Gu_1 + Hu_2 + Ku_3 \end{array} \right\} \\ \mathcal{D}(\mathcal{A}) = \Big(\mathcal{D}(A) \cap \mathcal{D}(D) \cap \mathcal{D}(G)\Big) \times \Big(\mathcal{D}(B) \cap \mathcal{D}(E) \cap \mathcal{D}(H)\Big) \\ \quad \times \Big(\mathcal{D}(C) \cap \mathcal{D}(F) \cap \mathcal{D}(K)\Big). \end{cases}$$

We denote by

$$\mathcal{T} := \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}, \qquad \mathcal{S} := \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & F \\ G & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{K} := \begin{pmatrix} 0 & 0 & C \\ D & 0 & 0 \\ 0 & H & 0 \end{pmatrix}.$$

Then, it is clear that $\mathcal{A} = \mathcal{T} + \mathcal{S} + \mathcal{K}$.

Proposition 4.1. *We have,*

(i) $Q_{\mathcal{T}}\mathcal{T} := \begin{pmatrix} Q_A(A) & 0 & 0 \\ 0 & Q_E(E) & 0 \\ 0 & 0 & Q_K(K) \end{pmatrix}$,

(ii) $Q_{\mathcal{S}}\mathcal{S} := \begin{pmatrix} 0 & Q_B(B) & 0 \\ 0 & 0 & Q_F(F) \\ Q_G(G) & 0 & 0 \end{pmatrix}$, *and*

(iii) $Q_{\mathcal{K}}\mathcal{K} := \begin{pmatrix} 0 & 0 & Q_C(C) \\ Q_D(D) & 0 & 0 \\ 0 & Q_H(H) & 0 \end{pmatrix}$.

Proof. The proof of (ii) and (iii) can be done in a similar fashion as that of (i).

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{D}(A) \times \mathcal{D}(E) \times \mathcal{D}(K)$ and $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $u \in Ax, v \in Ey$ and $w \in Kz$. This yields,

$$Q_{\mathcal{T}}\mathcal{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Q_{\mathcal{T}}\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Now, let us find the expression of $Q_{\mathcal{T}}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

Notice that

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \in Q_{\mathcal{T}}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

if and only if

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} - \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \begin{pmatrix} \overline{A(0)} \\ \overline{E(0)} \\ \overline{K(0)} \end{pmatrix}.$$

Then,

$$\begin{cases} \tilde{u} - u \in \overline{A(0)} \\ \tilde{v} - v \in \overline{E(0)} \\ \tilde{w} - w \in \overline{K(0)}. \end{cases}$$

This is equivalent to

$$\begin{cases} \tilde{u} \in Q_A(u), \\ \tilde{v} \in Q_E(v), \\ \tilde{w} \in Q_K(w). \end{cases}$$

This shows that

$$Q_{\mathcal{T}}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Q_A(u) \\ Q_E(v) \\ Q_K(w) \end{pmatrix} = \begin{pmatrix} Q_A(Ax) \\ Q_E(Ey) \\ Q_K(Kz) \end{pmatrix}.$$

$$Q_{\mathcal{T}}\mathcal{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} Q_A A & 0 & 0 \\ 0 & Q_E E & 0 \\ 0 & 0 & Q_K K \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\text{for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{D}(A) \times \mathcal{D}(E) \times \mathcal{D}(K). \quad \square$$

Lemma 4.1. *We have*

(i) $\mathcal{T} = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}$ *is closed if and only if* A, E, *and* K *are closed.*

(ii) *If* A, E *and* K *are Fredholm linear relation, then* $\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}$ *is a Fredholm linear relation and if* A, E *are everywhere defined, then* $i(\mathcal{T}) = i(A) + i(E) + i(K)$.

Proof. Suppose that $\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}$ is closed.

By Lemma 2.1(i) and Proposition 4.1, it follows that $\begin{pmatrix} Q_A(A) & 0 & 0 \\ 0 & Q_E(E) & 0 \\ 0 & 0 & Q_K(K) \end{pmatrix}$ is a closed linear operator and that $\begin{pmatrix} \overline{A(0)} \\ \overline{E(0)} \\ \overline{K(0)} \end{pmatrix} = \begin{pmatrix} A(0) \\ E(0) \\ K(0) \end{pmatrix}$ and thus, $Q_A(A)$, $Q_E(E)$ and $Q_K(K)$ are closed. $\overline{A(0)} = A(0)$, $E(0) = \overline{E(0)}$ and $\overline{K(0)} = K(0)$. Consequently, A, E, and K are closed linear relations. Conversely, if we suppose that A, E, and K are closed linear relations, then using Lemma 2.1(i) it follows that $Q_A(A)$, $Q_E(E)$ and $Q_K(K)$ are closed operators and that $\overline{A(0)} = A(0)$, $E(0) = \overline{E(0)}$ and $\overline{K(0)} = K(0)$. Thus $Q_{\mathcal{T}}\mathcal{T}$ is a closed linear operator and $\mathcal{T}(0)$ is closed. Hence applying Lemma 2.1(i) we infer that $\mathcal{T}$ is closed.

(ii) Suppose that A, E and K are Fredholm linear relations. Using Lemma 2.1(i) it follows that $Q_A(A)$, $Q_E(E)$ and $Q_K(K)$ are Fredholm operators. Then

$$\begin{pmatrix} Q_A(A) & 0 & 0 \\ 0 & Q_E(E) & 0 \\ 0 & 0 & Q_K(K) \end{pmatrix}$$

is a Fredholm operator.

Now using (i) and Lemma 2.1(i), we deduce that

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}$$

is a Fredholm linear relation.

On the other hand,

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & K \end{pmatrix}$$
$$= \begin{pmatrix} I & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I \end{pmatrix} \mathcal{M}$$

where

$$\mathcal{M} = \begin{pmatrix} I & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & K \end{pmatrix}.$$

Then by Lemma 2.2, we get

$$\begin{aligned} i(\mathcal{T}) &= i(A) + i(\mathcal{M}) - \dim\left(\mathcal{M}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \bigcap \begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= i(A) + i(\mathcal{M}) - \dim\left(\begin{pmatrix} 0 \\ E(0) \\ K(0) \end{pmatrix} \bigcap \begin{pmatrix} A^{-1}(0) \\ 0 \\ 0 \end{pmatrix}\right) \\ &= i(A) + i(\mathcal{M}). \end{aligned}$$

A similar proof shows that $i(\mathcal{M}) = i(E) + i(K)$. Consequently, $i(\mathcal{T}) = i(A) + i(E) + i(K)$. $\square$

Lemma 4.2. (a) *If the following conditions are satisfied*

(i) *B is E-bounded with E-bound δ_1,*

(ii) *F is K-bounded with K-bound δ_2,*

(ii) *G is A-bounded with A-bound δ_3.*

Then, $\mathcal{S}$ is $\mathcal{T}$-bounded with T-bound $\delta = \max\{\delta_1, \delta_2, \delta_3\}$.

(b) *If the following conditions are satisfied*

(i) *C is F-bounded with F-bound δ_1,*

(ii) *D is G-bounded with G-bound δ_2,*

(ii) *H is A-bounded with A-bound δ_3.*

Then, $\mathcal{K}$ is $\mathcal{S}$-bounded with S-bound $\delta = \max\{\delta_1, \delta_2, \delta_3\}$.

Proof. Let $\varepsilon > 0$. By the above assumptions and Remark 2.1 there exist constants, $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \geq 0$ such that $\delta_1 \leq b_1 < \delta_1+\varepsilon$, $\delta_2 \leq b_2 < \delta_2+\varepsilon$, $\delta_3 \leq b_3 < \delta_3+\varepsilon$ and

$$\|Bg\|^2 \leq a_1^2\|g\|^2 + b_1^2\|Eg\|^2 \quad \text{for all } g \in \mathcal{D}(E) \subset \mathcal{D}(B).$$
$$\|Fh\|^2 \leq a_2^2\|h\|^2 + b_2^2\|Kh\|^2 \quad \text{for all } h \in \mathcal{D}(K) \subset \mathcal{D}(F).$$
$$\|Gf\|^2 \leq a_3^2\|f\|^2 + b_3^2\|Af\|^2 \quad \text{for all } f \in \mathcal{D}(A) \subset \mathcal{D}(G).$$

Hence we obtain, for $(f, g, h) \in \mathcal{D}(A) \times \mathcal{D}(E) \times \mathcal{D}(K)$

$$\begin{aligned}\left\| \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & F \\ G & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} 0 & Q_B B & 0 \\ 0 & 0 & Q_F F \\ Q_G G & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 . \\ &= \|Q_B B g\|^2 + \|Q_F F h\|^2 + \|Q_G G f\|^2 \\ &= \|Bg\|^2 + \|Fh\|^2 + \|Gf\|^2 \end{aligned}$$

and

$$\begin{aligned}\left\| \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} Q_A A & 0 & 0 \\ 0 & Q_E E & 0 \\ 0 & 0 & Q_K K \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 . \\ &= \|Q_A A f\|^2 + \|Q_E F g\|^2 + \|Q_K K h\|^2 \\ &= \|Af\|^2 + \|Eg\|^2 + \|Kh\|^2. \end{aligned}$$

Then,

$$\begin{aligned}\left\| \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & F \\ G & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 &= \|Bg\|^2 + \|Fh\|^2 + \|Gf\|^2 \\ &\leq a_3^2 \|f\|^2 + b_3^2 \|Gf\|^2 + a_1^2 \|g\|^2 \\ &\leq \eta \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 + \chi \left\| \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|^2 \end{aligned}$$

where $\eta = \max\{a_1, a_2, a_3\}^2$ and $\chi = \max\{b_1, b_2, b_3\}^2$ as

$$\max\{b_1, b_2, b_3\} = \{\delta_1 + \varepsilon, \ \ \delta_2 + \varepsilon, \ \ \delta_3 + \varepsilon\} = \delta + \varepsilon.$$

This shows that $\mathcal{S}$ is $\mathcal{T}-$ bounded with T-bound $< \delta$. The proof of (ii) can be done in a similar fashion as that of (i). $\square$

Theorem 4.1. *If the following conditions are satisfied:*
$\mathcal{D}(E) \subset \mathcal{D}(B)$, $\mathcal{D}(K) \subset \mathcal{D}(F)$, $\mathcal{D}(A) \subset \mathcal{D}(G)$, $\mathcal{D}(F) \subset \mathcal{D}(C)$, $\mathcal{D}(G) \subset \mathcal{D}(D)$, $\mathcal{D}(B) \subset \mathcal{D}(H)$, $\mathcal{K}(0) \subset \mathcal{S}(0) \subset \overline{\mathcal{T}(0)}$ *and*

$$\begin{aligned} &\|B\varphi\| \leq a_1\|\varphi\| + b_1\|E\varphi\| \quad \textit{for all } \varphi \in \mathcal{D}(E) \\ &\|F\varphi\| \leq a_2\|\varphi\| + b_2\|K\varphi\|^2 \quad \textit{for all } \varphi \in \mathcal{D}(K) \\ &\|G\varphi\| \leq a_3\|\varphi\| + b_3\|A\varphi\| \quad \textit{for all } \varphi \in \mathcal{D}(A) \\ &\|C\varphi\| \leq a_4\|\varphi\| + b_4\|F\varphi\| \quad \textit{for all } \varphi \in \mathcal{D}(F) \\ &\|D\varphi\| \leq a_5\|\varphi\| + b_5\|G\varphi\|^2 \quad \textit{for all } \varphi \in \mathcal{D}(G) \\ &\|H\varphi\| \leq a_6\|\varphi\| + b_6\|B\varphi\| \quad \textit{for all } \varphi \in \mathcal{D}(B) \end{aligned}$$

with

$$\max\{b_1,\ b_2,\ b_3\}\big(1+\max\{a_4,\ a_5,\ a_6\}\big)<1.$$

Then, $\mathcal{A}:=\begin{pmatrix} A & B & C\\ D & E & F\\ G & H & K\end{pmatrix}$ *is closed if and only if* $A,\ E,\ K$ *are closed.*

Proof. Consider

$$\mathcal{T}:=\begin{pmatrix} A & 0 & 0\\ 0 & E & 0\\ 0 & 0 & K\end{pmatrix},\qquad \mathcal{S}:=\begin{pmatrix} 0 & B & 0\\ 0 & 0 & F\\ G & 0 & 0\end{pmatrix}\quad \text{and}\quad \mathcal{K}:=\begin{pmatrix} 0 & 0 & C\\ D & 0 & 0\\ 0 & H & 0\end{pmatrix}.$$

Since,

$$\begin{cases}\mathcal{D}(\mathcal{T})=\mathcal{D}(A)\times\mathcal{D}(E)\times\mathcal{D}(K),\\ \mathcal{D}(\mathcal{S})=\mathcal{D}(G)\times\mathcal{D}(B)\times\mathcal{D}(F),\\ \mathcal{D}(\mathcal{K})=\mathcal{D}(D)\times\mathcal{D}(H)\times\mathcal{D}(C).\end{cases}$$

Then,

$$\mathcal{D}(\mathcal{T})\subset\mathcal{D}(\mathcal{S})\subset\mathcal{D}(\mathcal{K}).$$

On the other hand we have,

$$\begin{aligned}&\|S\varphi\|^2\le\eta_1^2\,\|\varphi\|^2+\chi_1^2\,\|T\varphi\|^2\,,\quad \varphi\in\mathcal{D}(T)\\ &\|K\varphi\|^2\le\eta_2^2\,\|\varphi\|^2+\chi_2^2\,\|S\varphi\|^2\,,\quad \varphi\in\mathcal{D}(S),\end{aligned}$$

where

$$\begin{aligned}\eta_1&=\max\left\{\sqrt{a_1^2+a_1b_1},\ \sqrt{a_2^2+a_2b_2},\ \sqrt{a_3^2+a_3b_3}\right\},\\ \chi_1&=\max\left\{\sqrt{b_1^2+a_1b_1},\ \sqrt{b_2^2+a_2b_2},\ \sqrt{b_3^2+a_3b_3}\right\},\\ \eta_2&=\max\left\{\sqrt{a_4^2+a_4b_4},\ \sqrt{a_5^2+a_5b_5},\ \sqrt{a_6^2+a_6b_6}\right\},\\ \chi_2&=\max\left\{\sqrt{b_4^2+a_4b_4},\ \sqrt{b_5^2+a_5b_5},\ \sqrt{b_6^2+a_6b_6}\right\}.\end{aligned}$$

From Remark 2.1 it follows that

$$\begin{aligned}\|S\varphi\|\ &\le\ \max\{a_1,a_2,a_3\}\,\|\varphi\|+\max\{b_1,b_2,b_3\}\,\|T\varphi\|\,,\quad \varphi\in\mathcal{D}(T)\\ \|K\varphi\|\ &\le\ \max\{a_4,a_5,a_6\}\,\|\varphi\|+\max\{b_4,b_5,b_6\}\,\|S\varphi\|\,,\quad \varphi\in\mathcal{D}(S)\end{aligned}$$

where

$$\max\{b_1,b_2,b_3\}\big(1+\max\{a_4,a_5,a_6\}\big)<1.$$

Using Theorem 3.2 we deduce that

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}$$

is closed if and only if

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}$$

is closed, which in turn is equivalent to the fact A, E and K are closed (Lemma 4.1). $\square$

Theorem 4.2. *Let $\widehat{\mathcal{T}}$ be the bijection associated with $\mathcal{T}$. If we suppose that the conditions of Theorem 4.1 are satisfied and we suppose that $\alpha_1(1+\beta_1) < 1$ and $(1+\beta_1) < \gamma(\hat{\mathcal{T}})$ where*

$$\alpha_1 = \max\left\{\max\{a_1, a_2, a_3\}, \max\{b_1, b_2, b_3\}\right\}$$

and

$$\beta_1 = \max\left\{\max\{a_4, a_5, a_6\}, \max\{b_4, b_5, b_6\}\right\}.$$

Then, if A, E and K are Fredholm linear relations then $\mathcal{A}$ is Fredholm linear relation, which satisfies the following properties
(i) $\alpha(\mathcal{A}) \le \alpha(\mathcal{T})$
(ii) $\beta(\mathcal{A}) \le \beta(\mathcal{T})$
and if $\mathcal{A}$ is single valued then, $i(\mathcal{A}) = i(A) + i(E) + i(K)$.

Proof. The proofs follows from Theorems 3.3 and 4.1. Now if $\mathcal{A}$ is single valued, then by Corollary 3.1, we have $i(\mathcal{A}) = i(\mathcal{T})$.

On the other hand,

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & K \end{pmatrix}$$

and $i(\mathcal{A}) = i(\mathcal{T}) = i(A) + i(E) + i(K)$. $\square$

5. Conclusion

Sufficient conditions are obtained for the Fredholmness of the algebraic sum of three linear relations (Theorem 3.3). Such a result generalizes that for single valued linear operators (see for example Theorem 4.2 in [19, Theorem 4.2, Chapter XVII]). However, if A_1 is a Fredholm linear relation and $A_2,\ A_3, \ldots, A_n$ for $n \ge 4$ are (possible unbounded) linear relations such that

$\mathcal{D}(A_1) \subset \mathcal{D}(A_2) \subset ...\mathcal{D}(A_{n-1}) \subset \mathcal{D}(A_n)$ and $A_n(0) \subset A_{n-1}(0) \subset ... \subset A_1(0)$

$$\|A_{k+1}\varphi\| \le \alpha_k\Big(\|\varphi\| + \|A_k\varphi\|\Big) \quad \text{for } k = 1, \ldots, n-1 \text{ for } \varphi \in \mathcal{D}(A_1),$$

it is unclear which additional conditions should be put on the linear relations A_k and the scalars α_k for $k = 1, \dots, n$, so that the algebraic sum $\sum_{k=1}^{n} A_k$ is a Fredholm linear relation. This question will be left as an open question, even in the case of single valued operators. The answer to this question is of a great interest as it helps investigate some properties of the $n \times n$ matrix of linear relations:

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \quad n \geq 4.$$

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