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# Periodic solutions for a Cauchy problem on time scales

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**Abstract.** This paper firstly shows that there does not exist a nonzero periodic solution for a nonhomogeneous Cauchy problem by using the Laplace transformation on time scales. Secondly, two new Gronwall inequalities, which play an important role in the qualitative analysis of differential and integral equations, are established. Thirdly, by employing the contraction mapping principle, existence and uniqueness results of weighted $S$-asymptotically $\omega$-periodic solutions for nonlinear Cauchy problem on time scales are obtained in an asymptotically periodic function space. Finally, some examples are presented to illustrate some of the results described here.

Keywords: Asymptotically periodic solutions; Cauchy problem; Time scales; Gronwall inequality

2010 Mathematics Subject Classification: 34A12; 34K13; 34N05

## 1. INTRODUCTION

In 1990, the theory of dynamic equations on time scales was introduced by Stefan Hilger [9] in order to unify continuous and discrete calculus. This theory not only brought equations leading to new applications but also allowed one to get a better understanding of the subtle differences between discrete and continuous systems. One can find in Bohner and Peterson's books [2,3] most of the material needed to read this paper.

The question of existence and uniqueness of periodic solutions for differential and difference equations has attracted much attention during the last decades, because periodic motion is a very important and special phenomenon in both the natural and social sciences such as climate, food supplies, sustainable development and insect population [13,19,4,20,12].

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Meanwhile, periodic behavior of solutions for Cauchy problems arises from many fields of applied science involving celestial mechanics, biology and finance [5,6,17].

The literature concerning $S$-asymptotically $\omega$-periodic solutions, as a very recent result, has been paid increasing attention [6,17,14]. Consequently, various methods have been developed for the study of the existence of periodic solutions, $S$-asymptotically $\omega$-periodic solutions and weighted $S$-asymptotically $\omega$-periodic solutions.

Most of the discussions in the earlier literature are devoted to the existence of periodic solutions for either differential or difference equations. In recent years, as time scales theory became established, attention has been given to study the periodic solutions on time scales [19,8,11,15,18,10,1,16]. However, to the best of the authors' knowledge, there are few papers concerning the $S$-asymptotically $\omega$-periodic solutions and weighted $S$-asymptotically $\omega$-periodic solutions on time scales.

Motivated by the above works, in this paper, we firstly consider the nonexistence of nonzero periodic solutions for the following nonhomogeneous Cauchy problem on time scales

$$\begin{cases} u^{\Delta}(t) = pu(t) + q(t), \quad t \in \mathbb{T}_0^k, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\mathbb{T}_0$ is a periodic time scale, $p \neq 0$ is a regressive constant number, $q \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$.

Secondly, we investigate the existence of $S$-asymptotically $\omega$-periodic solutions and weighted $S$-asymptotically $\omega$-periodic solutions for the following nonlinear differential equation

$$\begin{cases} u^{\Delta}(t) = r(t)u(t) + f(t, u), \quad t \in \mathbb{T}_0^k, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $r \in \mathcal{R}^+$ is regressive, $f(t, u) \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R})$.

Wang, Fečkan and Zhou in [17] have studied fractional differential Cauchy problems with order $\alpha \in (0, 1)$. Many nice and interesting results were obtained.

The organization of this paper is as follows. In Section 2, some advanced topics in the theory of dynamic equations on time scales are recalled, meanwhile, the definitions of periodic time scales and asymptotically periodic functions are given. In Section 3, we first find the Laplace transform of a function, and prove a theorem which is exclusively for the periodic functions. Then, the nonexistence nonzero periodic solution for the problem (1.1) is studied. In Section 4, two Gronwall inequalities on time scales are obtained, existence and uniqueness results of weighted $S$-asymptotically $\omega$-periodic solutions for the problem (1.2) are researched. Finally, two examples are presented to illustrate some of the results described here.

## 2. Preliminaries

We suppose that the reader is familiar with the basic concepts and calculus on time scales for dynamic equations. So, in this section, we first recall some advanced topics in the theory of dynamic equations on time scales which are used in what follows. For more details one can see references [2,3].

Let $\mathbb{T}$ be a time scale. A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0\}$.

If $r \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$, then the generalized exponential function $e_r$ is defined by

$$e_r(t,s) = \exp\left\{ \int_s^t \frac{1}{\mu(\tau)} Log(1+\mu(\tau)r(\tau))\Delta\tau \right\}.$$

It is obvious that if $r \in \mathcal{R}^+$ then $e_r(t,s) > 0$ for $s,t \in \mathbb{R}$.

**Lemma 2.1** (*Lemma 2.36, [2]*). *Assume that* $p \in \mathcal{R}$. *Then*
$e_0(t,s) \equiv 1$, $e_p(t,t) \equiv 1$ *and* $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,
$e_p^{\Delta}(\cdot,s) = pe_p(\cdot,s)$, $e_p(t,\sigma(s)) = \frac{e_p(t,s)}{1+\mu(s)p(s)}$ *and* $e_p(t,s)e_p(s,r) = e_p(t,r)$.

Throughout this paper, we assume that the time scale $\mathbb{T}_0$ is such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = \infty$.

**Definition 2.1** (*Definition 3.82, [2]*). Assume that $u : \mathbb{T}_0 \to \mathbb{R}$ is regulated. Then the Laplace transform of $u$ is defined by

$$\mathcal{L}\{u\}(\lambda) := \int_0^{\infty} u(t)e^{\sigma}_{\ominus\lambda}(t,0)\Delta t$$

for $\lambda \in \mathcal{D}\{u\}$, where $\mathcal{D}\{u\}$ consists of all complex numbers $\lambda \in \mathcal{R}$ for which the improper integral exists.

**Remark 2.1.** For any time scale $\mathbb{T}$ that is unbounded above, assume that $u : \mathbb{T} \to \mathbb{R}$ is regulated. For any fixed $t_0 \in \mathbb{T}$, we can also define the Laplace transform of $u$ by

$$\mathcal{L}\{u\}(\lambda) := \int_{t_0}^{\infty} u(t)e^{\sigma}_{\ominus\lambda}(t,t_0)\Delta t$$

for $\lambda \in \mathcal{D}\{u\}$ which is the same as above.

**Lemma 2.2** (*Lemma 2.77, [2]*). *Assume* $p, q \in C_{rd}(\mathbb{T},\mathbb{R})$ *are regressive. Let* $u_0 \in \mathbb{R}$. *Then the unique solution of the Cauchy problem*

$$u^{\Delta}(t) = p(t)u(t) + q(t), \quad u(t_0) = u_0$$

*is given by*

$$u(t) = e_p(t,t_0)u_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau))q(\tau)\Delta\tau.$$

Now, we turn our attention to the $\omega$-periodic time scales and the definition of asymptotically periodic functions. Without lost of generality, we suppose that $\omega > 0$.

**Definition 2.2.** [20,21]. We say that a time scale $\mathbb{T}$ is $\omega$-periodic if there exists $\omega > 0$ such that if $t \in \mathbb{T}$, then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive $\omega$ is called the period of the time scale $\mathbb{T}$.

**Definition 2.3.** [20,21]. Let $\mathbb{T}$ be an $\omega$-periodic time scale, we say that $u : \mathbb{T} \to \mathbb{R}$ is an $\omega$-periodic function if $u(t) = u(t+\omega)$ for all $t \in \mathbb{T}$.

**Lemma 2.3** (*Theorem 2.1, [1]*). *Let $\mathbb{T}$ be an $\omega$-periodic time scale and $r \in C_{rd}(\mathbb{T}, \mathbb{R})$ an $\omega$-periodic function on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have*

$\mu(t) = \mu(\omega + t), \quad e_r(t, s) = e_r(t + \omega, s + \omega)$ *if* $r \in \mathcal{R}$,
$k_r := e_r(t + \omega, t) - 1 > 0$ *is independent of* $t \in \mathbb{T}$ *whenever* $r \in \mathcal{R}$.

Define $C_b(\mathbb{T}, X) = \{\varphi : \varphi \in C_{rd}(\mathbb{T}, X)$ and $\varphi \in B(\mathbb{T}, X)\}$ with the norm of the uniform convergence, then $C_b(\mathbb{T}, X)$ is a Banach space with the norm of the uniform convergence. The following definitions are similar to the definitions in [6].

**Definition 2.4.** Let $\mathbb{T}$ be an $\omega$-periodic time scale. A function $u \in C_b(\mathbb{T}, X)$ is called an $S$-asymptotically $\omega$-periodic function if $\lim_{t\to\infty}\big(u(t + \omega) - u(t)\big) = 0$ for all $t \in \mathbb{T}$. In this case, we say that $\omega$ is an asymptotic period of $u$ and that $u$ is $S$-asymptotically $\omega$-periodic.

$SAP_\omega(X)$ denotes the space formed for all the $X$-valued $S$-asymptotically $\omega$-periodic functions endowed with the uniform convergence norm denoted $\|\cdot\|_\infty$. Then $SAP_\omega(X)$ is a Banach space.

**Definition 2.5.** Let $\mathbb{T}$ be an $\omega$-periodic time scale. A function $u \in C_b(\mathbb{T}, X)$ is called a weighted $Sv$-asymptotically $\omega$-periodic function if $\lim_{t\to\infty} \frac{u(t+\omega)-u(t)}{v(t)} = 0$ for all $t \in \mathbb{T}$, where $v(t) \in C_b(\mathbb{T}, \mathbb{R}^+)$.

Let $SAP_{v,\omega}(X)$ represent the space formed for all the weighted $Sv$-asymptotically $\omega$-periodic functions endowed with the norm $\|\cdot\|_{SAP_{v,\omega}(X)} = \sup_{t\geq 0}\|u(t)\|_X + \sup_{t\geq 0}\frac{\|u(t+\omega)-u(t)\|_X}{v(t)}$. Then $SAP_{v,\omega}(X)$ is a Banach space.

## 3. Nonexistence of periodic solutions

Throughout this section, we suppose that $\mathbb{T}_0$ is an $\omega$-periodic time scale with constant graininess, i.e. $\mathbb{T} = c\mathbb{Z}$, where $c \neq 0$ is a constant number.

**Lemma 3.1.** *Let* $f(t) := \int_0^t e_p(t, \sigma(s))q(s)\Delta s$ *for* $t \in \mathbb{T}_0$, $p \neq 0$ *is regressive,* $q \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$. *Then*

$$\widehat{f}(\lambda) := \mathcal{L}\{f\}(\lambda) = k(\lambda, p, c)\widehat{q}(\lambda) := k(\lambda, p, c)\mathcal{L}\{q\}(\lambda),$$

*for those regressive* $\lambda \in \mathbb{C} \setminus \{p\}$ *satisfying*

$$\lim_{t\to\infty} e_{p\ominus\lambda}(t, 0) = 0, \tag{3.1}$$

*where* $k(\lambda, p, c) = \frac{1+c\lambda}{(\lambda-p)(1+cp)}$.

**Proof.** By using Lemma 2.1 and Definition 2.1, for $t \in \mathbb{T}_0$, we arrive at

$$\begin{aligned}\widehat{f}(\lambda) := \mathcal{L}\{f\}(\lambda) &= \int_0^\infty f(t)e^\sigma_{\ominus\lambda}(t, 0)\Delta t \\ &= \int_0^\infty \int_0^t e_p(t, \sigma(s))q(s)e^\sigma_{\ominus\lambda}(t, 0)\Delta s\Delta t\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty q(s) \int_s^\infty e_p(t,\sigma(s)) e^\sigma_{\ominus\lambda}(t,0)\Delta t\Delta s \\
&= \int_0^\infty q(s) e_p(0,\sigma(s)) \int_s^\infty \frac{1}{1+\mu(t)\lambda} e_p(t,0) e_{\ominus\lambda}(t,0)\Delta t\Delta s \\
&= \int_0^\infty \frac{q(s) e_p(0,\sigma(s))}{p-\lambda} \int_s^\infty \frac{p-\lambda}{1+\mu(t)\lambda} e_{p\ominus\lambda}(t,0)\Delta t\Delta s.
\end{aligned}
$$

Combining the above equality and condition (3.1), we obtain

$$
\begin{aligned}
\widehat{f}(\lambda) &= \int_0^\infty \frac{q(s) e_p(0,\sigma(s))}{\lambda-p} e_{p\ominus\lambda}(s,0)\Delta s \\
&= \frac{1}{\lambda-p} \int_0^\infty e_p(s,\sigma(s)) e_{\ominus\lambda}(s,0) q(s)\Delta s \\
&= \frac{1}{\lambda-p} \int_0^\infty \frac{e_p(s,s)}{1+\mu(s)p} e_{\ominus\lambda}(s,0) q(s)\Delta s \\
&= \frac{1}{\lambda-p} \int_0^\infty \frac{1+\mu(s)\lambda}{1+\mu(s)p} \frac{e_{\ominus\lambda}(s,0)}{1+\mu(s)\lambda} q(s)\Delta s \\
&= k(\lambda,p,c) \int_0^\infty q(s) e^\sigma_{\ominus\lambda}(s,0)\Delta s = k(\lambda,p,c)\mathcal{L}\{q\}(\lambda),
\end{aligned}
$$

which gives us the desired result. ■

**Lemma 3.2.** *Let $u(t)$ be $\omega$-periodic and regulated, $t \in \mathbb{T}_0$. Then*

$$
\mathcal{L}\{u\}(\lambda) = \frac{\int_0^\omega u(t) e^\sigma_{\ominus\lambda}(t,0)\Delta t}{\omega - \int_0^\omega e_{\ominus\lambda}(t+\omega,t)\Delta t},
$$

*for all complex values of $\lambda \in \mathcal{R}$ such that*

$$
|e_{\ominus\lambda}(t+\omega,t)| < 1, \quad t \in \mathbb{T}_0. \tag{3.2}
$$

**Proof.** We first show that $e_\lambda(t+n\omega,t) = (e_\lambda(t+\omega,t))^n$. In fact, by using Lemma 2.3 we get

$$
e_\lambda(t+2\omega,t) = e_\lambda(t+2\omega,t+\omega) e_\lambda(t+\omega,t) = (e_\lambda(t+\omega,t))^2.
$$

The result can be obtained by using induction.

Condition (3.2), Lemma 2.1 and Definition 2.1 indicate that

$$
\begin{aligned}
\widehat{u}(\lambda) := \mathcal{L}\{u\}(\lambda) &= \int_0^\infty u(t) e^\sigma_{\ominus\lambda}(t,0)\Delta t \\
&= \int_0^\omega u(t) e^\sigma_{\ominus\lambda}(t,0)\Delta t + \int_0^\omega u(t) e^\sigma_{\ominus\lambda}(t+\omega,0)\Delta t \\
&\quad + \int_0^\omega u(t) e^\sigma_{\ominus\lambda}(t+2\omega,0)\Delta t + \cdots \\
&= \int_0^\omega u(t) e^\sigma_{\ominus\lambda}(t,0)\{1 + e_{\ominus\lambda}(t+\omega,t) + e_{\ominus\lambda}(t+2\omega,t) + \cdots\}\Delta t
\end{aligned}
$$

$$= \int_0^\omega u(t)e^\sigma_{\ominus\lambda}(t,0)\{1 + e_{\ominus\lambda}(t+\omega,t) + (e_{\ominus\lambda}(t+\omega,t))^2 + \cdots\}\Delta t$$
$$= \frac{\int_0^\omega u(t)e^\sigma_{\ominus\lambda}(t,0)\Delta t}{\omega - \int_0^\omega e_{\ominus\lambda}(t+\omega,t)\Delta t},$$

which proves our desired equality. ■

Now, we give our main result about the nonexistence of a periodic solution for problem (1.1).

**Theorem 3.1.** *Let $q \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ be $\omega$-periodic, $u$ be regulated. Then problem (1.1) has not nonzero $\omega$-periodic solution on $\mathbb{T}_0^k$ if (3.1) and (3.2) hold.*

**Proof.** Lemma 2.2 tells us that the solution of (1.1) is given by

$$u(t) = e_p(t,0)u_0 + \int_0^t e_p(t,\sigma(\tau))q(\tau)\Delta\tau.$$

Taking the Laplace transformation of the above equality, it follows from Example 3.91 in [2], Lemmas 3.1 and 3.2 that

$$\frac{\int_0^\omega u(t)e^\sigma_{\ominus\lambda}(t,0)\Delta t}{\omega - \int_0^\omega e_{\ominus\lambda}(t+\omega,t)\Delta t} = \frac{1}{\lambda - p}u_0 + k(p,c,\lambda)\frac{\int_0^\omega q(t)e^\sigma_{\ominus\lambda}(t,0)\Delta t}{\omega - \int_0^\omega e_{\ominus\lambda}(t+\omega,t)\Delta t},$$

that is

$$(\lambda - p)\int_0^\omega u(t)e^\sigma_{\ominus\lambda}(t,0)\Delta t$$
$$= u_0\left(\omega - \int_0^\omega e_{\ominus\lambda}(t+\omega,t)\Delta t\right) + (\lambda - p)k(p,c,\lambda)\int_0^\omega q(t)e^\sigma_{\ominus\lambda}(t,0)\Delta t. \quad (3.3)$$

Letting $\lambda \to 0^+$, we have

$$p\int_0^\omega u(t)\Delta t = pk(p,c,0)\int_0^\omega q(t)\Delta t,$$

which implies

$$u(t) = k(p,c,0)q(t). \quad (3.4)$$

Letting $\lambda \to p$ in (3.3), we have $\lim_{\lambda\to p} u_0\big(\omega - \int_0^\omega e_{\ominus\lambda}(t+\omega,t)\Delta t\big) = 0$, since $\omega p \neq 0$, so $u_0 = 0$, that is

$$u(t) = \int_0^t e_p(t,\sigma(\tau))q(\tau)\Delta\tau. \quad (3.5)$$

By using (1.1), (3.4) and (3.5) we get $u = q \equiv 0$. Therefore, the problem (1.1) has no nonzero $\omega$-periodic solution on $\mathbb{T}_0^k$. This concludes the proof. ■

**Remark 3.1.** In the following, we check the result of Theorem 3.1 by using the classical method.

Assume that $u(t)$ is an arbitrary $\omega$-periodic solution of (1.1). We equivalently rewrite $u^{\Delta}(t) = pu(t) + q(t)$ as

$$u^{\Delta}(t) = p(u^{\sigma}(t) - \mu(t)u^{\Delta}(t)) + q(t),$$

that is

$$u^{\Delta}(t) + (\ominus p)u^{\sigma}(t) = \frac{q(t)}{1+\mu(t)p}.$$

Multiplying both sides of the above equation by the integrating factor $e_{\ominus p}(t,0)$, we get

$$(u(t)e_{\ominus p}(t,0))^{\Delta} = e_{\ominus p}(t,0)\frac{q(t)}{1+\mu(t)p}.$$

Integrating both sides of this equation from $t$ to $t+\omega$ produces

$$u(t+\omega)e_{\ominus p}(t+\omega,0) - u(t)e_{\ominus p}(t,0) = \int_t^{t+\omega} \frac{e_{\ominus p}(s,0)}{1+\mu(s)p}q(s)\Delta s,$$

which implies

$$\begin{aligned} u(t) &= \frac{1}{(e_{\ominus p}(t+\omega,t)-1)e_{\ominus p}(t,0)} \int_t^{t+\omega} \frac{e_{\ominus p}(s,0)}{1+\mu(s)p}q(s)\Delta s \\ &= \int_t^{t+\omega} \frac{e_p(t,\sigma(s))}{e_p(0,\omega)-1}q(s)\Delta s. \end{aligned} \tag{3.6}$$

It is not difficult to find that $u(t)$ is indeed an $\omega$-periodic function. By using the method of variable substitution in (3.6), we arrive at

$$u(t) = \int_0^{\omega} \frac{e_p(t,\sigma(s))}{e_p(0,\omega)-1}q(s)\Delta s = e_p(t,0)\int_0^{\omega} \frac{e_p(0,\sigma(s))}{e_p(0,\omega)-1}q(s)\Delta s. \tag{3.7}$$

Letting $\lambda \to 0^+$ in condition (3.1), we get $\lim_{t\to\infty} e_p(t,0) = 0$. So, letting $t \to \infty$ in both sides of (3.7), we achieve that $\lim_{t\to\infty} u(t) = 0$. Since $u(t)$ is a periodic function, therefore $u(t) \equiv 0$ for $t \in \mathbb{T}_0^k$. This coincides with the result of Theorem 3.1.

## 4. WEIGHTED $S$-ASYMPTOTICALLY $\omega$-PERIODIC SOLUTIONS

In this section, we first give two Gronwall inequalities on time scales, then study the existence of weighted $S$-asymptotically $\omega$-periodic solutions for the problem (1.2).

**Theorem 4.1.** *Let $m \in C_{rd}(\mathbb{T}_0,\mathbb{R}^+)$, $n \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+,\mathbb{R}^+)$. Assume in addition that $n(t,s)$ is nondecreasing with respect to $t$ with $1 - \int_0^t n(t,s)\Delta s > 0$. If $u \in C_{rd}(\mathbb{T}_0,\mathbb{R}^+)$ satisfies*

$$u(t) \le m(t) + \int_0^t n(t,s)u(s)\Delta s, \tag{4.1}$$

*then*

$$u(t) \le m(t) + \frac{\int_0^t n(t,s)m(s)\Delta s}{1-\int_0^t n(t,s)\Delta s}, \quad \textit{for all } t \in \mathbb{T}_0.$$

*If $m(t)$ is nondecreasing, then*

$$u(t) \le \frac{m(t)}{1-\int_0^t n(t,s)\Delta s}, \quad \textit{for all } t \in \mathbb{T}_0.$$

**Proof.** Let $z(t) = \int_0^t n(t,s)u(s)\Delta s$, $t \in \mathbb{T}_0$. Then $z(t)$ is nondecreasing on $\mathbb{T}_0$. By using Theorem 1.117 in [2] we get

$$z^\Delta(t) = \int_0^t n_t^\Delta(t,s)u(s)\Delta s + n(\sigma(t),t)u(t).$$

From the above equality and (4.1), we arrive at

$$z^\Delta(t) \le \int_0^t n_t^\Delta(t,s)(m(s)+z(s))\Delta s + n(\sigma(t),t)(m(t)+z(t)),$$

that is

$$z^\Delta(t) - \left(\int_0^t n(t,s)z(s)\Delta s\right)^\Delta \le \left(\int_0^t n(t,s)m(s)\Delta s\right)^\Delta.$$

An integration of the above inequality with respect to $t$ from 0 to $t$ yields

$$z(t) - \int_0^t n(t,s)z(s)\Delta s \le \int_0^t n(t,s)m(s)\Delta s.$$

Since $1-\int_0^t n(t,s)\Delta s > 0$, and $z(t)$ is nondecreasing on $\mathbb{T}_0$, we obtain

$$z(t) \le \frac{\int_0^t n(t,s)m(s)\Delta s}{1-\int_0^t n(t,s)\Delta s}.$$

The above inequality and (4.1) prove the desired result.

In particular, if $m(t)$ is nondecreasing, then

$$u(t) \le m(t) + \frac{\int_0^t n(t,s)m(s)\Delta s}{1-\int_0^t n(t,s)\Delta s} \le \frac{m(t)}{1-\int_0^t n(t,s)\Delta s}.$$

This completes the proof. ■

Thanks to Theorem 3.1 in [7], we have the following result.

**Theorem 4.2.** *Let $m \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ be nondecreasing, $n(t,s), n_t^\Delta(t,s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R}^+)$. If $u \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ satisfies*

$$u(t) \le m(t) + \int_0^t n(t,s)u(s)\Delta s,$$

*then*

$$u(t) \le G^{-1}\left(G(m(t)) + \int_0^t n(t,s)\Delta s\right), \quad \textit{for all } t \in \mathbb{T}_0,$$

*where $G$ is an increasing bijective function and its $\Delta$-derivative is given by*

$$\big(G(z(t))\big)^{\Delta} = \frac{z^{\Delta}(t)}{z(t)}. \tag{4.2}$$

**Proof.** The proof will be omitted since it is similar to that of Theorem 3.1 in [7]. ■

**Remark 4.1.** As far as the function $e_p(t,s)$ is concerned, $p$ is either a constant number or a single variable function, i.e. $e_{p(\cdot)}(t,s)$. However, up to now, $e_{p(\cdot,\cdot)}(t,s)$ is not well defined. Consequently, there does not exist Gronwall inequality involving exponential function with respect to (4.1). Therefore, Theorems 4.1 and 4.2 provide an effective tool to deal with these inequalities.

In the following, we investigate the existence and uniqueness of weighted $S$-asymptotically $\omega$-periodic solutions for the problem (1.2). We assume that $u_0 = 0$, and there exist two functions $\alpha$, $\beta \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ such that

(H$_1$) $|f(t,u_1) - f(t,u_2)| \le \alpha(t)|u_1 - u_2|$, for all $t \in \mathbb{T}_0,\ u_1, u_2 \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$.
(H$_2$) $|f(t+\omega,u) - f(t,u)| \le \beta(t)(|u|+1)$, for all $t \in \mathbb{T}_0,\ u \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$.
(H$_3$) $\lim_{t\to\infty} \alpha(t) \int_0^{t+\omega} e_r(t+\omega, \sigma(s))\Delta s = 0$, for all $t, s \in \mathbb{T}_0$, $r \in \mathcal{R}^+$.

It is not difficult to find that $\lim_{t\to\infty} \alpha(t) \int_0^{t+\omega} e_r(t+\omega, \sigma(s))\Delta s = 0$ for $r \in \mathcal{R}^+$ implies

$$\lim_{t\to\infty} \alpha(t) \int_0^{t} e_r(t+\omega, \sigma(s))\Delta s = 0, \qquad \lim_{t\to\infty} \alpha(t) \int_0^{t+\omega} e_r(t, \sigma(s))\Delta s = 0.$$

Notice that $f(t,u) \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R})$, so we can define $f_m := \sup_{s\in[0,\omega]_{\mathbb{T}_0}} |f(s,0)|$, where $[0,\omega]_{\mathbb{T}_0}$ denotes $[0,\omega] \cap \mathbb{T}_0$. If $r \in \mathcal{R}^+$, we also have the following inequalities

$$\int_0^t e_r(t,s)\Delta s \le \int_0^{t+\omega} e_r(t,s)\Delta s \le \int_0^{t+\omega} e_r(t+\omega,s)\Delta s, \quad \text{for } t \in \mathbb{T}_0.$$

**Theorem 4.3.** *Assume that conditions* (H$_1$), (H$_3$) *are satisfied. Suppose further that*

$$\int_0^t e_r(t,\sigma(s))\alpha(s)\Delta s < 1, \quad \text{for } t \in \mathbb{T}_0. \tag{4.3}$$

*Then, problem (1.2) has a unique $S$-asymptotically $\omega$-periodic solution.*

**Proof.** Define the operator $F : SAP_\omega(\mathbb{R}^+) \to C_b(\mathbb{T}_0, \mathbb{R}^+)$ by

$$Fu = e_r(t,0)u_0 + \int_0^t e_r(t,\sigma(\tau)) f(\tau, u(\tau))\Delta\tau.$$

Clearly, $u(t)$ is a solution of problem (1.2) if and only if $u(t)$ is the fixed point of operator $F$.
We first show that $F : SAP_\omega(\mathbb{R}^+) \to SAP_\omega(\mathbb{R}^+)$. For $t \in \mathbb{T}_0$, we get

$$|Fu(t+\omega) - Fu(t)|$$
$$= \bigg| u_0(e_r(t+\omega,0) - e_r(t,0)) + \int_0^{t+\omega} e_r(t+\omega,\sigma(s)) f(s,u(s))\Delta s$$

$$
\begin{aligned}
&\quad - \int_0^t e_r(t,\sigma(s))f(s,u(s))\Delta s\Bigg| \\
&\le \int_0^t |e_r(t+\omega,\sigma(s))f(s,u(s)) - e_r(t,\sigma(s))f(s,u(s))|\Delta s \\
&\quad + \int_t^{t+\omega} |e_r(t+\omega,\sigma(s))f(s,u(s))|\Delta s \\
&\le k_r \int_0^t |e_r(t,\sigma(s))f(s,u(s))|\Delta s + \int_t^{t+\omega} |e_r(t+\omega,\sigma(s))f(s,u(s))|\Delta s \\
&\le k_r \int_0^t e_r(t,\sigma(s))|f(s,u(s)) - f(s,0)|\Delta s \\
&\quad + \int_0^{t+\omega} e_r(t+\omega,\sigma(s))|f(s,u(s)) - f(s,0)|\Delta s \\
&\quad + \int_0^t e_r(t+\omega,\sigma(s))|f(s,u(s)) - f(s,0)|\Delta s + \int_0^t e_r(t+\omega,\sigma(s))|f(s,0)|\Delta s \\
&\quad + k_r \int_0^t e_r(t,\sigma(s))|f(s,0)|\Delta s + \int_0^{t+\omega} e_r(t+\omega,\sigma(s))|f(s,0)|\Delta s \\
&\le (k_r+2)(\|u\|_\infty + f_m)\alpha(t) \int_0^{t+\omega} e_r(t+\omega,\sigma(s))\Delta s.
\end{aligned}
$$

Taking $t \to \infty$ in the above inequality, we obtain that $F : SAP_\omega(\mathbb{R}^+) \to SAP_\omega(\mathbb{R}^+)$.

Then, we prove that $F : SAP_\omega(\mathbb{R}^+) \to SAP_\omega(\mathbb{R}^+)$ is a contraction mapping. In fact, for any $u, v \in SAP_\omega(\mathbb{R}^+)$, we get

$$
\begin{aligned}
|Fu(t) - Fv(t)| &= \left| \int_0^t e_r(t+\omega,\sigma(s))(f(s,u(s)) - f(s,v(s)))\Delta s \right| \\
&\le \int_0^t e_r(t+\omega,\sigma(s))\alpha(s)\Delta s \|u-v\|_\infty < \|u-v\|_\infty.
\end{aligned}
$$

Therefore, the contraction mapping principle tells us that problem (1.2) has a unique $S$-asymptotically $\omega$-periodic solution on $\mathbb{T}_0$. ■

**Theorem 4.4.** *Assume that conditions of Theorem 4.3 and* $(\mathrm{H}_2)$ *are satisfied. Suppose further that* $\varphi(t)$ *is nondecreasing on* $\mathbb{T}_0$*, and* $\psi(t,s) := \alpha(t)e_r(t,\sigma(s))$ *is nondecreasing on* $\mathbb{T}_0$ *with respect to* $t$ *with* $1 - \int_0^t \psi(t,s)\Delta s > 0$*. Then, problem (1.2) has a unique weighted* $Sv$*-asymptotically* $\omega$*-periodic solution if*

$$
\lim_{t\to\infty} \frac{\varphi(t)}{\alpha(t)v(t)(1 - \int_0^t \psi(t,s)\Delta s)} = 0, \tag{4.4}
$$

*where*

$$
\begin{aligned}
\varphi(t) := \alpha(t)\bigg( & u_0 k_r e_r(t,0) + \int_{-\omega}^0 e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
& + \int_0^t e_r(t,\sigma(s))\beta(s)(|u|+1)\Delta s \bigg),
\end{aligned}
$$

$v \in C_b(\mathbb{T}_0, \mathbb{R}^+)$.

**Proof.** Theorem 4.3 indicates that there exists a unique $S$-asymptotically $\omega$-periodic solution $u$ for (1.2), in the following, we only show that $u$ is a weighted $Sv$-asymptotically $\omega$-periodic function.

Letting $t \in \mathbb{T}_0$, we observe that

$$\begin{aligned}
\chi(t) &:= \frac{u(t+\omega)-u(t)}{v(t)} \\
&= \frac{1}{v(t)}(e_r(t+\omega,0)-e_r(t,0))u_0 + \frac{1}{v(t)}\bigg(\int_0^{t+\omega} e_r(t+\omega,\sigma(s))f(s,u(s))\Delta s \\
&\quad - \int_0^t e_r(t,\sigma(s))f(s,u(s))\Delta s\bigg) \\
&= \frac{u_0k_r}{v(t)}e_r(t,0) + \frac{1}{v(t)}\int_{-\omega}^0 e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
&\quad + \frac{1}{v(t)}\bigg(\int_0^t e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
&\quad - \int_0^t e_r(t,\sigma(s))f(s,u(s))\Delta s\bigg) \\
&= \frac{u_0k_r}{v(t)}e_r(t,0) + \frac{1}{v(t)}\int_{-\omega}^0 e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
&\quad + \frac{1}{v(t)}\bigg(\int_0^t e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
&\quad - \int_0^t e_r(t,\sigma(s))f(s,u(s+\omega))\Delta s \\
&\quad + \int_0^t e_r(t,\sigma(s))f(s,u(s+\omega))\Delta s - \int_0^t e_r(t,\sigma(s))f(s,u(s))\Delta s\bigg).
\end{aligned}$$

It follows from above equality, $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$ that

$$\begin{aligned}
|\chi(t)| &\le \frac{u_0k_r}{v(t)}e_r(t,0) + \frac{1}{v(t)}\int_{-\omega}^0 e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
&\quad + \frac{1}{v(t)}\int_0^t \Big(e_r(t,\sigma(s))\beta(s)(|u|+1) + e_r(t,\sigma(s))\alpha(t)|u(s+\omega)-u(s)|\Big)\Delta s \\
&= \frac{u_0k_r}{v(t)}e_r(t,0) + \frac{1}{v(t)}\int_{-\omega}^0 e_r(t,\sigma(s))f(s+\omega,u(s+\omega))\Delta s \\
&\quad + \frac{1}{v(t)}\int_0^t \Big(e_r(t,\sigma(s))\beta(s)(|u|+1) + e_r(t,\sigma(s))\alpha(t)|\mu(t)|v(t)\Big)\Delta s.
\end{aligned}$$

Setting $\theta(t) = \alpha(t)v(t)|\chi(t)|$, we have

$$\theta(t) \le \varphi(t) + \int_0^t \psi(t,s)\theta(s)\Delta s.$$

Using Theorem 4.1, we derive

$$\theta(t) \le \frac{\varphi(t)}{1 - \int_0^t \psi(t,s)\Delta s},$$

that is

$$|\chi(t)| \le \frac{\varphi(t)}{\alpha(t)v(t)(1 - \int_0^t \psi(t,s)\Delta s)}.$$

Due to the condition (4.4), we obtain

$$\lim_{t\to\infty} \frac{u(t+\omega) - u(t)}{v(t)} = \lim_{t\to\infty} \chi(t) = 0,$$

and this completes the proof. ■

**Theorem 4.5.** *Assume that the conditions of Theorem 4.3 and* $(\mathrm{H}_2)$ *are satisfied. Assume that* $\varphi(t)$, $\psi(t,s)$ *are the same as in Theorem 4.4 with* $\varphi \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ *nondecreasing,* $\psi, \psi_t^{\Delta} \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R}^+)$. *Then, problem (1.2) has a unique weighted* $Sv$*-asymptotically* $\omega$*-periodic solution if*

$$\lim_{t\to\infty} \frac{G^{-1}\big(G(m(t)) + \int_0^t n(t,s)\Delta s\big)}{\alpha(t)v(t)} = 0, \tag{4.5}$$

*where* $v \in C_b(\mathbb{T}_0, \mathbb{R}^+)$, $G$ *is defined in (4.2).*

**Proof.** Repeating the process of proof in Theorem 4.4, we have $\theta(t) = \alpha(t)v(t)|\chi(t)|$ such that

$$\theta(t) \le \varphi(t) + \int_0^t \psi(t,s)\theta(s)\Delta s.$$

Obviously, $\theta \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$. From Theorem 4.2, we find

$$\theta(t) \le G^{-1}\left(G(m(t)) + \int_0^t n(t,s)\Delta s\right).$$

Since $\alpha(t)v(t) > 0$, this means that

$$|\chi(t)| \le \frac{G^{-1}\big(G(m(t)) + \int_0^t n(t,s)\Delta s\big)}{\alpha(t)v(t)}.$$

Hence the proof is complete by condition (4.5). ■

In the following, we give some examples to illustrate our main results.

**Example 1.** Setting $\mathbb{T} = c\mathbb{Z}$, $c \ne 0$ is a constant, and $r \in \mathbb{C}\backslash\{-\frac{1}{c}\}$ is a constant, we consider the following equation

$$\begin{cases} u^{\Delta}(t) = e_{\ominus 2r}(t,t_0)u(t), & t_0 < t \in c\mathbb{Z}^k, \\ u(t_0) = 0. \end{cases} \tag{4.6}$$

Since $e_{\ominus 2r}(t+\omega,t)-1>0$ is independent of $t\in c\mathbb{Z}$ whenever $r\in\mathcal{R}$, and $e_r(t,s)=(1+cr)^{\frac{t-s}{c}}$, $\int_a^b f(t)\Delta t=\sum_{k=\frac{a}{c}}^{\frac{b}{c}-1} f(kc)c$ for $t\in c\mathbb{Z}$, then we have

(1)

$$\begin{aligned}&|f(t,u_1)-f(t,u_2)|\le e_{\ominus 2r}(t,t_0)|u_1-u_2|:=\alpha(t)|u_1-u_2|,\\ &\quad \text{for all } t\in c\mathbb{Z},\ u_1,u_2\in C_{rd}(c\mathbb{Z},\mathbb{R}^+);\end{aligned}$$

(2)

$$\begin{aligned}|f(t+\omega,u)-f(t,u)| &= \big(e_{\ominus 2r}(t+\omega,t_0)-e_{\ominus 2r}(t,t_0)\big)|u|\\ &\le e_{\ominus 2r}(t,t_0)\big(e_{\ominus 2r}(t+\omega,t)-1\big)(|u|+1)\\ &:=\beta(t)(|u|+1);\end{aligned}$$

(3)

$$\begin{aligned}&\lim_{t\to\infty}\alpha(t)\int_0^{t+\omega} e_r(t+\omega,\sigma(s))\Delta s=\lim_{t\to\infty}e_{\ominus 2r}(t,t_0)\int_0^{t+\omega} e_r(t+\omega,\sigma(s))\Delta s\\ &\quad=\lim_{t\to\infty}e_{\ominus 2r}(t,t_0)\int_0^{t+\omega}\frac{1}{1+cr}e_r(t+\omega,s)\Delta s\\ &\quad=\lim_{t\to\infty}(1+2cr)^{\frac{t_0-t}{c}}\int_0^{t+\omega}(1+cr)^{\frac{t+\omega-s-1}{c}}\Delta s\\ &\quad=\lim_{t\to\infty}(1+2cr)^{\frac{t_0-t}{c}}(1+cr)^{\frac{t+\omega-1}{c}}\int_0^{t+\omega}(1+cr)^{\frac{-s}{c}}\Delta s\\ &\quad=\lim_{t\to\infty}(1+2cr)^{\frac{t_0-t}{c}}(1+cr)^{\frac{t+\omega-1}{c}}\sum_{k=0}^{\frac{t+\omega}{c}-1}(1+cr)^{-\frac{kc}{c}}c\\ &\quad=c\lim_{t\to\infty}(1+2cr)^{\frac{t_0-t}{c}}(1+cr)^{\frac{t+\omega-1}{c}}\sum_{k=0}^{\frac{t+\omega}{c}-1}(1+cr)^{-k}=0.\end{aligned}$$

Similarly, we get

(4)

$$\int_0^t e_r(t,\sigma(s))\alpha(s)\Delta s=\frac{1}{1+cr}\int_0^t e_r(t,s)e_{\ominus 2r}(t,s)\Delta s<1\quad \text{for } t\in c\mathbb{Z}.$$

So all assumptions of Theorem 4.3 hold for Eq. (4.6), and thus Eq. (4.6) has a unique $S$-asymptotically $\omega$-periodic solution on $c\mathbb{Z}$.

Careful readers may find that Example 1, in fact, is a linear equation, now we give a nonlinear equation based on it. First, we recall the Mean Value Theorem and Trigonometric Functions on time scales.

**Theorem 4.6** (*Theorem 1.14. [3]*)**.** *Let $f$ be a continuous function on $[a,b]$ that is differentiable on $[a,b)$. Then there exist $\xi,\tau\in[a,b)$ such that*

$$f^{\Delta}(\tau)\le\frac{f(b)-f(a)}{b-a}\le f^{\Delta}(\xi).$$

Let

$$f^{\Delta}(\eta_{\tau\xi}) := \max\left\{|f^{\Delta}(\tau)|, |f^{\Delta}(\xi)|\right\}.$$

Then based on the conditions of Theorem 4.6, we get

$$|f(b) - f(a)| \leq |f^{\Delta}(\eta_{\tau\xi})||b - a|. \tag{4.7}$$

From Lemma 3.26 in [2], for $t, t_0 \in \mathbb{T}$, $p \in C_{rd}$, $\mu p^2 \in \mathcal{R}$, we have the trigonometric functions

$$\cos_p(t, t_0) = \frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2}, \qquad \sin_p(t, t_0) = \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2i},$$

and they are $\Delta$-derivatives given by, respectively,

$$\cos_p^{\Delta}(t, t_0) = -p(t)\sin_p(t, t_0), \qquad \sin_p^{\Delta}(t, t_0) = p(t)\cos_p(t, t_0). \tag{4.8}$$

**Example 2.** Let $\mathbb{T}$ be a time scale as in Example 1. Suppose that $\mu u^2 \in \mathcal{R}, \mu r^2 \in \mathcal{R}$. Consider the following equation

$$\begin{cases} u^{\Delta}(t) = e_{\ominus 2r}(t, t_0)\sin_u\left(\dfrac{2\pi t}{\omega}, t_0\right) + 1 + \cos_r\left(\dfrac{2\pi t}{\omega}, t_0\right), \quad t_0 < t \in c\mathbb{Z}^k, \\ u(t_0) = 0. \end{cases} \tag{4.9}$$

From (4.7) and (4.8), we have

$$\begin{aligned} |f(t, u_1) - f(t, u_2)| &= e_{\ominus 2r}(t, t_0)\left|\sin_{u_1}\left(\frac{2\pi t}{\omega}, t_0\right) - \sin_{u_2}\left(\frac{2\pi t}{\omega}, t_0\right)\right| \\ &\leq e_{\ominus 2r}(t, t_0)\left|u(\eta_{\tau\xi})\cos_u\left(\frac{2\pi\eta_{\tau\xi}}{\omega}, t_0\right)\right||u_1 - u_2| \\ &= c_{\tau\xi}\alpha(t)|u_1 - u_2|, \quad \text{for all } t \in c\mathbb{Z},\ u_1, u_2 \in C_{rd}(c\mathbb{Z}, \mathbb{R}^+), \end{aligned}$$

where $c_{\tau\xi} = \left|u(\eta_{\tau\xi})\cos_u\left(\frac{2\pi\eta_{\tau\xi}}{\omega}, t_0\right)\right|$ is a constant.

$$\begin{aligned} |f(t+\omega, u) - f(t, u)| = \Bigg| & e_{\ominus 2r}(t+\omega, t_0)\sin_u\left(\frac{2\pi(t+\omega)}{\omega}, t_0\right) \\ & - e_{\ominus 2r}(t, t_0)\sin_u\left(\frac{2\pi t}{\omega}, t_0\right) + \cos_r\left(\frac{2\pi(t+\omega)}{\omega}\right) - \cos_r\left(\frac{2\pi t}{\omega}\right)\Bigg| \\ = \Bigg| & e_{\ominus 2r}(t+\omega, t_0)\sin_u\left(\frac{2\pi t}{\omega}, t_0\right) - e_{\ominus 2r}(t, t_0)\sin_u\left(\frac{2\pi t}{\omega}, t_0\right) \\ & + \cos_r\left(\frac{2\pi t}{\omega}\right) - \cos_r\left(\frac{2\pi t}{\omega}\right)\Bigg| \end{aligned}$$

$$
\begin{aligned}
&= \left| e_{\ominus 2r}(t+\omega, t_0)\sin_u\left(\frac{2\pi t}{\omega}, t_0\right) - e_{\ominus 2r}(t, t_0)\sin_u\left(\frac{2\pi t}{\omega}, t_0\right) \right| \\
&= e_{\ominus 2r}(t, t_0)(e_{\ominus 2r}(t+\omega, t) - 1)\left| \sin_u\left(\frac{2\pi t}{\omega}, t_0\right) \right| \\
&= \frac{e_{\ominus 2r}(t, t_0)}{k_{2r}+1}\left| \sin_u\left(\frac{2\pi t}{\omega}, t_0\right) \right| \\
&\le \frac{e_{\ominus 2r}(t, t_0)}{k_{2r}+1}(|u|+1) := \widetilde{\beta}(t)(|u|+1).
\end{aligned}
$$

Combining Eqs. (3) and (4) in Example 1, we get that all assumptions of Theorem 4.3 hold for Eq. (4.9), and thus Eq. (4.9) has a unique $S$-asymptotically $\omega$-periodic solution on $c\mathbb{Z}$.

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