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Partial answers of the Asadi et al.'s open question on M-metric spaces with numerical results

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Abstract. In 2014, Asadi et al. introduced the concept of an M-metric space which is a generalization of a partial metric space and established Banach and Kannan fixed point theorems on M-metric spaces. In this work, we prove two fixed point theorems for Chatterjea contraction mappings in the framework of M-metric spaces. These provide partial answers to a question posed by Asadi et al. concerning a fixed point for Chatterjea contraction mappings. We also give some examples and numerical results which can be obtained from our results.

Keywords: Banach contraction principle; Partial metric spaces; M-metric spaces

Mathematics Subject Classification: 47H09; 47H10

1. INTRODUCTION

The famous metric fixed point theorem, generally known as the Banach contraction principle, is one of the essential results in the metric fixed point theory. This principle states that, if (X, d) is a complete metric space, then a mapping $T : X \to X$ satisfying the Banach contractive condition, that is, there exists $k \in [0, 1)$ such that

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$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$, has a unique fixed point.

In fact, the origin of this principle appeared in the methods for solving nonlinear differential equations via successive approximations. However, since the Banach contraction principle is remarkable in its simplicity, a wide range of applications has been given in many different frameworks. In the last decades, a number of fixed point results have been obtained in attempts to generalize this principle. In parallel with the Banach contraction principle, Kannan [3] and Chatterjea [2] proved, respectively, the following fixed point theorems.

Theorem 1.1 (*[3]*). *Let (X, d) be a complete metric space and $T : X \to X$ satisfies the Kannan contractive condition, that is, there exists $k \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 1.2 (*[2]*). *Let (X, d) be a complete metric space and $T : X \to X$ satisfies the Chatterjea contractive condition, that is, there exists $k \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)] \tag{1.3}$$

for all $x, y \in X$. Then T has a unique fixed point.

On the other hand, Matthews [5] introduced the concept of a partial metric space and created an extension of the celebrated Banach contraction principle to the partial metric framework. Based on the results of Matthews [5], Asadi et al. [1] presented a new concept of an M-metric space and studied the topology on M-metric spaces. They also established the fixed point theorems, which are generalizations of Banach and Kannan fixed point theorems in the setup of partial metric spaces. Here we will state the mentioned fixed point results in [1].

Theorem 1.3 (*[1]*). *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in [0, 1) \text{ such that } \quad m(Tx, Ty) \leq km(x, y) \text{ for all } x, y \in X. \tag{1.4}$$

Then T has a unique fixed point.

Theorem 1.4 (*[1]*). *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in \left[0, \frac{1}{2}\right) \text{ such that } \quad m(Tx, Ty) \leq k[m(x, Tx) + m(y, Ty)] \text{ for all } x, y \in X. \tag{1.5}$$

Then T has a unique fixed point.

They also posed the following open problem.

Problem 1.5 (*[1]*)**.** Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:

$$\exists k \in \left[0, \frac{1}{2}\right) \text{ such that } \quad m(Tx, Ty) \leq k[m(x, Ty) + m(Tx, y)] \text{ for all } x, y \in X. \tag{1.6}$$

Does T have a unique fixed point?

Later, many authors have studied this subject and their topological properties. In the sequel, Monfared et al. [6] proved the following theorem as a solution of Problem 1.5 for a smallest interval.

Theorem 1.6 (*[6]*)**.** *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in \left[0, \frac{\sqrt{3}-1}{2}\right) \text{ such that } \quad m(Tx, Ty) \leq k[m(x, Tx) + m(y, Ty)] \quad \text{for all } x, y \in X. \tag{1.7}$$

Then T has a unique fixed point.

In this work, we give two other partial answers to Problem 1.5. These results are the existence theorems of a unique fixed point for Chatterjea contraction mappings in the framework of M-metric spaces. The presented results improve, extend and unify Theorem 1.2 and the Chatterjea fixed point result in partial metric spaces. Furthermore, we give some illustrative examples which support our results, and leave the open questions for those who might be interested.

2. Preliminaries

In this paper, the letter $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of all non-negative real numbers and the set of all positive integers, respectively. The following definitions, notations and lemma are needed in the sequel.

Definition 2.1 (*[5]*)**.** Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is called a partial metric (briefly, p-metric) if the following conditions are satisfied for all $x, y, z \in X$:

(p_1) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$ (equality);
(p_2) $p(x, x) \leq p(x, y)$ (small self-distances);
(p_3) $p(x, y) = p(y, x)$ (symmetry);
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (triangularity).

Also, the pair (X, p) is called a partial metric space (briefly, p-metric space).

It is obvious that the class of partial metric spaces is effectively larger than that of metric spaces. The following examples show that, in general, a p-metric space need not necessarily be a metric space.

Example 2.2. Let $X = [0, \infty)$ and $p : X \times X \to \mathbb{R}_+$ be defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a p-metric space but it is not a metric space. Indeed, $p(1, 1) = 1 \neq 0$.

Example 2.3. Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and $p : X \times X \to \mathbb{R}_+$ be defined by

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$$

for all $[a, b], [c, d] \in X$. Then (X, p) is a p-metric space but it is not a metric space. Indeed, $p([1, 3], [1, 3]) = 2$.

Each p-metric p on a nonempty set X generates a T_0-topology τ_p on X which has as a base the family of open p-ball $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) := \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

Definition 2.4 (*[5]*). Let (X, p) be a p-metric space.

- A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. In this case, we write $x_n \to x$ as $n \to \infty$.
- A sequence $\{x_n\}$ in X is called Cauchy whenever $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
- The p-metric space (X, p) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

Note that the limit of a sequence in a partial metric space is not necessarily unique as the following example shows.

Example 2.5 (*[5]*). Let $X = [0, \infty)$ and $p : X \times X \to \mathbb{R}_+$ be defined by

$$p(x, y) = \max\{x, y\}$$

for all $x, y \in X$. Then (X, p) is a p-metric space. Consider the sequence $\{x_n\} = \{1 + \frac{1}{n}\}$ in X. Note that

$$\lim_{n\to\infty} p(x_n, 1) = \lim_{n\to\infty} \max\{1 + \frac{1}{n}, 1\} = \lim_{n\to\infty} (1 + \frac{1}{n}) = 1 = p(1, 1)$$

and

$$\lim_{n\to\infty} p(x_n, 2) = \lim_{n\to\infty} \max\{1 + \frac{1}{n}, 2\} = \lim_{n\to\infty} 2 = 2 = p(2, 2).$$

This implies that $x_n \to 1$ as $n \to \infty$ and $x_n \to 2$ as $n \to \infty$. Moreover, for any $a \geq 1$ we have $\lim_{n\to\infty} p(x_n, a) = a$. It yields that $x_n \to a$ as $n \to \infty$ for all $a \geq 1$.

Let X be a nonempty set and $m : X \times X \to \mathbb{R}_+$ be a given function. For every $x, y \in X$, the following notation are useful in the sequel:

1. $m_{x,y} := \min\{m(x, x), m(y, y)\}$;
2. $M_{x,y} := \max\{m(x, x), m(y, y)\}$.

Very recently, Asadi et al. [1] gave the notion of an M-metric space which is a generalization of a p-metric space. Next, we recall the definition and some properties of M-metric spaces.

Definition 2.6 (*[1]*). Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}_+$ is called an m-metric if the following conditions are satisfied for all $x, y, z \in X$:

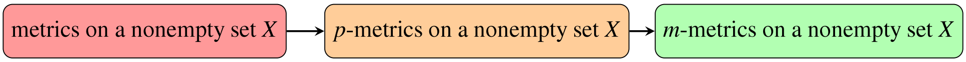

Fig. 1. Relationship between metrics, p-metrics and M-metrics on a nonempty set X.

(m_1) $m(x,x) = m(y,y) = m(x,y) \Leftrightarrow x = y$;
(m_2) $m_{x,y} \leq m(x,y)$;
(m_3) $m(x,y) = m(y,x)$;
(m_4) $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$.

Also, the pair (X, m) is called an M-metric space.

Lemma 2.7 (*[1]*)**.** *Every partial metric space is an M-metric space but the converse is not true.*

From Lemma 2.7, we refer the reader to the relation between metrics, p-metrics and M-metrics in Fig. 1.

From the fact in Fig. 1, it is interesting that we have to focus on m-metric spaces. Some examples of an m-metric which show that it is a real generalization of a partial metric are the following.

Example 2.8. Let $X = [0, \infty)$ and $m : X \times X \to \mathbb{R}_+$ be defined by $m(x,y) = \frac{x+y}{2}$ for all $x, y \in X$. Then m is an m-metric on X, but it is not a p-metric. Indeed, $m(4,4) = 4 \not\leq m(4,2) = 3$.

Example 2.9. Let $X = \{1, 2, 3\}$ and $m : X \times X \to \mathbb{R}_+$ be defined by

$$m(x,y) = \begin{cases} 1, & x = y = 1, \\ 9, & x = y = 2, \\ 5, & x = y = 3, \\ 10, & x, y \in \{1,2\} \text{ and } x \neq y, \\ 7, & x, y \in \{1,3\} \text{ and } x \neq y, \\ 8, & x, y \in \{2,3\} \text{ and } x \neq y. \end{cases}$$

Then m is an m-metric but it is not a p-metric. Indeed, $m(2,2) = 9 \not\leq m(2,3) = 8$.

Remark 2.10 (*[1]*)**.** For every x, y in an M-metric space (X, m), the following assertions hold:

1. $0 \leq M_{x,y} + m_{x,y} = m(x,x) + m(y,y)$;
2. $0 \leq M_{x,y} - m_{x,y} = |m(x,x) - m(y,y)|$;
3. $M_{x,y} - m_{x,y} \leq (M_{x,z} - m_{x,z}) + (M_{z,y} - m_{z,y})$.

Definition 2.11 (*[1]*)**.** Let (X, m) be an m-metric space.

1. A sequence $\{x_n\}$ in X converges to point $x \in X$ if and only if

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0. \tag{2.1}$$

2. A sequence $\{x_n\}$ in X is called an m-Cauchy sequence if

$$\lim_{n,m\to\infty}(m(x_n, x_m) - m_{x_n,x_m}) \text{ and } \lim_{n,m\to\infty}(M_{x_n,x_m} - m_{x_n,x_m}) \tag{2.2}$$

exist and are finite.

3. An M-metric space (X, m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n\to\infty}(m(x_n, x) - m_{x_n,x}) = 0 \text{ and } \lim_{n\to\infty}(M_{x_n,x} - m_{x_n,x}) = 0.$$

Example 2.12 ([4]). Let $X = [0, \infty)$ and an m-metric $m : X \times X \to \mathbb{R}_+$ be defined by $m(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. Then (X, m) is complete.

Lemma 2.13 ([1]). *Let (X, m) be an M-metric space and $\{x_n\}$, $\{y_n\}$ be sequences in X. Assume that $x_n \to x \in X$ and $y_n \to y \in X$ as $n \to \infty$. Then*

$$\lim_{n\to\infty}(m(x_n, y_n) - m_{x_n,y_n}) = m(x, y) - m_{x,y}.$$

Lemma 2.14 ([1]). *Let (X, m) be an M-metric space and $\{x_n\}$ be a sequence in X. Assume that $x_n \to x \in X$ as $n \to \infty$. Then*

$$\lim_{n\to\infty}(m(x_n, y) - m_{x_n,y}) = m(x, y) - m_{x,y}$$

for all $y \in X$.

Lemma 2.15 ([1]). *Let (X, m) be an M-metric space and $\{x_n\}$ be a sequence in X. Assume that $x_n \to x \in X$ and $x_n \to y \in X$ as $n \to \infty$. Then $m(x, y) = m_{x,y}$. Furthermore, if $m(x, x) = m(y, y)$, then $x = y$.*

Lemma 2.16 ([1]). *Let $\{x_n\}$ be a sequence in an M-metric space (X, m) and there exists $r \in [0, 1)$ such that*

$$m(x_{n+2}, x_{n+1}) \le r m(x_{n+1}, x_n), \tag{2.3}$$

for all $n \in \mathbb{N}$. Then the following assertions hold:

(A) $\lim_{n\to\infty} m(x_{n+1}, x_n) = 0$;
(B) $\lim_{n\to\infty} m(x_n, x_n) = 0$;
(C) $\lim_{n,m\to\infty} m_{x_m,x_n} = 0$;
(D) $\{x_n\}$ *is an m-Cauchy sequence.*

3. Main results

In this section, we give partial answers of Problem 1.5 and also furnish some illustrative examples to demonstrate the validity of the hypotheses and degree of utility of our results. The main result in this section is a generalization of the classical Chatterjia fixed point result in metric spaces and the Chatterjia fixed point result in partial metric spaces.

Theorem 3.1. *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in \left[0, \frac{1}{2}\right) \text{ such that } \quad m(Tx, Ty) \leq k[m(x, Ty) + m(Tx, y)] \text{ for all } x, y \in X. \tag{3.1}$$

If there is $x_0 \in X$ such that

$$m(T^n x_0, T^n x_0) \leq m(T^{n-1} x_0, T^{n-1} x_0) \tag{3.2}$$

for all $n \in \mathbb{N}$, then T has a unique fixed point. Moreover, if the Picard sequence $\{x_n\}$ in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ such that x_0 is an initial point in the condition (3.2), then $\{x_n\}$ converges to a fixed point of T.

Proof. Starting from $x_0 \in X$ in the hypothesis, we will construct the sequence $\{x_n\}$ in X such that

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. From (3.1), (3.2) and the condition (m_4), we get

$$\begin{aligned} m(x_{n+1}, x_n) &= m(Tx_n, Tx_{n-1}) \\ &\leq k[m(x_n, x_n) + m(x_{n+1}, x_{n-1})] \\ &\leq k[m(x_n, x_n) + (m(x_{n+1}, x_n) - m_{x_{n+1}, x_n} + m(x_n, x_{n-1}) - m_{x_n, x_{n-1}}) \\ &\quad + m_{x_{n+1}, x_{n-1}}] \\ &= k[m(x_n, x_n) + m(x_{n+1}, x_n) - m(x_{n+1}, x_{n+1}) + m(x_n, x_{n-1}) \\ &\quad - m(x_n, x_n) + m(x_{n+1}, x_{n+1})] \\ &= k[m(x_{n+1}, x_n) + m(x_n, x_{n-1})] \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that

$$m(x_{n+1}, x_n) \leq rm(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$, where $0 \leq r := \frac{k}{1-k} < 1$. By using Lemma 2.16, we get (A), (B), (C) and (D) in Lemma 2.16 hold. It follows from (D) that $\{x_n\}$ is an M-Cauchy sequence X. From the completeness of X, we get $x_n \to x$ as $n \to \infty$ for some $x \in X$. So

$$m(x_n, x) - m_{x_n, x} \to 0 \text{ as } n \to \infty \tag{3.3}$$

and

$$M_{x_n, x} - m_{x_n, x} \to 0 \text{ as } n \to \infty. \tag{3.4}$$

From (B), we get $m(x_n, x_n) \to 0$ as $n \to \infty$ and so

$$m_{x_n, x} = \min\{m(x_n, x_n), m(x, x)\} \to 0 \text{ as } n \to \infty \tag{3.5}$$

and

$$m_{x_n, Tx} = \min\{m(x_n, x_n), m(Tx, Tx)\} \to 0 \text{ as } n \to \infty. \tag{3.6}$$

From (3.3), (3.4), (3.5), we obtain

$$m(x_n, x) \to 0 \text{ as } n \to \infty \tag{3.7}$$

and

$$M_{x_n,x} \to 0 \text{ as } n \to \infty. \tag{3.8}$$

By Remark 2.10, we have

$$M_{x_n,x} + m_{x_n,x} = m(x_n, x_n) + m(x, x) \tag{3.9}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above equation and using (3.5), (3.8) and (B), we have

$$m(x, x) = 0. \tag{3.10}$$

This implies that

$$m_{x,Tx} = \min\{m(x, x), m(Tx, Tx)\} = 0. \tag{3.11}$$

Next, we will show that $m(x, Tx) = 0$. From (m_4), we get

$$m(x, Tx) = m(x, Tx) - m_{x,Tx} \leq m(x, x_n) - m_{x,x_n} + m(x_n, Tx) - m_{x_n,Tx} \tag{3.12}$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \to \infty$ in (3.12) and using (3.3), (3.4), (3.6), (3.11), we get

$$\begin{aligned}
m(x, Tx) &\leq \limsup_{n\to\infty} \left[m(x, x_n) - m_{x,x_n} + m(x_n, Tx) - m_{x_n,Tx}\right] \\
&\leq \limsup_{n\to\infty} \left[m(x, x_n) - m_{x,x_n} + m(x_n, Tx)\right] \\
&\leq \limsup_{n\to\infty} [m(x, x_n) - m_{x,x_n}] + \limsup_{n\to\infty} m(x_n, Tx) \\
&= \limsup_{n\to\infty} m(x_n, Tx) \\
&\leq \limsup_{n\to\infty} [k(m(x_{n-1}, Tx) + m(Tx_{n-1}, x))] \\
&\leq k\left[\limsup_{n\to\infty} m(x_{n-1}, Tx) + \limsup_{n\to\infty} m(x_n, x)\right] \\
&= k\left[\limsup_{n\to\infty} m(x_{n-1}, Tx)\right] \\
&\leq k\left[\limsup_{n\to\infty} \left[m(x_{n-1}, x) - m_{x_{n-1},x} + m(x, Tx) - m_{x,Tx} + m_{x_{n-1},Tx}\right]\right] \\
&\leq km(x, Tx).
\end{aligned}$$

This implies that

$$m(x, Tx) = 0. \tag{3.13}$$

By the contractive condition (3.1), we have

$$m(Tx, Tx) \leq 2km(x, Tx) = 0$$

and hence

$$m(Tx, Tx) = 0. \tag{3.14}$$

From (3.5), (3.13), (3.14), we obtain

$$m(x,x) = m(Tx,Tx) = m(x,Tx).$$

Using the property (m_1), we get $x = Tx$.

Finally, we will show that T has a unique fixed point. Assume that y is an another fixed point of T. From (3.1), we get

$$\begin{aligned} m(x,y) &= m(Tx,Ty) \\ &\leq k(m(x,Ty) + m(y,Tx)) \\ &= k(m(x,y) + m(y,x)) \\ &\leq 2km(x,y) \\ &< m(x,y), \end{aligned}$$

which is a contraction. Then T has a unique fixed point. This completes the proof. □

Theorem 3.2. *Theorem 3.1 remains true if the assumption (3.2) is replaced by the following condition:*

$$m(T^{n-1}x_0, T^{n-1}x_0) \leq m(T^n x_0, T^n x_0) \tag{3.15}$$

for all $n \in \mathbb{N}$.

Proof. Let $x_0 \in X$ be a starting point in the hypothesis and let $\{x_n\}$ be the Picard iteration defined by

$$x_n = Tx_{n-1}, \quad n \in \mathbb{N}.$$

From the condition (3.15), (m_4) and the contractive condition (3.1), we deduce that

$$\begin{aligned} m(x_{n+1},x_n) &= m(Tx_n, Tx_{n-1}) \\ &\leq k[m(x_n,x_n) + m(x_{n+1},x_{n-1})] \\ &= k[m_{x_n,x_{n+1}} + m(x_{n+1},x_{n-1})] \\ &\leq k[m(x_{n+1},x_n) + m(x_n,x_{n-1}) - m_{x_n,x_{n-1}} + m_{x_{n+1},x_{n-1}}] \\ &= k[m(x_{n+1},x_n) + m(x_n,x_{n-1}) - m(x_{n-1},x_{n-1}) + m(x_{n-1},x_{n-1})] \\ &= k[m(x_{n+1},x_n) + m(x_n,x_{n-1})] \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that

$$m(x_{n+1},x_n) \leq rm(x_n,x_{n-1})$$

for all $n \in \mathbb{N}$, where $0 \leq r := \frac{k}{1-k} < 1$. The rest of the proof is similar to that of Theorem 3.1 and hence it is omitted. □

4. Examples and Numerical Results

In this section, we give some examples to illustrate Theorem 3.1 and also give some numerical results.

Example 4.1. Let $X = [0, \infty)$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by $m(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. From Example 2.12, we get (X, m) is a complete M-metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} 0, & 0 \le x < 3, \\ \dfrac{x}{1+x}, & x \ge 3. \end{cases}$$

We will show that the condition (3.1) is satisfied with $k = \frac{1}{4}$. Suppose that $x, y \in X$. Then there are three possibilities:

Case 1: If $x, y \in [0, 3)$, the claim is obvious.
Case 2: If $x, y \in [3, \infty)$, we get

$$\begin{aligned} m(Tx, Ty) &= \frac{1}{2}\left(\frac{x}{1+x} + \frac{y}{1+y}\right) \\ &\le \frac{1}{2}\left(\frac{x}{4} + \frac{y}{4}\right) \\ &\le \frac{1}{4}\left(\frac{x}{2} + \frac{y}{2(1+y)} + \frac{y}{2} + \frac{x}{2(1+x)}\right) \\ &= k\left[m(x, Ty) + m(y, Tx)\right]. \end{aligned}$$

Case 3: Assume that $(x, y) \in [3, \infty) \times [0, 3) \cup [0, 3) \times [3, \infty)$. Without loss of generality, we may assume that $x \in [0, 3)$ and $y \in [3, \infty)$. Then we obtain

$$\begin{aligned} m(Tx, Ty) &= \frac{1}{2}\left(\frac{y}{1+y}\right) \\ &\le \frac{1}{2}\left(\frac{y}{4}\right) \\ &\le \frac{1}{4}\left(0 + \frac{y}{2(1+y)} + \frac{y}{2} + 0\right) \\ &= k\left[m(x, Ty) + m(y, Tx)\right]. \end{aligned}$$

Then T satisfies the condition (3.1) for all $x, y \in X$ with $k = \frac{1}{4}$. Also, T satisfies the condition (3.2) for all $x_0 \in X$. Thus, all conditions of Theorem 3.1 are satisfied and so the existence of a unique fixed point of T follows from Theorem 3.1. In this case, a unique fixed point of T is a point 0.

We can see some numerical experiments for approximating the unique fixed point of T in Fig. 2. Furthermore, the convergence behavior of some iterations are shown in Fig. 3.

Example 4.2. Let $X = [0, \infty)$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by $m(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. From Example 2.12, we get (X, m) is a complete M-metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} x^2, & 0 \le x < \dfrac{1}{2}, \\ \dfrac{1}{4}, & x \ge \dfrac{1}{2}. \end{cases}$$

Now we shall claim that T satisfies the condition (3.1) with $k = \frac{1}{3}$. Suppose that $x, y \in X$. We will divide the proof into three cases.

x_0	3.00000000	5.00000000	7.00000000	9.00000000
x_1	0.75000000	0.83333333	0.87500000	0.90000000
x_2	0.00000000	0.00000000	0.00000000	0.00000000
x_3	0.00000000	0.00000000	0.00000000	0.00000000
x_4	0.00000000	0.00000000	0.00000000	0.00000000
x_5	0.00000000	0.00000000	0.00000000	0.00000000
x_6	0.00000000	0.00000000	0.00000000	0.00000000
x_7	0.00000000	0.00000000	0.00000000	0.00000000
x_8	0.00000000	0.00000000	0.00000000	0.00000000
x_9	0.00000000	0.00000000	0.00000000	0.00000000
x_{10}	0.00000000	0.00000000	0.00000000	0.00000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fig. 2. Iterates of Picard iterations in Example 4.1.

Case 1: If $x, y \in \left[0, \frac{1}{2}\right)$, then we get

$$\begin{aligned} m(Tx, Ty) = \frac{x^2}{2} + \frac{y^2}{2} &\leq \frac{x}{6} + \frac{y^2}{6} + \frac{y}{6} + \frac{x^2}{6} \\ &= \frac{1}{3}\left(\frac{x}{2} + \frac{y^2}{2} + \frac{y}{2} + \frac{x^2}{2}\right) \\ &= \frac{1}{3}[m(x, Ty) + m(y, Tx)]. \end{aligned}$$

Case 2: If $x, y \in \left[\frac{1}{2}, \infty\right)$, then we get

$$\begin{aligned} m(Tx, Ty) = \frac{1}{8} + \frac{1}{8} &\leq \frac{x}{6} + \frac{1}{24} + \frac{y}{6} + \frac{1}{24} \\ &= \frac{1}{3}\left(\frac{x}{2} + \frac{1}{8} + \frac{y}{2} + \frac{1}{8}\right) \\ &= \frac{1}{3}[m(x, Ty) + m(y, Tx)]. \end{aligned}$$

Case 3: Assume that $(x, y) \in \left[0, \frac{1}{2}\right) \times \left[\frac{1}{2}, \infty\right) \cup \left[\frac{1}{2}, \infty\right) \times \left[0, \frac{1}{2}\right)$. Without loss of generality, we may assume that $x \in \left[0, \frac{1}{2}\right)$ and $y \in \left[\frac{1}{2}, \infty\right)$. Since $\frac{x^2}{2} \leq \frac{x}{6} + \frac{x^2}{6}$ and $\frac{1}{8} \leq \frac{1}{24} + \frac{y}{6}$. Then we obtain that

$$\begin{aligned} m(Tx, Ty) = \frac{x^2}{2} + \frac{1}{8} &\leq \frac{x}{6} + \frac{1}{24} + \frac{y}{6} + \frac{x^2}{6} \\ &= \frac{1}{3}\left(\frac{x}{2} + \frac{1}{8} + \frac{y}{2} + \frac{x^2}{2}\right) \\ &= \frac{1}{3}[m(x, Ty) + m(y, Tx)]. \end{aligned}$$

This implies that T satisfies the condition (3.1) with $k = \frac{1}{3}$. It is easy to see that T satisfies the condition (3.2) for all $x_0 \in X$. Therefore, all conditions in Theorem 3.1 are satisfied. So T has a unique fixed point. In this example, a unique fixed point of T is a point 0.

We can see some numerical experiments for approximating the unique fixed point of T in Fig. 4. Furthermore, the convergence behavior of some iterations are shown in Fig. 5.

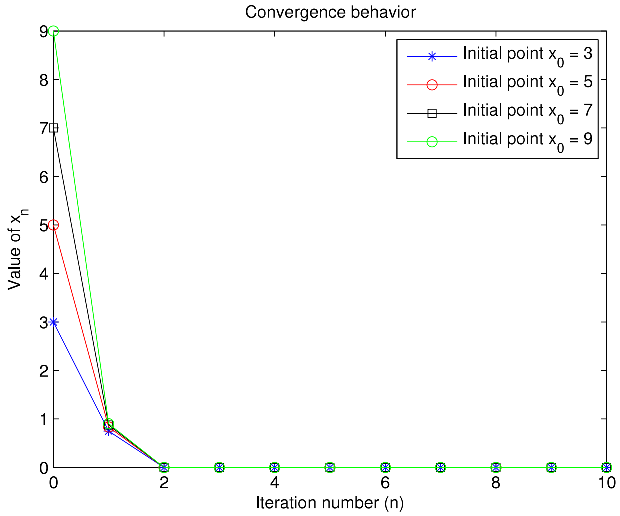

Fig. 3. Convergence behavior for Example 4.1.

x_0	0.20000000	0.30000000	0.40000000	0.50000000
x_1	0.04000000	0.09000000	0.16000000	0.25000000
x_2	0.00160000	0.00810000	0.02560000	0.06250000
x_3	0.00000256	0.00006561	0.00065536	0.00390625
x_4	0.00000000	0.00000000	0.00000043	0.00001526
x_5	0.00000000	0.00000000	0.00000000	0.00000000
x_6	0.00000000	0.00000000	0.00000000	0.00000000
x_7	0.00000000	0.00000000	0.00000000	0.00000000
x_8	0.00000000	0.00000000	0.00000000	0.00000000
x_9	0.00000000	0.00000000	0.00000000	0.00000000
x_{10}	0.00000000	0.00000000	0.00000000	0.00000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fig. 4. Iterates of Picard iterations in Example 4.2.

5. Open problems

In this section, we give some challenging questions for the readers.

- **Problem 1.**: Can Theorem 3.1 be proved without the condition (3.2) (or (3.15) for Theorem 3.2)?
- **Problem 2.**: Does there exist any other condition (3.2) to prove Theorem 3.1 (or (3.15) for Theorem 3.2)?

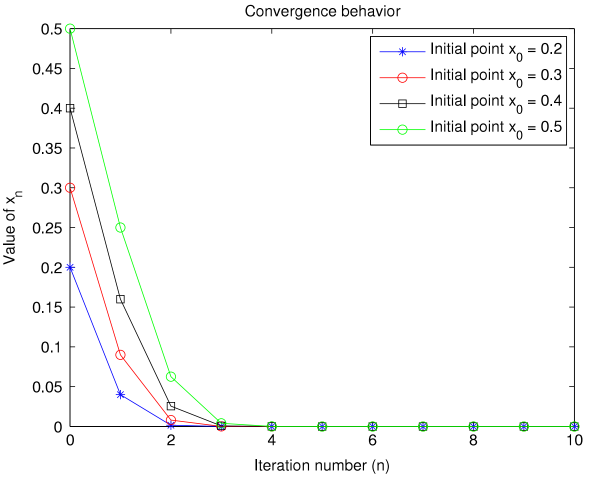

Fig. 5. Convergence behavior for Example 4.2.

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References

[1] M. Asadi, E. Karapinar, P. Salimi, New extension of p-metric spaces with some fixed-point results on M-metric spaces, J. Inequal. Appl. 18 (2014).
[2] S.K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci. 25 (1972) 727–730.
[3] R. Kannan, Some results on fixed points II, Amer. Math. Monthly 76 (1969) 405–408.
[4] P. Kumrod, W. Sintunavarat, An improvement of recent results in M-metric spaces with numerical results, J. Math. Anal. 8 (2017) 202–213.
[5] S. Matthews, Partial metric topology, Ann. New York Acad. Sci. 728 (1994) 183–197.
[6] H. Monfared, M. Azhini, M. Asadi, Fixed point results on M-metric spaces, J. Math. Anal. 7 (5) (2016) 85–101.