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Parametric evaluations of Ramanujan's singular moduli

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Abstract. At scattered places of his notebooks, Ramanujan recorded over 30 values of singular moduli. In this paper, we offer some general formulas for the explicit evaluations of Ramanujan's singular moduli by parameterizations of Ramanujan's theta-functions and give examples.

Mathematics subject classification: 11F20; 33D10; 33E05

Keywords: Ramanujan's singular moduli; Ramanujan's theta-functions

1. Introduction

As usual, for positive integers n and any complex number a, we write

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1. \tag{1.1}$$

In Chapter 16 of his second notebook [4,11], Ramanujan defined his general theta-function $f(a,b)$ as

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \tag{1.2}$$

where $|ab| < 1$. If we set $a = q^{2iz}$, $b = q^{-2iz}$, and $q = e^{\pi i\tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical

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theta-functions in its standard notation [16, p. 464]. Three special cases of Ramanujan's general theta-function are given by

$$\phi(q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.3}$$

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$f(-q) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty. \tag{1.5}$$

If $q = e^{2\pi iz}$ with $\mathrm{Im}(z) > 0$, then $f(-q) = q^{-1/24}\eta(z)$, where $\eta(z)$ denotes the classical Dedekind eta-function.

If, as usual in the theory of elliptic functions, $k = k(q)$ denotes the modulus, then the singular moduli k_n is defined by $k_n = k\left(e^{-\pi\sqrt{n}}\right)$, where n is a positive integer. In terms of Ramanujan, set $\alpha = k^2$ and $\alpha_n = k_n^2$. In his notebooks, Ramanujan recorded without proofs the values of over 30 singular moduli. In his second letter to Hardy [12, p. xxix], Ramanujan asserted the value of k_{210}^2 which was first proved by Watson [15] by using a remarkable formula found in Ramanujan's first notebook [11, Vol. 1, p. 320]. The same formula can also be used to find values of α_n for even values of n. On page 82 of his first notebook, Ramanujan offers three additional theorems for calculating α_n for even values of n. Particularly, he offered formulas for α_{4p}, α_{8p}, and α_{16p}. Ramanujan also recorded several values of α_n for odd n in his first notebook [11, p. 80] and second notebook [11, p. 262, 345, 346]. All these results were proved by Berndt, Chan, and Zhang [7] by employing Ramanujan's class invariants G_n and g_n [5, p. 34] defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \text{ and } g_n := 2^{-1/4}q^{-1/24}\chi(-q); \tag{1.6}$$

where $\chi(q) = (-q;q^2)_\infty, q = e^{-\pi\sqrt{n}}$ and n is a positive integer. An account of these can be found in Chapter 34 of Berndt's book [5].

Ramanathan [10] and J. M. and P. B. Borwein [8] previously determined some of Ramanujan's singular moduli α_n. Recently, Bairy [1] also calculated some values of singular moduli α_n by establishing some general theorems involving Ramanujan's class invariants G_n and g_n.

Since from [4, p. 124], $\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}$ and $\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}$, it follows from (1.6) that

$$G_n = \{4\alpha(1-\alpha)\}^{-1/24} \text{ and } g_n = 2^{-1/12}(1-\alpha)^{1/12}\alpha^{-1/24}. \tag{1.7}$$

Several values of class invariants G_n and g_n are calculated by Ramanujan. We refer to Chapter 34 of Berndt's book [5]. It is clear from (1.7) that if we know g_n or G_n the corresponding values of α_n can be determined by solving a quadratic equation. However, the expressions one obtain are generally unattractive and so better algorithm is sought for evaluations of α_n.

In this paper, we offer some general formulas for explicit evaluations of Ramanujan's singular moduli α_n by considering certain parameterizations of Ramanujan's theta-functions for special values of q and give examples. It is worth to mention here that all formulas of α_n offered in this paper can be used to find α_n for both even and odd values of n whereas in the literature there were different formulas for calculation of α_n for odd and even values of n. We consider the following parameterizations of Ramanujan's theta-functions:

$$J_n := \frac{f(-q)}{\sqrt{2}q^{1/8}f(-q^4)}; \quad q = e^{-\pi\sqrt{n}}. \tag{1.8}$$

$$s_{4,n} := \frac{f(q)}{\sqrt{2}q^{1/8}f(-q^4)}; \quad q = e^{-\pi\sqrt{n}/2}, \tag{1.9}$$

$$h_{2,n} := \frac{\phi(q)}{2^{1/4}\phi(q^2)}; \quad q = e^{-\pi\sqrt{n/2}}, \tag{1.10}$$

and

$$h_{4,n} := \frac{\phi(q)}{\sqrt{2}\phi(q^4)}; \quad q = e^{-\pi\sqrt{n/4}}. \tag{1.11}$$

The parameter J_n is a particular case $k = 4$ of the parameter $r_{k,n}$ introduced by Yi [17] and defined by

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}; \quad q = e^{-2\pi\sqrt{n/k}} \tag{1.12}$$

where n and k are positive real numbers. Baruah and Saikia [2] and Yi [17] evaluated several values of J_n. The parameter $s_{4,n}$ is a particular case $k = 4$ of the parameter $s_{k,n}$ due to Berndt [6, p. 9, (4.7)], and defined by

$$s_{k,n} := \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(-(-1)^k q^k)}; \quad q = e^{-\pi\sqrt{n/k}}, \tag{1.13}$$

where k and n being positive real numbers. Baruah and Saikia [3] evaluated several values of the parameter $s_{4,n}$. The parameters $h_{2,n}$ and $h_{4,n}$ are the particular cases $k = 2$ and 4, respectively of the general parameter $h_{k,n}$ introduced by Yi [18] (also see [17]), and defined by

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \tag{1.14}$$

where n and k are positive real numbers. For explicit values of $h_{2,n}$ and $h_{4,n}$ see [3], [9], [13], and [18].

We end this introduction by recording some transformation formulas for Ramanujan's theta-functions for ready references in next sections.

$$f(q) = \sqrt{z}2^{-1/6}(\alpha(1-\alpha))^{1/24}q^{-1/24}, \tag{1.15}$$
$$f(-q) = \sqrt{z}2^{-1/6}(1-\alpha)^{1/6}\alpha^{1/24}q^{-1/24}, \tag{1.16}$$
$$f(-q^4) = \sqrt{z}2^{-2/3}(1-\alpha)^{1/24}\alpha^{1/6}q^{-1/6}, \tag{1.17}$$
$$\chi(q) = 2^{1/6}(\alpha(1-\alpha))^{-1/24}q^{1/24}, \tag{1.18}$$
$$\phi(q) = \sqrt{z}, \tag{1.19}$$
$$\phi(q^2) = \sqrt{z}\left(\left(1+\sqrt{1-\alpha}\right)/2\right)^{1/2}, \tag{1.20}$$
$$\phi(q^4) = \sqrt{z}(1+(1-\alpha)^{1/4})/2. \tag{1.21}$$

For (1.15)–(1.18) see [4, p. 124, Entry 12(i), (ii), (iv), & (v)]. For (1.19)–(1.21) see [4, p. 122, Entry 10(i), (iv), & (v)].

2. General formulas for explicit evaluations of A_N

In this section we prove some general formulas for the explicit evaluations of Ramanujan's singular moduli α_n and give examples.

Theorem 2.1. *If J_n is as defined in (1.8), then*

$$\alpha_n = \frac{1}{1+J_n^8}.$$

Proof. Transcribing using (1.16) and (1.17) and simplifying, we find that

$$\frac{f(-q)}{\sqrt{2}q^{1/8}f(-q^4)} = \left(\frac{\alpha}{1-\alpha}\right)^{1/8}. \tag{2.1}$$

Setting $q := e^{-\pi\sqrt{n}}$ and $\alpha_n := \alpha\left(e^{-\pi\sqrt{n}}\right)$ and employing the definition of J_n in (2.1), we find that

$$J_n = \left(\frac{\alpha_n}{1-\alpha_n}\right)^{1/8}. \tag{2.2}$$

Simplifying (2.2), we easily arrive at the desired result. □

Corollary 2.2. *We have*

$$\alpha_{1/n} = J_n^8\alpha_n.$$

Proof. From [2, p. 9, Theorem 6.1], we note that

$$J_{1/n} = 1/J_n. \tag{2.3}$$

Replacing n by $1/n$ in Theorem 2.1 and simplifying using (2.3), we obtain

$$\alpha_{1/n} = \frac{J_n^8}{1+J_n^8}. \tag{2.4}$$

Employing Theorem 2.1 in (2.4), we complete the proof. □

Corollary 2.3. *We have*

$$1 - \alpha_n = J_n^8 \alpha_n.$$

Proof. This follows from Theorem 2.1. □

Corollary 2.4. *We have*

$$\alpha_{1/n} = 1 - \alpha_n.$$

Proof. This follows directly from Corollaries 2.3 and 2.4. □

From Theorem 2.1 it is obvious that if we know the values of J_n then the corresponding values of α_n can easily be calculated. Baruah and Saikia [2] evaluated the explicit values of J_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 23, 25, 31, 36, 49$. Yi [17] also calculated J_n for $n = 1, 2, 3, 4, 5, 8, 9$ and 25 in terms of $r_{k,n}$. In Theorem 2.6 below, we find the explicit values of α_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9,$ and 10. Similarly, we can find α_n for remaining values of n for which values of J_n are available.

Lemma 2.5. [2, p. 9 and 10, Theorem 6.3 & 6.5] *If $J_n$is as defined in(1.8), then*

$$J_1 = 1,$$

$$J_2 = 2^{1/8}\left(1+\sqrt{2}\right)^{1/8},$$

$$J_3 = \left(2+\sqrt{3}\right)^{1/4},$$

$$J_4 = 2^{5/16}\left(1+\sqrt{2}\right)^{1/4},$$

$$J_5 = \frac{1}{\sqrt{2}}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/2},$$

$$J_6 = r_{4,6} = \left(1+\sqrt{2}\right)^{3/8}\left(2(1+\sqrt{2}+\sqrt{6})\right)^{1/8},$$

$$J_7 = \left(8+3\sqrt{7}\right)^{1/4},$$

$$J_8 = 2^{1/4}\left(1+\sqrt{2}\right)^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8},$$

$$J_9 = \frac{1}{2}+\frac{3^{1/4}}{\sqrt{2}}+\frac{\sqrt{3}}{2},$$

$$J_{10} = \frac{(1+\sqrt{5})^{9/8}(2+3\sqrt{2}+\sqrt{5})^{1/8}}{2}.$$

Theorem 2.6. *We have*

$$\begin{aligned}
\alpha_1 &= 1/2,\\
\alpha_2 &= 3-2\sqrt{2} = \left(\sqrt{2}-1\right)^2,\\
\alpha_3 &= \left(2-\sqrt{3}\right)/4,\\
\alpha_4 &= \left(17-12\sqrt{2}\right) = \left(\sqrt{2}-1\right)^4,\\
\alpha_5 &= 16\Bigg/\left(16+\left(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}\right)^4\right),\\
\alpha_6 &= 35-14\sqrt{6}+24\sqrt{2}-20\sqrt{3} = \left(2-\sqrt{3}\right)^2\left(\sqrt{3}-\sqrt{2}\right)^2,\\
\alpha_7 &= \left(8-3\sqrt{7}\right)/16,\\
\alpha_8 &= 113+80\sqrt{2}-28\sqrt{2+10\sqrt{2}}-20\sqrt{2\left(2+10\sqrt{2}\right)},\\
\alpha_9 &= 1\Big/\left(2\left(97+52\sqrt{2}\,3^{1/4}+56\sqrt{3}+30\sqrt{2}\,3^{3/4}\right)\right),\\
\alpha_{10} &= 323+144\sqrt{5}-228\sqrt{2}-102\sqrt{10}.
\end{aligned}$$

Proof. We set $n = 1, 2, 3, 4, 5, 6, 8, 9,$ and 10 in Theorem 2.1 and use the corresponding values of J_n from Lemma 2.5 to complete the proof. □

The values of $\alpha_{1/n}$ can be determined if we know α_n, J_n or both by appealing to Corollaries 2.3, 2.4, or Eq. (2.4). For example, we calculate some values $\alpha_{1/n}$ in Theorem 2.7.

Theorem 2.7. *We have*

$$\begin{aligned}
\alpha_{1/2} &= 2\sqrt{2}-2,\\
\alpha_{1/3} &= \left(2+\sqrt{3}\right)/4,\\
\alpha_{1/4} &= 12\sqrt{2}-16,\\
\alpha_{1/5} &= \left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^4\Bigg/\left(16+\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^4\right),\\
\alpha_{1/6} &= 20\sqrt{3}+14\sqrt{6}-34-24\sqrt{2},\\
\alpha_{1/7} &= \left(8+3\sqrt{7}\right)\Big/16.
\end{aligned}$$

Proof. We employ the values of α_n for $n = 2, 3, 4, 5, 6,$ and 7, respectively from Theorem 2.6 in Corollary 2.4 and simplify to complete the proof. □

Theorem 2.8. *If $s_{4,n}$ is as defined in (1.9), then*

$$\alpha_{4/n} = 1/s_{4,n}^8.$$

Proof. Transcribing using (1.15) and (1.17) and simplifying, we find that

$$\frac{f(q)}{\sqrt{2}q^{1/8}f(-q^4)} = 1/\alpha_n^{1/8}. \tag{2.5}$$

Setting $q := e^{-\pi\sqrt{n/4}}$ and $\alpha_{n/4} := \alpha\left(e^{-\pi\sqrt{n/4}}\right)$, employing the definition of $s_{4,n}$ in (2.5) and simplifying, we complete the proof. □

From Theorem 2.8 it is clear that if we know the values of $s_{4,n}$ then the explicit values of $\alpha_{n/4}$ can easily be calculated. Baruah and Saikia [3, Corollary 3.3, 3.5, & 3.7] evaluated the values of $s_{4,n}$ for n = 1, 2, 4, 8, 9, 10, 12, 16, 18, 28, 32, 36, 72, 20, 28, 40, 64, 52, 60, 68, 100, 108, 144, 196, 2/3, 1/2, 4/7, 4/5, 4/9, 4/25, 4/49, 2/5, 3, 1/3, 1/5, 7, 1/7, 25, 1/25, 13, 1/13, and 15. In next lemma we list some values $s_{4,n}$ from [3, Corollary 3.3(ix), (xiii), Corollary 3.5(v), (viii), & Corollary 3.9(i)].

Lemma 2.9. *If $s_{4,n}$ is as defined in (1.9), then*

$$s_{4,18} = 2^{-1/8}\left(-1+35\sqrt{2}+28\sqrt{3}\right)^{-1/8}\left(\sqrt{3}+\sqrt{2}\right),$$

$$s_{4,72} = \left(\sqrt{3}+\sqrt{2}\right)\left(1+35\sqrt{2}-28\sqrt{3}\right)^{1/4},$$

$$s_{4,52} = 2^{-1/4}\left(18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}}\right)^{1/8}\left(3+\sqrt{13}\right)^{3/8},$$

$$s_{4,100} = 2^{-19/8}\left(1+\sqrt{5}\right)^{2}\left(1+5^{1/4}\right),$$

$$s_{4,3} = 2^{-3/8}\left(2+\sqrt{3}\right)^{-1/8}\left(9+4\sqrt{3}+4\sqrt{2+\sqrt{3}}\right)^{1/8}.$$

Theorem 2.10. *We have*

$$\alpha_{9/2} = 2\left(\sqrt{3}-\sqrt{3}\right)^{8}\left(-1+35\sqrt{2}+28\sqrt{3}\right),$$

$$\alpha_{18} = \left(\sqrt{3}-\sqrt{2}\right)^{8}\left(1+35\sqrt{2}-28\sqrt{3}\right)^{-2},$$

$$\alpha_{13} = \left(\sqrt{13}-3\right)^{3}\left(18+5\sqrt{13}-6\sqrt{18+5\sqrt{13}}\right),$$

$$\alpha_{25} = 2^{-13}\left(1+\sqrt{5}\right)^{16}\left(1+5^{1/4}\right)^{-8},$$

$$\alpha_{3/4} = 8\left(9+4\sqrt{3}-4\sqrt{2+\sqrt{3}}\right)\left(97-56\sqrt{3}\right)\left(2+\sqrt{3}\right)^{1/2}.$$

Proof. We set n = 18, 72, 52, 100, and 3 in Theorem 2.8, used the corresponding values of $s_{4,n}$ from Lemma 2.9, and simplify to complete the proof. □

Many other values of α_n can be calculated similarly by appealing to the known values of $s_{4,n}$ from [3].

Theorem 2.11. *If $h_{2,n}$ is as defined in (1.10), then*

$$\alpha_{n/2} = 1 - \left(\frac{h_{2,n}^2 - \sqrt{2}}{h_{2,n}^2}\right)^2.$$

Proof. Transcribing using (1.19) and (1.20), we find that

$$\frac{\phi(q)}{2^{1/4}\phi(q^2)} = \frac{2^{1/4}}{\left(1+\sqrt{1-\alpha}\right)^{1/2}}. \tag{2.6}$$

Setting $q := e^{-\pi\sqrt{n/2}}$ and $\alpha_{n/2} := \alpha\left(e^{-\pi\sqrt{n/2}}\right)$, employing the definition of $h_{2,n}$, and simplifying, we complete the proof. □

The Theorem 2.11 implies that the values of $\alpha_{n/2}$ can easily be calculated if we know the corresponding values of $h_{2,n}$. Yi [17, p. 142, Theorem 9.1.6] (also see [18]) calculated $h_{2,n}$ for n = 1, 2, 1/2, 4, 1/4, 8, and 1/8. Baruah and Saikia [3] evaluated $h_{2,n}$ for n = 3, 1/3, 9, 1/9, 5, and 1/5. Naika, Chandankumar, and Manjunatha [9, p. 2891–2894, Corollary 4.2, 4.4, & 4.6] evaluated $h_{2,n}$ for n = 6, 1/6, 2/3, 3/2, 24, 1/24, 10, 1/10, 14, 7/2, 1/10, and 2/7. Recently, Saikia [14] also evaluated $h_{2,n}$ for n = 6, 1/6, 3/2, 2/3, 10, 5/2, 1/10, 2/5, 50, 25/2, 1/50, 2/25, 14, 7/2, 1/14, 2/7, 98, 47/2, 1/98, and 2/49 by establishing some new theta-function identities. We give some examples now.

Lemma 2.12. *If $h_{2,n}$ is as defined in (1.10), we have*

$$h_{2,3} = \left(1+\sqrt{2}\right)^{1/2}\left(\sqrt{3}-\sqrt{2}\right)^{1/2},$$

$$h_{2,9} = \left(\sqrt{3}+\sqrt{2}\right)\left(2-\sqrt{3}\right),$$

$$h_{2,5} = \sqrt{\frac{1+\sqrt{2}+\sqrt{5}}{1+\sqrt{5}+\sqrt{10}}},$$

$$h_{2,1/14} = \frac{\sqrt{3+4\sqrt{2}+2\sqrt{7}}}{2\sqrt{2}}.$$

For values of $h_{2,3}$ and $h_{2,9}$ see [3, Theorem 4.4(i), (iii)], for $h_{2,5}$ see [3, Theorem 4.5(i)], and for value $h_{2,1/14}$ see [9, p. 2894, Corollary 4.6, (46)].

Theorem 2.13. *We have*

$$\alpha_{3/2} = 2\left(-3-2\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)\left(-1+\sqrt{2}\right)^2\left(\sqrt{2}+\sqrt{3}\right)^2,$$

$$\alpha_{9/2} = 2\left(7-5\sqrt{2}\right)\left(-7+4\sqrt{3}\right)\left(49-20\sqrt{6}\right)\left(2+\sqrt{3}\right)^4,$$

$$\alpha_{5/2} = \left(\sqrt{5}-2\right)\left(1+\sqrt{2}-\sqrt{5}\right)^2\left(1+\sqrt{2}\right)^3,$$

$$\alpha_{1/28} = 16\sqrt{2}\left(3+2\sqrt{7}\right)\Big/\left(3+4\sqrt{2}+2\sqrt{7}\right)^2.$$

Proof. We employ the values of $h_{2.n}$ from Lemma 2.12 in Theorem 2.11.

Theorem 2.14. *If $h_{4,n}$ is as defined in (1.11), then*

$$\alpha_{n/4} = 1 - \left(\frac{\sqrt{2} - h_{4,n}}{h_{4,n}} \right)^4.$$

Proof. Transcribing using (1.19) and (1.21) and simplifying, we find that

$$\frac{\phi(q)}{\sqrt{2}\phi(q^4)} = \frac{\sqrt{2}}{1 + (1 - \alpha)^{1/4}}. \tag{2.7}$$

Setting $q := e^{-\pi\sqrt{n/4}}$ and $\alpha_{n/4} := \alpha\left(e^{-\pi\sqrt{n/4}}\right)$ in (2.7), using the definition of $h_{4,n}$, and simplifying, we complete the proof. □

Yi [18] evaluated $h_{4,n}$ for $n = 1, 2$, and 4. Saikia [13, p. 174, Theorem 4.3] also evaluated $h_{4,n}$ for $n = 1, 2, 1/2, 4, 1/4, 8, 1/8, 3, 1/3, 9, 1/9, 16$, and $1/16$ in terms of the parameter A_n which is equivalent to $h_{4,n}$. Saikia also evaluated $h_{4,n}$ for $n = 5, 1/5, 25, 1/25, 7, 1/7, 49$, and $1/49$ in [14] by establishing two new theta-function identities.

The Theorem 2.14 implies that if we know the explicit values of $h_{4,n}$ then the explicit values of $\alpha_{n/4}$ can easily be determined. For example, setting $n = 1/4$ in Theorem 2.14 and employing the value

$$h_{4,1/4} = \left(-3 + 2\sqrt{2}\right) \Big/ \left(2 - 2\sqrt{2} + \sqrt{2\left(-4 + 3\sqrt{2}\right)}\right) \tag{2.8}$$

in Theorem 2.14, we obtain

$$\alpha_{1/16} = 8\left(16 - 12\sqrt{2} - 15\sqrt{-4 + 3\sqrt{2}} + 12\sqrt{-8 + 6\sqrt{2}}\right)\left(3 + 2\sqrt{2}\right)^4. \tag{2.9}$$

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