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Oscillation criterion for first-order linear differential equations with several delay arguments

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Abstract. By using iterated estimates involving all delay arguments, we establish an oscillation criterion for first-order linear differential equations with several delay arguments. This criterion is focused on the interaction among the delay arguments, instead of converting the original equation into a single delay equation and using existing results. Several examples illustrate the results obtained.

Keywords: Differential equation; Oscillation; Multiple delay arguments

Mathematics Subject Classification: 34k11; 34C10

1. Introduction

In this article we study the oscillation of solutions to the first-order linear differential equation with multiple delays,

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(\tau_i(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

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where $p_i(t)$ and $\tau_i(t)$ are real valued continuous functions, the delay arguments $\tau_i(t)$ are non-decreasing and satisfy $\tau_i(t) < t$ and $\lim_{t\to\infty}\tau_i(t) = \infty$ for $i = 1, 2, \dots, n$.

Let

$$\tau(t) = \min_{1\le i\le n}\{\tau_i(t)\}.$$

For $t_1 \ge t_0$, let $\phi(t) \in C([\tau(t_1), t_1], \mathbb{R})$. Then by a solution of (1.1) with initial function ϕ, we mean a nontrivial function $x(t) \in C([\tau(t_1), \infty), \mathbb{R})$ such that $x(t) = \phi(t)$ for $t \in [\tau(t_1), t_1]$, $x(t)$ is continuously differentiate for $t \ge t_1$, and $x(t)$ satisfies (1.1) for all $t \ge t_1$. As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *non-oscillatory*.

The problem of obtaining sufficient conditions to ensure that all solutions of (1.1) are oscillatory has been studied by a number of authors, see for example [1–5] and the references therein.

First, we present a summary of the known results for the case $p_i(t) \ge 0$. In 1972, Ladas et al. [2] proved that if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}\sum_{i=1}^{n} p_i(s)ds > 1, \tag{1.2}$$

then all solutions of (1.1) are oscillatory. In 1978, Ladde [4] proved that if

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}\sum_{i=1}^{n} p_i(s)ds > \frac{1}{e}, \tag{1.3}$$

then all solutions of (1.1) are oscillatory. In 1982, Ladas and Stavroulakis [3] study the delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \tag{1.4}$$

where p_i, τ_i for $i = 1, 2, \dots, n$ are positive constants. They proved that all solutions are oscillatory under each one of the following two conditions:

$$\Big(\prod_{i=1}^{n} p_i\Big)^{1/n}\Big(\sum_{i=1}^{n}\tau_i\Big) > \frac{1}{e}, \tag{1.5}$$

$$\frac{1}{n}\Big(\sum_{i=1}^{n}(p_i\tau_i)^{1/2}\Big)^2 > \frac{1}{e}. \tag{1.6}$$

In 1984, Hunt and Yorke [1] proved that if there exists a positive constant τ_0 such that $t - \tau_i(t) \le \tau_0$ for $1 \le i \le n$, and

$$\liminf_{t\to\infty}\sum_{i=1}^{n} p_i(t)(t-\tau_i(t)) > \frac{1}{e}, \tag{1.7}$$

then all solutions of (1.1) are oscillatory.

Our objective is to establish new criteria for the oscillation of all solutions to (1.1), which are not covered by the conditions (1.2), (1.3), (1.5), (1.6) and (1.7). We focus on the relationship among the different delay arguments, rather than converting the original equation into a first-order delay differential equation whose oscillatory character is known. In our view, each of the delay arguments will affect the oscillation of solutions to (1.1).

Our main tool is obtaining an estimate for $x(\tau_i(t))/x(t)$ by integrating (1.1); then use this estimate in (1.1) and integrate again. We further develop the idea used in articles such as [6–9]. The criterion obtained here can be applied even if the coefficient p_i take negative values. To illustrate our results we present some examples.

2. Main results

Note that if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution. Therefore, when $x(t)$ does not have zeros, we assume that $x(t)$ is positive. Iterates of the delay functions are denoted as follows $\tau^0(t) = t$, $\tau^1(t) = \tau(t)$, $\tau^2(t) = \tau(\tau(t))$, etc.

Following an idea in [7,8], we define the multiple integral functions

$$\begin{aligned}\phi_{\tau_i,p_{n_1},p_{n_2},\dots,p_{n_j}}(t) = \int_{\tau_i(t)}^{t} p_{n_1}(s_1) \int_{\tau_{n_1}(s_1)}^{\tau_{n_1}(t)} p_{n_2}(s_2) \int_{\tau_{n_2}(s_2)}^{\tau_{n_2}(\tau_{n_1}(t))} p_{n_3}(s_3) \\ \cdots \int_{\tau_{n_{j-1}}(s_{j-1})}^{\tau_{n_{j-1}}(\cdots \tau_{n_1}(t))} p_{n_j}(s_j)\, ds_j \cdots ds_3\, ds_2\, ds_1,\end{aligned} \tag{2.1}$$

where $i, n_1, n_2, \dots, n_j$ are integers in $\{1, 2, \dots, n\}$, and $j = 1, 2, \dots$. If $p_i \geq 0$ for $i = 1, \dots, n$, then $\phi \geq 0$. To illustrate the above definition, we consider the first-order linear differential equation

$$x'(t) + px(t-\tau) + qx(t-\delta) = 0, \quad t \geq t_0, \tag{2.2}$$

where p, q, τ, δ are positive constants. From (2.1), by simple calculations, we have

$$\phi_{\tau,p_{n_1},p_{n_2},\dots,p_{n_j}}(t) = \frac{\tau^j}{j!} p^k q^{j-k}, \quad \phi_{\delta,p_{n_1},p_{n_2},\dots,p_{n_j}}(t) = \frac{\delta^j}{j!} p^k q^{j-k}, \tag{2.3}$$

where $j = 1, 2, \dots$, and k is the number of occurrences of p, while $j - k$ is the number of occurrences of q. Next we prove a lemma to be used in the proof of our main result.

Lemma 2.1. *Let B be an $n \times n$ matrix with $b_{ij} \leq 0$ for $i \neq j$, and $c_i \geq 1$ for $i = 1, \dots, n$. If there exists a solution $\vec{y}$ of $B\vec{y} = \vec{c}$ with $y_i > 0$, then B is invertible, and $y_i \geq f_i > 0$ for $i = 1, \dots, n$, where $B\vec{f} = (1, 1, \dots, 1)^T$.*

Proof. Since $y > 0$, from the ith row of the system $B\vec{y} = \vec{c}$, we have $b_{ii} > 0$. First, using row operations, we decompose B as the product of a lower triangular matrix L, and an upper triangular matrix U.

Multiply row 1 of B by $l_{21} = b_{21}/b_{11} \leq 0$, and subtract it from row 2. Then Multiply row 1 of B by $l_{31} = b_{31}/b_{11} \leq 0$, and subtract it from row 3. Continuing this elimination process, we have zeros below b_{11}, and the system

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & \widetilde{b}_{22} & & \widetilde{b}_{2n} \\ 0 & \widetilde{b}_{32} & & \widetilde{b}_{3n} \\ \vdots & & & \\ 0 & \widetilde{b}_{n2} & & \widetilde{b}_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & 0 \\ -l_{31} & 0 & \ddots & 0 \\ \vdots & & & \\ -l_{n1} & 0 & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 - c_1 \frac{b_{21}}{b_{11}} \\ c_3 - c_1 \frac{b_{21}}{b_{11}} \\ \vdots \\ c_n - c_1 \frac{b_{21}}{b_{11}} \end{bmatrix} \ge \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}.$$

Here $\widetilde{b}_{22} = b_{22} - b_{12}\frac{b_{21}}{b_{11}}$, $\widetilde{b}_{23} = b_{23} - b_{13}\frac{b_{21}}{b_{11}} \le b_{23} \le 0, \dots, \widetilde{b}_{32} = b_{32} - b_{12}\frac{b_{21}}{b_{11}} \le b_{32} \le 0$, $\dots$.

Since $y_i > 0$, $\widetilde{b}_{23} \le 0, \dots$, and $\widetilde{b}_{2n} \le 0$, from the second row, we have $\widetilde{b}_{22}y_2 + \widetilde{b}_{23}y_3 + \dots + \widetilde{b}_{2n}y_n \ge 1$. Then $\widetilde{b}_{22} > 0$.

Continuing with the row operations we have zeros below the diagonal, and we obtain a system that is equivalent to the original system:

$$U \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = L^{-1} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \ge \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

The inequality of vectors is in the componentwise sense, and happens because the main diagonal of L^{-1} consists of 1's, and the other entries are non-negative. Note that matrix B has the decomposition $B = LU$, and that $\det B = \det U > 0$; therefore B is invertible.

From the definition of vectors $\vec{y}$ and $\vec{f}$, we have

$$B[\vec{y} - \vec{f}] = \vec{c} - \vec{1},$$
$$U[\vec{y} - \vec{f}] = L^{-1}[\vec{c} - \vec{1}] \ge \vec{c} - \vec{1} \ge \vec{0}.$$

From the bottom row of this system, we have: $u_{nn} > 0$, $u_{nn}(y_n - f_n) \ge 0$; thus $y_n \ge f_n$. From the second row from the bottom, we have: $u_{n-1,n-1} > 0$, $u_{n-1,n} \le 0$, $y_n \ge f_n$ and $u_{n-1,n-1}(y_{n-1} - f_{n-1}) + u_{n-1,n}(y_n - f_n) \ge 0$; thus $y_{n-1} - f_{n-1} \ge 0$ and $y_{n-1} \ge f_{n-1}$. Doing this back substitution $n-1$ times yields $y_i \ge f_i$ for $i = 1, 2, \dots, n$.

As above, for the vectors $\vec{1}$ and $\vec{0}$, we have

$$B[\vec{f} - \vec{0}] = \vec{1} - \vec{0},$$
$$U[\vec{f} - \vec{0}] = L^{-1}[\vec{1} - \vec{0}] \ge \vec{1} - \vec{0} > \vec{0}.$$

Then the back substitution process yields $f_i > 0$, and completes the proof. □

Next, using (2.1), we define the $n \times n$ matrix $A_1(t)$ with entries

$$a_{1;in_1}(t) = \phi_{\tau_i, p_{n_1}}(t), \tag{2.4}$$

and the n dimensional vector $\vec{f}_1$ as the solution of $(I - A_1)\vec{f}_1 = \vec{1} = (1, 1, \ldots, 1)^T$, when it exists.

Recursively for $m = 2, 3, \ldots$, we define the $n \times n$ matrices $A_m(t)$ with entries

$$\begin{aligned}
&a_{m;in_1}(t) \\
&= \phi_{\tau_i, p_{n_1}}(t) + \sum_{n_2=1}^{n} \phi_{\tau_i, p_{n_1}, p_{n_2}} f_{m-1;n_2}(\tau_{n_2}(\tau_{n_1}(t))) \\
&\quad + \sum_{n_2=1}^{n}\sum_{n_3=1}^{n} \phi_{\tau_i, p_{n_1}, p_{n_2}, p_{n_3}}(t) f_{m-2;n_3}(\tau_{n_2}(\tau_{n_1}(t))) f_{m-1;n_2}(\tau_{n_1}(t)) \\
&\quad + \cdots + \\
&\quad + \sum_{n_2=1}^{n} \cdots \sum_{n_m=1}^{n} \phi_{\tau_i, p_{n_1}, p_{n_2}, \ldots, p_{n_m}}(t) \prod_{k=1}^{m-1} f_{m-k;n_{1+k}}(\tau_{n_m} \cdots (\tau_{n_1}(t)))\,.
\end{aligned}$$

As a short notation we use

$$\begin{aligned}
&a_{m;in_1}(t) \\
&= \sum_{j=1}^{m} \sum_{(n_2, n_3, \ldots, n_j)} \Big[\phi_{\tau_i, p_{n_1}, p_{n_2}, \ldots, p_{n_j}}(t) \prod_{k=1}^{j-1} f_{m-k;n_{1+k}}(\tau_{n_j} \cdots (\tau_{n_1}(t)))\Big].
\end{aligned} \tag{2.5}$$

Here and in the sequel, we use the convention that for $j > n$, $\sum_{i=j}^{n} p = 0$ and $\prod_{i=j}^{n} p = 1$.

The vector $\vec{f}_m$ is defined as the solution to the equation

$$(I - A_m)\vec{f}_m = \vec{1} = (1, 1, \ldots, 1)^T, \tag{2.6}$$

and will be used as a lower bound for $x(\tau_i(t))/x(t)$.

Lemma 2.2. *Let $x(t)$ be a solution of (1.1). Assume that there exist $m \geq 1$ and $t_1^* \geq t_1 > t_0$ such that $\tau^{m+2}(t_1) \geq t_0$, $p_i(t) \geq 0$ for $t \in [\tau^{m+1}(t_1), t_1^*]$ and $x(t) > 0$ for $t \in [\tau^{m+2}(t_1), t_1^*]$. Then*

$$\frac{x(\tau_i(t))}{x(t)} \geq f_{m;i}(t) \quad \forall t \in [t_1, t_1^*], \tag{2.7}$$

where $i = 1, 2, \ldots, n$, $m \geq 1$, and $f_{m;i}(t)$ is defined by (2.6).

Proof. Since $p_i(\cdot)$ and $x(\tau_i(\cdot))$ are non-negative on the interval $[\tau^{m+1}(t_1), t_1^*]$, from (1.1), we have

$$x'(t) = -\sum_{i=1}^{n} p_i(t) x(\tau_i(t)) \leq 0, \qquad \text{for } t \in [\tau^{m+1}(t_1), t_1^*]. \tag{2.8}$$

Therefore $x(t)$ is non-increasing, and thus $x(\tau_i(t)) \geq x(t)$. Integrating (1.1) from $\tau_i(t)$ to t, we have

$$x(\tau_i(t)) = x(t) + \sum_{n_1=1}^{n} \int_{\tau_i(t)}^{t} p_{n_1}(s_1) x(\tau_{n_1}(s_1)) ds_1 \,. \tag{2.9}$$

Integrating (1.1) from $\tau_{n_1}(s_1)$ to $\tau_{n_1}(t)$, and substituting in the above expression, we have

$$\begin{aligned} x(\tau_i(t)) = x(t) + \sum_{n_1=1}^{n} \int_{\tau_i(t)}^{t} & p_{n_1}(s_1) \\ \times \Big[x(\tau_{n_1}(t)) + & \sum_{n_2=1}^{n} \int_{\tau_{n_1}(s_1)}^{\tau_{n_1}(t)} p_{n_2}(s_2) x(\tau_{n_2}(s_2)) ds_2 \Big] ds_1 \,. \end{aligned} \tag{2.10}$$

Integrating (1.1) from $\tau_{n_2}(s_2)$ to $\tau_{n_2}(\tau_{n_1}(t))$, and substituting in the above expression, we have

$$\begin{aligned} x(\tau_i(t)) = x(t) + \sum_{n_1=1}^{n} \int_{\tau_i(t)}^{t} p_{n_1}(s_1) \Big[x(\tau_{n_1}(t)) + \sum_{n_2=1}^{n} \int_{\tau_{n_1}(s_1)}^{\tau_{n_1}(t)} p_{n_2}(s_2) \\ \times \Big[x(\tau_{n_2}(\tau_{n_1}(t))) + \sum_{n_3=1}^{n} \int_{\tau_{n_2}(s_2)}^{\tau_{n_2}(\tau_{n_1}(t))} p_{n_3}(s_3) x(\tau_{n_3}(s_3)) ds_3 \Big] ds_2 \Big] ds_1 \,. \end{aligned} \tag{2.11}$$

For $t \in [\tau^{m-1}(t_1), t_1^*]$, from (2.9), using that $x(\tau_{n_1}(t)) \leq x(\tau_{n_1}(s_1))$, and (2.1), we obtain

$$\begin{aligned} x(\tau_i(t)) &\geq x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \int_{\tau_i(t)}^{t} p_{n_1}(s_1) ds_1 \\ &= x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \phi_{\tau_i, p_{n_1}}(t), \end{aligned} \tag{2.12}$$

where $i = 1, 2, \ldots, n$. Dividing by $x(t)$ which is positive, we have a system of inequalities (in the componentwise sense)

$$\vec{y} \geq \vec{1} + A_1 \vec{y},$$

where $y_i = \frac{x(\tau_i(t))}{x(t)}$, $\vec{1} = (1, 1, \ldots, 1)^T$, and A_1 is defined by (2.4). That is from the inequality (2.12), we define the matrix $A_1(t)$ with entries

$$a_{1;in_1}(t) = \phi_{\tau_i, p_{n_1}}(t),$$

where $i = 1, \ldots, n$ and $n_1 = 1, \ldots, n$. Since $\phi_{\tau_i, p_{n_1}}(t) \geq 0$ and $x(t) > 0$, we have the necessary conditions for applying Lemma 2.1 with $B = I - A_1$. Therefore

$$\frac{x(\tau_i(t))}{x(t)} \geq f_{1;i}(t) > 0 \quad \text{for } t \in [\tau^{m-1}(t_1), t_1^*],\ i = 1, 2, \ldots, n, \tag{2.13}$$

where $(I - A_1)\vec{f}_1 = \vec{1}$.

For $t \in [\tau^{m-2}(t_1), t_1^*]$, using that

$$x(\tau_{n_2}(s_2)) \geq x(\tau_{n_2}(\tau_{n_1}(t))) \geq f_{1;n_2}(\tau_{n_1}(t)) x(\tau_{n_1}(t)),$$

(2.10), and (2.1), we obtain

$$
\begin{aligned}
x(\tau_i(t)) &\geq x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \int_{\tau_i(t)}^{t} p_{n_1}(s_1)ds_1 \\
&\quad + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \int_{\tau_i(t)}^{t} p_{n_1}(s_1) \sum_{n_2=1}^{n} f_{1;n_2}(\tau_{n_1}(t)) \int_{\tau_{n_1}(s_1)}^{\tau_{n_1}(t)} p_{n_2}(s_2)ds_2ds_1 \\
&= x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t))\phi_{\tau_i,p_{n_1}}(t) \\
&\quad + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \sum_{n_2=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2}}(t) f_{1;n_2}(\tau_{n_1}(t)) \\
&= x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t))\Big[\phi_{\tau_i,p_{n_1}}(t) + \sum_{n_2=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2}}(t) f_{1;n_2}(\tau_{n_1}(t))\Big].
\end{aligned}
$$

From the above inequality, we define the matrix $A_2(t)$ with entries

$$
a_{2;in_1}(t) = \phi_{\tau_i,p_{n_1}}(t) + \sum_{n_2=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2}}(t) f_{1;n_2}(\tau_{n_1}(t)),
$$

where $i = 1, \ldots, n$ and $n_1 = 1, \ldots, n$. Setting $y_i = x(\tau_i(t))/x(t)$ and applying Lemma 2.1 with $B = I - A_2$, we obtain

$$
\frac{x(\tau_i(t))}{x(t)} \geq f_{2;i}(t) > 0 \quad \text{for } t \in [\tau^{m-2}(t_1), t_1^*],\ i = 1, 2, \ldots, n, \tag{2.14}
$$

where $(I - A_2)\vec{f}_2 = \vec{1}$.

For $t \in [\tau^{m-3}(t_1), t_1^*]$, using

$$
x(\tau_{n_3}(s_3)) \geq x(\tau_{n_3}(\tau_{n_2}(\tau_{n_1}(t)))) \geq f_{1;n_3}(\tau_{n_2}(\tau_{n_1}(t))) f_{2;n_2}(\tau_{n_1}(t)) x(\tau_{n_1}(t)),
$$

(2.11) and (2.1), we obtain

$$
\begin{aligned}
&x(\tau_i(t)) \\
&\geq x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \int_{\tau_i(t)}^{t} p_{n_1}(s_1)\,ds_1 \\
&\quad + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \sum_{n_2=1}^{n} \int_{\tau_i(t)}^{t} p_{n_1}(s_1) \int_{\tau_{n_1}(s_1)}^{\tau_{n_1}(t)} p_{n_2}(s_2) f_{2;n_2}(\tau_{n_1}(t))\,ds_2\,ds_1 \\
&\quad + \sum_{n_1=1}^{n} x(\tau_{n_1}(t)) \sum_{n_2=1}^{n} f_{2;n_2}(\tau_{n_1}(t)) \sum_{n_3=1}^{n} f_{1;n_3}(\tau_{n_2}(\tau_{n_1}(t))) \\
&\quad \times \int_{\tau_i(t)}^{t} p_{n_1}(s_1) \int_{\tau_{n_1}(s_1)}^{\tau_{n_1}(t)} p_{n_2}(s_2) \int_{\tau_{n_2}(s_2)}^{\tau_{n_2}(\tau_{n_1}(t))} p_{n_3}(s_3)ds_3\,ds_2\,ds_1
\end{aligned}
$$

$$= x(t) + \sum_{n_1=1}^{n} x(\tau_{n_1}(t))\Big[\phi_{\tau_i,p_{n_1}}(t) + \sum_{n_2=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2}}(t) f_{1;n_2}(\tau_{n_1}(t))$$
$$+ \sum_{n_2=1}^{n}\sum_{n_3=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2},p_{n_3}}(t) f_{1;n_3}(\tau_{n_2}(\tau_{n_1}(t))) f_{2;n_2}(\tau_{n_1}(t))\Big].$$

Let $A_3(t)$ be the matrix with entries

$$a_{3;in_1}(t) = \phi_{\tau_i,p_{n_1}}(t) + \sum_{n_2=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2}}(t) f_{1;n_2}(\tau_{n_1}(t))$$
$$+ \sum_{n_2=1}^{n}\sum_{n_3=1}^{n} \phi_{\tau_i,p_{n_1},p_{n_2},p_{n_3}}(t) f_{1;n_3}(\tau_{n_2}(\tau_{n_1}(t))) f_{2;n_2}(\tau_{n_1}(t)),$$

where $i = 1, \dots, n$ and $n_1 = 1, \dots, n$. Setting $y_i = x(\tau_i(t))/x(t)$ and applying Lemma 2.1 with $B = I - A_3$, we obtain

$$\frac{x(\tau_i(t))}{x(t)} \geq f_{3;i}(t) > 0, \quad \text{for } [\tau^{m-3}(t_1), t_1^*],\ i = 1, 2, \dots, n \tag{2.15}$$

where $(I - A_3)\vec{f}_3 = \vec{1}$.

Repeating the above procedure, in general, we have

$$a_{m;in_1}(t) = \sum_{j=1}^{m} \sum_{(n_2,n_3,\dots,n_j)} \phi_{\tau_i,p_{n_1},\dots,p_{n_j}}(t) \prod_{k=1}^{j-1} f_{m-k;n_{1+k}}(\tau_{n_j} \cdots (\tau_{n_1}(t))),$$

where $t \in [t_1, t_1^*]$, $\sum_{(n_2,n_3,\dots,n_j)}$ is defined in (2.5). Repeating the above process m times we obtain the statement of the lemma. □

Remark 2.3. Note that matrix A_m requires computing $\phi_{\tau_i,p_{n_1},\dots,p_{n_m}}$, i.e. m integrals, which in turn require t being large enough so that $\tau^{m+1}(t) \geq t_0$. For some functions p_i, as m increases the matrix A_m may become nearly singular which leads to some components of f_m being large. Also if some p_i's having negative values, f_m may have negative components. This is the case in the next statement.

Theorem 2.4. *Assume that there exist some increasing sequences $\{t_k\}_{k=1}^{\infty}$ approaching ∞ and some bounded sequences of positive integers $\{m_k\}_{k=1}^{\infty}$, such that for each k, $\tau^{m_k+2}(t_k) \geq t_0$, and $p_i(t) \geq 0$ for $t \in [\tau^{m_k+1}(t_k), t_k]$. If for each t_k, there exists $i \in \{i = 1, 2, \dots, n\}$ such that $f_{1;i}(t_k) > 0$, $f_{2;i}(t_k) > 0, \dots, f_{m_{k-1}-1;i}(t_k) > 0$, and $f_{m_{k,i}}(t_k)$ is not defined or is negative. Then every solution of (1.1) is oscillatory.*

Proof. Let $x(t)$ be a solution to (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \in [\tau^{m_k+2}(t_k), t_k]$. By Lemma 2.2 (in the case $t_k = t_k^*$), we get $f_{m_{k,i}}(t_k) > 0$, which is a contradiction that completes the proof. □

Remark 2.5. For single delay equations, upper bounds for $x(\tau^2(t))/x(\tau(t))$ was obtained in [7]. These bounds lead to another criterion for oscillation of solutions. However, for the multiple delay Eq. (1.1), we are unable to find an upper bound for $x(\tau_{n_1}(\tau_{n_2}(t)))/x(\tau_{n_2}(t))$ (without reducing (1.1) to an inequality of a single delay).

To illustrate the construction of the lower bounds $f_{m;i}$, we consider the differential equation (2.2). From (2.3), we define the matrix A_1 with entries:

$$a_{1;11} = \phi_{\tau,p} = p\tau, \quad a_{1;12} = \phi_{\tau,q} = q\tau,$$
$$a_{1;21} = \phi_{\delta,p} = p\delta, \quad a_{1;22} = \phi_{\delta,q} = q\delta\,.$$

Then by Lemma 2.2 with $B = I - A_1$, the lower bounds are

$$f_{1;1} = \frac{1 - a_{1;22} + a_{1;12}}{(1 - a_{1;11})(1 - a_{1;22}) - a_{1;12}a_{1;21}} = \frac{1 - q\delta + q\tau}{1 - p\tau - q\delta},$$
$$f_{1;2} = \frac{1 - a_{1;11} + a_{1;21}}{(1 - a_{1;11})(1 - a_{1;22}) - a_{1;12}a_{1;21}} = \frac{1 - p\tau + p\delta}{1 - p\tau - q\delta}\,.$$

The matrix A_2 has entries

$$\begin{aligned}
a_{2;11} &= \phi_{\tau,p} + f_{1;1}\phi_{\tau,p,p} + f_{2;1}\phi_{\tau,p,q}\\
&= p\tau + \frac{\tau^2}{2!}p^2 f_{1;1} + \frac{\tau^2}{2!}pqf_{2;1} = p\Big(\tau + \frac{\tau^2}{2!}(pf_{1;1} + qf_{2;1})\Big),\\
a_{2;12} &= \phi_{\tau,q} + f_{1;1}\phi_{\tau,q,p} + f_{2;1}\phi_{\delta,q,q}\\
&= q\tau + \frac{\tau^2}{2!}pqf_{1;1} + \frac{\tau^2}{2!}q^2 f_{2;1} = q\Big(\tau + \frac{\tau^2}{2!}(pf_{1;1} + qf_{2;1})\Big),\\
a_{2;21} &= \phi_{\delta,p} + f_{1;1}\phi_{\delta,p,p} + f_{2;1}\phi_{\delta,p,q}\\
&= p\delta + \frac{\delta^2}{2!}p^2 f_{1;1} + \frac{\delta^2}{2!}pqf_{2;1} = p\Big(\delta + \frac{\delta^2}{2!}(pf_{1;1} + qf_{2;1})\Big),\\
a_{2;22} &= \phi_{\delta,q} + f_{1;1}\phi_{\delta,q,p} + f_{2;1}\phi_{\delta,q,q}\\
&= q\delta + \frac{\delta^2}{2!}pqf_{1;1} + \frac{\delta^2}{2!}q^2 f_{2;1} = q\Big(\delta + \frac{\tau^2}{2!}(pf_{1;1} + qf_{2;1})\Big).
\end{aligned}$$

Then by Lemma 2.2 with $B = I - A_2$,

$$\begin{aligned}
f_{2;1} &= \frac{1 - a_{2;22} + a_{2;12}}{(1 - a_{2;11})(1 - a_{2;22}) - a_{2;12}a_{2;21}}\\
&= \frac{1 + q(\tau - \delta) + \frac{1}{2!}(\tau^2 - \delta^2)(pqf_{1;1} + q^2 f_{2;1})}{1 - p\tau - \frac{\tau^2}{2!}p^2 f_{1;1} - \frac{\tau^2}{2!}pqf_{2;1} - q\delta - \frac{\delta^2}{2!}pqf_{1;1} - \frac{\delta^2}{2!}q^2 f_{2;1}},\\
f_{2;2} &= \frac{1 - a_{2;11} + a_{2;21}}{(1 - a_{2;11})(1 - a_{2;22}) - a_{2;12}a_{2;21}}\\
&= \frac{1 + p(\delta - \tau) + \frac{1}{2!}(\delta^2 - \tau^2)(p^2 f_{1;1} + pqf_{2;1})}{1 - p\tau - \frac{\tau^2}{2!}p^2 f_{1;1} - \frac{\tau^2}{2!}pqf_{2;1} - q\delta - \frac{\delta^2}{2!}pqf_{1;1} - \frac{\delta^2}{2!}q^2 f_{2;1}}.
\end{aligned}$$

For $m > 2$, the matrix A_m has entries

$$a_{m;11} = p\sum_{i=1}^{m}\frac{\tau^i}{i!}\prod_{j=1}^{i-1}(pf_{1;m-j} + qf_{2;m-j}),$$
$$a_{m;12} = q\sum_{i=1}^{m}\frac{\tau^i}{i!}\prod_{j=1}^{i-1}(pf_{1;m-j} + qf_{2,m-j}),$$
$$a_{m;21} = p\sum_{i=1}^{m}\frac{\delta^i}{i!}\prod_{j=1}^{i-1}(pf_{1;m-j} + qf_{2,m-j}),$$

$$a_{m;22} = q \sum_{i=1}^{m} \frac{\delta^i}{i!} \prod_{j=1}^{i-1} (p f_{1;m-j} + q f_{2;m-j}) .$$

Note that $a_{m;11} a_{m;22} = a_{m;21} a_{m;12}$. Then by Lemma 2.2,

$$f_{m:1} = \frac{1 - a_{m;22} + a_{m;12}}{(1 - a_{m;11})(1 - a_{m;22}) - a_{m;12} a_{m;21}} = \frac{1 - a_{m;22} + a_{m;12}}{1 - a_{m;11} - a_{m;22}},$$

$$f_{m;2} = \frac{1 - a_{m;11} + a_{m;21}}{(1 - a_{m;11})(1 - a_{m;22}) - a_{m;12} a_{m;21}} = \frac{1 - a_{m;11} + a_{m;21}}{1 - a_{m;11} - a_{m;22}} .$$

For a large index m, the quantities $f_{1;m}$ and $f_{2;m}$ may not be defined (because of a division by zero), or may be negative. From Theorem 2.4 we have the following corollary.

Corollary 2.6. *For Eq. (2.2), the matrix A_m has entries*

$$a_{m;in_1} = \sum_{k=1}^{m} \frac{\tau_i^k}{k!} p_j \prod_{k^*=1}^{k-1} \sum_{k_*=1}^{n} p_{k_*} f_{k_*;m-k^*} .$$

If there exists an integer $m^* \geq 1$ such that $f_{1;i} > 0$, $f_{2;i} > 0, \dots, f_{m^*-1;i} > 0$, and $f_{m^*;i}$ not defined or negative, for some $i \in \{1, 2, \dots, n\}$, then all solutions of (2.2) are oscillatory.

3. Examples

We present numerical calculations obtained using the Matlab software. The improvement may not seem much. Nevertheless, as far as we know, such cases are not covered by previous publications.

Example 3.1. Consider the differential equation with constant delays

$$x'(t) + e^{-\frac{\pi}{2} a} x\big(t - \frac{\pi}{2}\big) + a e^{-2\pi a} x\big(t - 2\pi\big) = 0. \tag{3.1}$$

It is easy to verify that $x_1(t) = e^{-at} \sin t$ and $x_2(t) = e^{-at} \cos t$ for $a \in \mathbb{R}$ are oscillatory solutions of (3.1). The characteristic equation of (3.1) is $f(\lambda) = \lambda + e^{-\pi(\lambda+a)/2} + a e^{-2\pi(\lambda+a)} = 0$. It is easy to see that $f(-a) = 1 > 0$, and $f(-a + 0.49559) < 0$ provided with $a > 0.99908$. Therefore, there exist $\lambda_0 \in (-a, -a + 0.49559)$, such that $f(\lambda_0) = 0$. It follows that $x_3(t) = e^{\lambda_0 t}$ for $\lambda_0 \in (-a, -a + 0.49559)$ and $a > 0.99908$ is a non-oscillatory solution of (3.1).

Taking $a = \frac{111}{120}$ (see [3, Example 4.2]), it is easy to verify that none of the conditions (1.5) and (1.6) is satisfied. Taking $a = \frac{112}{120} \approx 0.9333$, none of the conditions (1.3), (1.5) and (1.6) is satisfied. Taking $a = \frac{115}{120} \approx 0.9583$, none of the conditions (1.3), (1.5), (1.6) and (1.7) is satisfied. To find the maximum value of a for which there are non-oscillatory solutions, we consider the limit as $m \to \infty$. Computations show that the maximum possible value of a is close to 0.99908. By Table 1 and Theorem 2.4 every solution of (3.1) has at least one zero on $[t_k - (2 + m_k), t_k]$, where m_k depends on a. So every solution of (3.1) is oscillatory.

Example 3.2. Consider the differential equation with unbounded delays

$$x'(t) + \frac{p_1}{t} x(\tau_1 t) + \frac{p_2}{t} x(\tau_2 t) = 0, \quad t \geq t_0, \tag{3.2}$$

where p_1, p_2, τ_1, τ_2 are positive constants with $0 < \tau_1, \tau_2 < 1$.

Table 1
Numerical results for different value of a.

$a \to 0.99908$	$f_{1;2} > 0$	$f_{2;2} > 0$	$\cdots$	$f_{m_k-1;2} > 0$	$f_{m_k;2} \leq 0$
0.9333	3.3634	7.4799	$\cdots$	$f_{5;2} \approx 54.991$	$f_{6;2} \approx -928.61$
0.9583	3.2129	6.6626	$\cdots$	$f_{8;2} \approx 68.972$	$f_{9;2} \approx -467.75$
0.9750	3.1203	6.2067	$\cdots$	$f_{2,10} \approx 91.27$	$f_{11,2} \approx -344.9$
0.9916	3.0332	5.8066	$\cdots$	$f_{20;2} \approx 104.5$	$f_{21;2} \approx -442.8$
0.9990	2.9964	5.6457	$\cdots$	$f_{209;2} \approx 428.6$	$f_{210;2} \approx -68.08$
0.99905	2.9962	5.6446	$\cdots$	$f_{327;2} \approx 290.3$	$f_{328;2} \approx -80.32$

From (2.1), by simple calculations we have

$$\phi_{\tau_1,p_1}(t) = \int_{\tau_1 t}^{t} \frac{p_1}{s_1} ds_1 = p_1\big(\ln t - \ln(\tau_1 t)\big) = p_1 \ln\Big(\frac{1}{\tau_1}\Big),$$

$$\phi_{\tau_1,p_2}(t) = \int_{\tau_1 t}^{t} \frac{p_2}{s_1} ds_1 = p_2\big(\ln t - \ln(\tau_1 t)\big) = p_2 \ln\Big(\frac{1}{\tau_1}\Big),$$

$$\begin{aligned}\phi_{\tau_1,p_1,p_2}(t) &= \int_{\tau_1 t}^{t} \frac{p_1}{s_1} \int_{\tau_2 s_1}^{\tau_2 t} \frac{p_2}{s_2} ds_2 ds_1 = p_1 p_2 \int_{\tau_1 t}^{t} \frac{1}{s_1} \ln\Big(\frac{t}{s_1}\Big) ds_1 \\ &= p_1 p_2 \int_{1/\tau_1}^{1} (-\ln u) d \ln u = p_1 p_2 \Big(-\frac{\ln^2 u}{2}\Big)\Big|_{1/\tau_1}^{1} \\ &= \frac{p_1 p_2}{2!} \ln^2\Big(\frac{1}{\tau_1}\Big),\end{aligned}$$

$$\begin{aligned}\phi_{\tau_1,p_2,p_1}(t) &= \int_{\tau_1 t}^{t} \frac{p_2}{s_1} \int_{\tau_1 s_1}^{\tau_1 t} \frac{p_1}{s_2} ds_2 ds_1 = p_1 p_2 \int_{\tau_1 t}^{t} \frac{1}{s_1} \ln\Big(\frac{t}{s_1}\Big) ds_1 \\ &= p_1 p_2 \int_{1/\tau_1}^{1} (-\ln u) d \ln u = p_1 p_2 \Big(-\frac{\ln^2 u}{2}\Big)\Big|_{1/\tau_1}^{1} \\ &= \frac{p_1 p_2}{2!} \ln^2\Big(\frac{1}{\tau_1}\Big),\end{aligned}$$

$$\begin{aligned}\phi_{\tau_1,p_1,p_1,p_1}(t) &= \int_{\tau_1 t}^{t} \frac{p_1}{s_1} \int_{\tau_1 s_1}^{\tau_1 t} \frac{p_1}{s_2} \int_{\tau_1 s_2}^{\tau_1^2 t} \frac{p_1}{s_3} ds_3 ds_2 ds_1 \\ &= p_1^3 \int_{\tau_1 t}^{t} \frac{1}{s_1} \int_{\tau_1 s_1}^{\tau_1 t} \frac{1}{s_2} \ln\Big(\frac{\tau_1 t}{s_2}\Big) ds_2 ds_1 \\ &= p_1^3 \int_{\tau_1 t}^{t} \frac{1}{s_1} \int_{t/s_1}^{1} (-\ln u) d \ln u ds_1 \\ &= \frac{p_1^3}{2!} \int_{\tau_1 t}^{t} \frac{1}{s_1} \ln^2\Big(\frac{t}{s_1}\Big) ds_1 = \frac{p_1^3}{3!} \ln^3\Big(\frac{1}{\tau_1}\Big)\end{aligned}$$

$$\begin{aligned}\phi_{\tau_1,p_1,p_1,\ldots,p_1}(t) &= \int_{\tau_1(t)}^{t} p_1(s_1) \int_{\tau_1(s_1)}^{\tau_1(t)} p_1(s_2) \cdots \int_{\tau_1(s_{j-1})}^{\tau_1^{i-1}(t)} p_1(s_j) ds_j \cdots ds_1 \\ &= \frac{p_1^j}{j!} \ln^j\Big(\frac{1}{\tau_1}\Big),\end{aligned}$$

and so on. In general, we have

$$\phi_{\tau_i, p_{n_1}, p_{n_2}, \ldots, p_{n_j}}(t) = \frac{1}{j!} p_1^k p_2^{j-k} \ln^j\left(\frac{1}{\tau_i}\right),$$

where i, n_1, n_2, $\ldots$, n_j are elements of $\{1, 2\}$, $j = 1, 2, \ldots$, and k is the number of occurrences of p_1.

Taking $p_1 = 0.888/e$, $p_2 = 1/(2e)$, $\tau_1 = 1/\sqrt{e}$, $\tau_2 = 1/e$, we get

$$\begin{aligned} f_{1;1} &\approx 1.3911, \quad f_{2;1} \approx 1.5561, \quad f_{3;1} \approx 1.6424, \ldots, \\ f_{107;1} &\approx 3.5957, \quad f_{108;1} \approx -3.3872; \\ f_{1;2} &\approx 1.7822, \quad f_{2;2} \approx 2.2942, \quad f_{3;2} \approx 2.6140, \ldots, \\ f_{107;2} &\approx 10.151, \quad f_{108;2} \approx -16.794. \end{aligned}$$

By Theorem 2.4, all solutions of (3.2) are oscillatory.

Example 3.3. Consider the differential equation with two constant delays and sign-changing coefficients

$$x'(t) + p(t)x(t-\tau) + q(t)x(t-\delta) = 0, \quad t \geq 0, \tag{3.3}$$

where $\tau = 0.5$, $\delta = 1$, $p(t)$ and $q(t)$ are the periodic functions

$$p(t) = \begin{cases} -\frac{1}{2e} + \frac{1}{e}t & \text{for } 0 \leq t < 1, \\ \frac{1}{2e} & \text{for } 1 \leq t < 6, \\ \frac{1}{2e} - \frac{1}{e}(t-6) & \text{for } 6 \leq t < 7, \\ -\frac{1}{2e} & \text{for } 7 \leq t < 8, \end{cases}$$

$$q(t) = \begin{cases} -\frac{1}{2e} + \frac{1}{e}t & \text{for } 0 \leq t < 1, \\ \frac{1}{2e} & \text{for } 1 \leq t < 4, \\ \frac{1}{2e} + (0.66 - \frac{1}{2e})(t-4) & \text{for } 4 \leq t < 5, \\ 0.66 & \text{for } 5 \leq t < 6, \\ 0.66 - (0.66 + \frac{1}{2e})(t-6) & \text{for } 6 \leq t < 7, \\ -\frac{1}{2e} & \text{for } 7 \leq t < 8. \end{cases}$$

Taking $t = t_k = 8k + 6$ and $m = 4$, we have

$$\begin{aligned} a_{1;11} &\approx 0.091, \quad a_{1;22} \approx 0.1839, \\ f_{1;1} &\approx 1.2540, \quad f_{1;2} \approx 1.5080, \\ a_{2;11} &\approx 0.103, \quad a_{2;22} \approx 0.230, \\ f_{2;1} &\approx 1.3114, \quad f_{2;2} \approx 1.6930, \\ a_{3;11} &\approx 0.1057 \quad a_{3;22} \approx 0.2433, \end{aligned}$$

$$f_{3;1} \approx 1.3249, \quad f_{3;2} \approx 1.7478,$$
$$a_{4;11} = p \sum_{i=1}^{4} \frac{\tau^i}{i!} \prod_{j=2}^{i-1} \big(pf_{4-j;1} + qf_{4-j;2}\big) \approx 0.2630,$$
$$a_{4;22} = q \sum_{i=1}^{4} \frac{\delta^i}{i!} \prod_{j=1}^{i-1} (pf_{4-j;1} + qf_{4-j;2}) \approx 0.8852,$$
$$1 - a_{4;11} - a_{4;22} \approx -0.1482 < 0\,.$$

Then the numerators of both $f_{4;1}$ and $f_{4;2}$ are positive while their common denominator is negative. By Theorem 2.4, all solutions of (3.3) are oscillatory.

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Appendix

In this section, for completeness, we give the algorithm on Matlab software used in Example 3.1. For other examples, the algorithm is similar and is omitted.

ALGORITHM: Data of the last line of Table 1:

```
m = 220; a = 0.99905;
p = exp(-a * pi/2); tau = pi/2;
q = a * exp(-a * pi * 2); delta = 2 * pi;
f = zeros(1, m); g = zeros(1, m);
A1 = zeros(1, m); A2 = zeros(1, m); A3 = zeros(1, m); A4 = zeros(1, m);
A1(1) = p * tau; A2(1) = q * tau; A3(1) = p * delta; A4(1) = q * delta;
f(1) = (1 - q * delta + q * tau)/(1 - p * tau - q * delta);
g(1) = (1 - p * tau + p * delta)/(1 - p * tau - q * delta);
for k = 2 : 1 : m
  b = 0; c = 0;
  for i = 2 : 1 : k
      fn0 = 1; ai2 = 1;
    for l = 1 : 1 : i
      ai2 = ai2 * l;
    end
    ai1 = (1/ai2) * tau^i; ai3 = (1/ai2) * delta^i;
      for j = 1 : 1 : i - 1
        fn0 = (p * f(k - j) + q * g(k - j)) * fn0;
      end
    b = ai1 * fn0 + b; c = ai3 * fn0 + c;
  end
  A1(k) = p * b + p * tau; A2(k) = q * b + q * tau;
  A3(k) = p * c + p * delta; A4(k) = q * c + q * delta;
  f(k) = (1 - A4(k) + A2(k))/(1 - A1(k) - A4(k));
  g(k) = (1 - A1(k) + A3(k))/(1 - A1(k) - A4(k));
end  f  g
```

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